DEMONSTRATIO MATHEMATICA
Vol. XXII No 1 1989

Ryszard Jerzy Pawlak

VARIATION OF DARBOUX FUNCTIONS

Introduction

In 1875 in paper [3], the first example of a discontinuous Darboux function was given. Since that time, there have appeared many papers devoted to the study of the properties of those functions. It has turned out that the family of Darboux functions contains many other important classes of functions such as, for instance, derivatives ([3]), functions being approximate derivatives ([5]), and even certain subfamilies of the classes of derivatives and approximate derivatives that can take infinite values ([7], [19]).

The proving of a number of interesting properties for real Darboux functions of a real variable accounted for seeking a generalization of the notion of a Darboux function to the case of transformations defined and taking their values in more abstract spaces. It should be stressed here that, both in the case of real functions of a real variable and in more general cases, there were also investigated functions whose definitions were "close to the definition of a Darboux function". Various generalizations of the notion of a Darboux function to the case of transformations defined and taking their values in more abstract spaces can be found, among others, in papers [2], [4], [9], [11], [13] and [14]. A list and an analysis of many such generalizations can be found in [6].

An essential difficulty in finding a generalization preserving the properties of real Darboux functions of a real variable as accurately as possible is the fact that those functions possess a number of interesting properties of the topological nature as well as many interesting properties connected with measure theory (cf. e.g. [20] and [1]). Analysing one of the better known definitions of the Darboux property, saying that a function $f: R \to R$ is a Darboux function if the image of any closed segment is connected, it can be noticed without difficulty that, while generalizing this notion, one should use the kind of sets whose topological nature as well as properties connected with measure theory are close to the properties of a closed segment on the line. Arcs seem to constitute such a family. In this context, throughout the paper, we shall adopt the following definition

D e f i n i t i o n ([15], [16]). We say that $f: X \to Y$ where X, Y are any topological spaces is a Darboux transformation (or possesses the Darboux property) if the image of any arc $L \subset X$ is a connected set.

The main aim of papers [15] and [16] was to show that the adoption of such a definition of a Darboux function allows one to obtain, for transformations defined and taking their values in more general spaces, results analogous to those for a Darboux function defined on the line. Whereas the present paper wears a somewhat different complexion. Its aim was to demonstrate that the adoption of the above definition creates new possibilities and problems which, many a time, have no analogue in the case of real functions of a real variable.

Throughout the paper, we apply the classical symbols and notations. However, in order to avoid any ambiguities, we shall now present those symbols used in the paper whose meanings are not explained in the main text. By the letter R we denote set of all real numbers with the natural topology, whereas R^2 stands for the plane (also with the natural topology). The symbol $\bar{R}$ stands for the extended set of real numbers i.e. $\bar{R} = R \cup \{-\infty\} \cup \{+\infty\}$. The letter N denotes the set of positive integers. The symbols (a,b), (a,b] etc. ... denote open intervals, those open at the endpoint a, etc. ... in the spaces R and R^2.

Let f be a real function. We adopt $E_{\alpha}(f) = f^{-1}((\alpha,+\infty))$ and $E^{\alpha}(f) = f^{-1}((-\infty,\alpha))$.

The symbols Fr A, $\bar{A}$ and Int A stand for the boundary, the closure and the interior of a set A, respectively. The two-dimensional Lebesgue measure of the set A is denoted by $m_2(A)$.

By an arc L we mean a subset of the topological space X which is homeomorphic (as a subspace) with $[0,1] \subset R$. If $h : [0,1] \xrightarrow{onto} L$ is a homeomorphism, then h(0) and h(1) are called endpoints of the arc L. The notation L(a,b) is understand as: an arc with endpoints a and b. Let $L \subset X$ be an arbitrary arc, and let $c,d \in L$. Then there exists exactly one arc $L' \subset L$ such that $L' = L(c,d)$. There arc L' will be denoted by $L_L(c,d)$.

