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SOME FIX-POINT THEOREMS FOR THE COMMUTATIVE ITERATION SEMIGROUPS

Many results concerning fixed point theorems of some families of self maps of a set X, are known. (Compare: [1], p.139) for a semi-flow on a polyhedron; ([1], p.98-101) for some semigroups of continuous affine maps; see also [1], p.109).

A various topological (or analytic) assumptions play an important role in the mentioned results. In particular, the continuity is important.

The purpose of this paper is to prove some fixed point theorems for some commutative iteration semigroups without any assumptions refering to continuity. The commutative families will be treated, too. We shall also give some examples.

1. Preliminaries

For an arbitrary set $X \neq \emptyset$, let $\overline{\overline{X}}$ be the cardinal number of X and let X^X denote the set $\{f: f$ is a map with domain X, whose range lies in $X\}$. The set of all bijections of X will be denoted by Per X.

Let $\mathfrak{F} \subset X^X$ and $\mathfrak{F} \neq \emptyset$. The family $\mathfrak{F}$ is called commutative if $f \circ h = h \circ f$ for all $f,h \in \mathfrak{F}$. A set $A \subset X$ is said to be $\mathfrak{F}$-invariant (strongly $\mathfrak{F}$-invariant) if $f(A) \subset A$ $(f(A) = A)$ for all $f \in \mathcal{F}$. A point $x_0 \in X$ is called a fixed point of $\mathcal{F}$ if $f(x_0) = x_0$ for every $f \in \mathcal{F}$ [1].

Throughout this paper R, Q, Z, N will denote the set of reals, rationals, integers, positive integers numbers, respectively.

The usual algebraic notation

$$uA = \{ua: a \in A\}, \quad A + B = \{a + b : a \in A, b \in B\},$$

will be used (for $u \in R$, $A \subset R$, $B \subset R$).

If $m, n \in N$, then $m|n$ denotes that n is divisible by m. Let $(P, +)$ be the commutative semigroup and let $n \in N$. It is said to be n-divisible if for every $p \in P$ there exists $t \in P$ such that $n \cdot t = p$, where $n \cdot p = \underbrace{t + \dots + t}_{n \text{ times}}$. For any $f \in X^X$ we define the iterates f^n by

$$f^0 = I, \quad f^{n+1} = f \circ f^n, \quad (n \in N \cup \{0\}),$$

where I is the identity map on X. A family $\{U_t: t \in P\}$ of self-mappings of a set X is called an iteration semigroup, if $U_t \circ U_s = U_{t+s}$ for all $t, s \in P$.

If $A \subset X$, then $f|_A$ denotes the restriction of the function $f \in X^X$ to the set A.

2. Auxiliary results

The following lemma is obvious.

Lemma 1. If $f \in X^X$, $A \subset X$ and A is f-invariant, then

(1) A is f^k-invariant for all $k \in N \cup \{0\}$ and $f^k(A)$ is f^j-invariant for every $k, j \in N \cup \{0\}$,

(2) $f^k|_A = (f|_A)^k$ for all $k \in N \cup \{0\}$.

Lemma 2. If $\overline{\overline{X}} = n < \infty$, $f \in X^X$ then

(3) there exists $k_0 = \min\{k \in N \cup \{0\}: f^k(X) = f^{k+1}(X)\}$, $k_0 \leq n-1$ and the set $A = f^{k_0}(X)$ is strongly f-invariant,

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1) If $\mathcal{F} = \{f\}$ then the curly brackets will be omitted.

(4) $f|_A$ is a bijection,

(5) there exists an l_0, $l_0 | n!$ such that

$$f^{l_0}|_A = (f|_A)^{l_0} = I_A.$$

P r o o f . Part (3) is an immediate consequence of the finitness of X.

The definition of A implies (4). Thus $f|_A$ is a bijection. Whence $B = \left\{(f|_A)^k\right\}_{k=1}^{\infty}$ is a cyclic subgroup of Per A and by Lagrange Theorem we obtain

$$\bar{\bar{B}} | \overline{\overline{\text{Per } A}} \quad \text{i.e.} \quad \bar{\bar{B}} | \bar{\bar{A}}!.$$

Let us define $l_0 = \bar{\bar{B}}$. Then $l_0 | \bar{\bar{A}}! | \bar{\bar{X}}!$ and so $l_0 | n!$. Moreover, $(f|_A)^{l_0} = I|_A$. The rest is a consequence of Lemma 1, (2).

Making use of Lemma 2 we obtain the following lemma.

L e m m a 3. If $\bar{\bar{X}} = n < \infty$, $f \in X^X$ and $n_0 = n!$ then the set $B = f^{n_0}(X)$ is strongly f-invariant and $f^{n_0}|_B = I|_B$.

L e m m a 4. Let $r \in \mathbb{N}$, $r \geqslant 2$ and let the family $\{g_1, \ldots, g_r\} \subset X^X$ be commutative. Then

$$(i) \qquad (g_1 \circ g_2 \circ \ldots \circ g_r)(X) \subset \bigcap_{j=1}^{r} g_j(X).$$

P r o o f . Let $i, j \in \{1, \ldots, r\}$, and define the maps G, $\tilde{g}_{ij}$ and $\tilde{G}_i$ by

$$G = g_1 \circ \ldots \circ g_r,$$

$$\tilde{g}_{ij} = \begin{cases} g_j, & j \neq i, \\ I, & j = i \end{cases}$$

and

$$\tilde{G}_i = \tilde{g}_{i1} \circ \tilde{g}_{i2} \circ \ldots \circ \tilde{g}_{ir},$$

respectively. Since $G = g_i \circ \tilde{G}_i$, then we have $G(X) = g_i(\tilde{G}_i(X)) \subset g_i(X)$ for $i \in \{1, \ldots, r\}$. These inclusions yield $G(X) \subset \bigcap_{i=1}^{r} g_i(X)$ which implies inclusion (i).

L e m m a 5. If $\bar{\bar{X}} = n < \infty$, $\mathcal{F} \subset X^X$, $\mathcal{F} \neq 0$ and $\mathcal{F}$ is commutative, then there exists a fixed point of the family $\{f^{n!}\}_{f \in \mathcal{F}}$.

P r o o f . If $f \in \mathcal{F}$, then by Lemma 3 the set $B_f = f^{n!}(X)$ is strongly f-invariant and $f^{n!}|_{B_f} = I|_{B_f}$. It is sufficient to show that $\bigcap_{f \in \mathcal{F}} B_f \neq \emptyset$. But $\overline{\overline{X^X}} = n^n < \infty$, then $\bar{\bar{\mathcal{F}}} = r < \infty$ and by Lemma 4 (i) we get

$$\bigcap_{f \in \mathcal{F}} B_f = \bigcap_{j=1}^{r} B_{f_j} = \bigcap_{j=1}^{r} f_j^{n!}(X) \supset f_1^{n!} \circ \dots \circ f_r^{n!}(X) \neq \emptyset,$$

whence $\bigcap_{f \in \mathcal{F}} B_f \neq \emptyset$. The assertion is proven.

3. Some fixed point theorems

We begin with the following statement.

T h e o r e m 1. Let X be a non-empty set. Assume:

(6) $\mathcal{F} \subset X^X$, $\mathcal{F} \neq \emptyset$,

(7) $\exists g \in \mathcal{F} \ \forall f \in \mathcal{F} : g \circ f = f \circ g$,

(8) there exists a unique fixed point x_0 of g.

Then the point x_0 is a fixed point of $\mathcal{F}$.

P r o o f . Since $(g \circ f)(x_0) = (f \circ g)(x_0)$ for all $f \in \mathcal{F}$, then $g(f(x_0)) = f(g(x_0)) = f(x_0)$ and the assertion follows by the definition of x_0.

T h e o r e m 2. Let X be a non-empty set, $n \in \mathbb{N}$. Assume:

(9) $\emptyset \neq \mathcal{F} \subset X^X$ and $\mathcal{F}$ is commutative,

(10) there is $g \in \mathcal{F}$ which has exactly n fixed points $x_1, \dots, x_n$.

Then

(11) there exists a fixed point x_0 of the family $\{f^{n!}\}_{f \in \mathcal{F}}$ and $x_0 \in \{x_1, \dots, x_n\}$.

P r o o f . Let $S = \{x_1, \dots, x_n\}$ and $x \in S$. By the assumptions $g(f(x)) = f(g(x)) = f(x)$ for all $f \in \mathcal{F}$. So, consequently, $f(x) \in S$ and $f|_S \in S^S$, $(f \in \mathcal{F})$. Using Lemma 5 to the family $\{f|_S\}_{f \in \mathcal{F}}$ we have the assertion.

T h e o r e m 3. Let $(P, +)$ be a commutative semigroup and let $\{U_t\}_{t\in P}$ be a iteration semigroup of self-mappings of a non-empty set X. Let $n \in \mathbb{N}$. Assume, that

(12) $\exists t_o \in P : U_{t_o}$ has exactly n fixed points $x_1, \dots, x_n$,

(13) P is $n!$-divisible.

Then

(14) there exists a fixed point x_o of the family $\{U_t\}_{t\in P}$ and $x_o \in \{x_1, \dots, x_n\}$.

P r o o f . By Theorem 2, there exists an $x_o \in \{x_1, \dots, x_n\}$ which is a fixed point of $U_{n!t}$ for every $t \in P$. Fix a $p \in P$. By (13) there exists a $t \in P$, such that $p = n!t$. Therefore $U_p(x_o) = U_{n!t}(x_o) = x_o$, which proves the assertion.

C o r o l l a r y 1. If we replace the assumption (12) of Theorem 3 by

(12′) $\exists t_o \in P \underset{C \neq \emptyset}{\exists C \subset X} : \bar{\bar{C}} = n < \infty$

and C is U_{t_o}-invariant, then the assertion (14) is valid.

P r o o f . The map $U_{t_o}|_C$ satisfies the assumptions of Lemma 4, with $X = C$. Hence $U_{n!t_o}|_B = I_{|B}$, with $B = U_{n!t_o}(C)$.

Consequently, $U_{n!t_o}$ has exactly $\bar{\bar{B}}$ fixed points. The corollary follows from Theorem 3.

The following corollary can now be proven in the same way as the above one.

C o r o l l a r y 2. If we replace the assumption (10) of Theorem 2 by

(10′) $\exists g \in \mathfrak{F} \underset{C \neq \emptyset}{\exists C \subset X} : \bar{\bar{C}} = n < \infty$

and C is g-invariant, then the assertion (11) is valid.

4. Remarks and examples

E x a m p l e 1. Let B be a Hamel basis of the reals containing the number 1, and $u: \mathbb{R} \to \mathbb{Q}$ the group homomorphism that maps each real x onto the coefficient of 1 in its Hamel basis expansion. In [2] Jürgen Weitkämper gave the following iteration semigroup on R: $U_t(x) = e^{u(t)}x$, $(t \in R,\ x \in R)$. He has proved that

(15) for $x \neq 0$ the set $\{t > 0:\ U_t(x) = x\}$ is different from R^+ and dense in R_0^+ (see [2]).

Let us consider for every $t \in R$ the following set

$$A(t) = \{x :\ U_t(x) = x\}.$$

By (15), Theorem 3 and Corollary 1 we infer that:

if $A(t) \neq \emptyset$ then $A(t)$ is infinite[2],

and

if C is non-empty and U_t-invariant, then C is infinite.

We shall give an example of the iteration semigroup $\{U_t\}_{t>0}$ for which

$$\overline{\overline{A(t)}} = \bar{\bar{R}} \ \text{ for all } \ t > 0 \ \text{ and } \ \bigcap_{t>0} A(t) = \emptyset.$$

E x a m p l e 2. Define the set X and semigroup $\{U_t\}_{t>0}$ by

$$X = \{u \cdot \mathfrak{X} + v \ :\ u, v > 0\},$$

$$U_t(x) = x + t \quad \text{for} \quad x \in X, \quad t > 0.$$

It is easy to verify, that

$$U_t(x) = x \quad \text{iff} \quad x = \frac{t}{k} \cdot \mathfrak{X} + v \quad \text{for some} \quad k \in N\ , \quad v > 0\ .$$

Hence

$$A(t) = \left\{\frac{t}{k} \cdot \mathfrak{X} + v \ :\ k \in N,\ v > 0\right\}, \quad \text{so} \quad \overline{\overline{A(t)}} = \bar{\bar{R}}.$$

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2) It is easy to check, that for each $t \in R$ either $A(t) = \emptyset$, or $A(t) = R$.

Moreover,

$$\bigcap_{t>0} A(t) = \emptyset, \quad \text{because} \quad A(1) \cap A(\sqrt{2}) = \emptyset.$$

E x a m p l e 3. Let $u: R \to Q$ be the group homeomorphism given in Example 1 and let $n \in N$, $n \geq 2$. Define

$S = \{(x,y,z) \in R^3 : x^2+y^2+z^2 = 1\} \cup (\{2,\dots,n\} \times \{0\} \times \{0\})$,

$l = R \times \{0\} \times \{0\}$.

Let T be the roration of S around the axis l by the angle π. Then, by Theorem 3, each semigroup $\{U_t\}_{t>0}$ with $u_1 = T$ has a fixed point x_0 and $x_0 \in \{-1,1,2,\dots,n\} \times \{0\} \times \{0\}$. For example, let U_t be the rotation of S around the axis l by the angle $\pi \cdot u(t)$, $(t > 0)$.

We observe that $U_1 = T$ and the sets

$$\{t > 0 : U_t = I\}, \quad \{t > 0 : U_t \neq I\}$$

are dense in R^+.

Let $S \supset C \neq \emptyset$, $V \in S^S$ and let C be finite and V-invariant. Under the above notations, by Theorem 2 and Corollary 2 we have

E x a m p l e 4. Each commutative family $\mathcal{F} \subset S^S$ for which $T \in \mathcal{F}$ or $V \in \mathcal{F}$ has a fixed point.

R e m a r k . The assumption that $\mathcal{F}$ is commutative cannot be omitted, i.e. Theorem 2 without it does not hold. Similar remarks are valid for Corollaries and Theorem 3. For example, the group of all isometries of the plane hasn't any fixed points.

E x a m p l e 5. Let X be a non-empty and finite set. If $f \in X^X$ and $f \circ f = f$ then the constant semigroup $\{f\}_{t \geq 0}$ has as much of fixed points as f.

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