If (X,ρ) is a metric space and $A \subset X$, then by dia A we denote the diameter of the set A, and by K(x,r) - the open ball with the centre at x and the radius r.

It is well known that if f is a continuous function on the interval [a,b] and N_f denotes the Banach indicatrix of the function f, then N_f is a measurable function, and the variation $\bigvee_a^b(f)$ of the function f on the interval [a,b] is equal to $\int_{-\infty}^{+\infty} N_f(y)\,dy$ ([10 , Theorem 3, p.254]). T. Salat proved ([17, Theorem 4]) that this fact also takes place in the case of Darboux functions.

In this paper we shall discuss, among other things, the problems of the measurability of the Banach indicatrix and the variation of Darboux functions mapping R^2 into R^2.

D e f i n i t i o n 1. By the Banach indicatrix of a function $f : E \to Y$ with respect to a set $D \subset E$ we mean a function $N_f^D : Y \to \bar{R}$ defined in the following manner: $N_f^D(p)$ is equal to the number of points of a set $f^{-1}(p) \cap D$ when this set is finite, or to $+\infty$ when the set is infinite (cf. e.g. [18, Definition 1, p.217] - in that definition, however, it was additionally assumed that f is a continuous function, which is a dispensable assumption in our considerations).

If $D = E$, then, instead of the notation N_f^E, we shall write N_f.

In paper [17, Theorem 2] it was proved that if X is a locally connected, second countable Hausdorff space and $f : X \to R$ is a connected function (i.e. f maps connected sets onto connected sets), then N_f is of the second class of Baire. In view of this result as well as those discussed above, the following question seems to be interesting: is the Banach indicatrix of a function $f : R^2 \to R^2$ with respect, for instance, to a closed segment always measurable. The answer to this question is negative (the constructing of such an example is very simple when f is a function whose Banach indicatrix takes the value 0 and $+\infty$ only; the example below will show that there exist Banach indicatrices of Darboux functions whose measurability "spoils" at finiteness).

E x a m p l e 2. Let $I = [(0,0), (1,0)]$ and let C denote some uncountable, closed and nowhere dense (in the topology of the segment I) subset of I. Let S be a Sierpiński set on the plane (i.e. S in non-disjoint from any closed set of positive measure and any three points of S do not lie on a common line - cf. [12, Theorem 14.4, p.97]).

Let $x_0 \in S$ and let, for any $x \in S \setminus \{x_0\}$, $I_x = [x_0, x]$. Note that if $x_1 \neq x_2$, then I_{x_1} and I_{x_2} are contained in different lines. Denote

$$\hat{S} = \Big(\Big(\bigcup_{x \in S \setminus \{x_0\}} I_x\Big) \setminus S\Big) \cup \{x_0\}.$$

Then card $\hat{S} = \mathfrak{c}$.

Let $g : I \to R^2$ be any function such that $g|_C$ is a one-to-one function mapping C onto $S \setminus \{x_0\}$ and each interval disjoint from C is mapped onto $\hat{S}$. Then, of course, g is a Darboux function. So, let us define the function $f : R^2 \to R^2$ in the following way:

$$f((x,y)) = \begin{cases} g((0,0)) & \text{when } x \leq 0 \text{ and } -\infty < y < +\infty, \\ g((x,0)) & \text{when } 0 \leq x \leq 1 \text{ and } -\infty < y < +\infty, \\ g((1,0)) & \text{when } x \geq 1 \text{ and } -\infty < y < +\infty. \end{cases}$$

Note that $f = g \circ h$, where h is some continuous function, which means that f is a Darboux function. It is not difficult to check that then the Banach indicatrix of this function with respect to I has the form:

$$N_f^I(p) = \begin{cases} 1 & \text{when } p \in S \setminus \{x_0\}, \\ 0 & \text{when } p \notin \hat{S} \cup S, \\ +\infty & \text{when } p \in \hat{S}. \end{cases}$$

Then, however, N_f^I is not a measurable function because $(N_f^I)^{-1}(1) = S \setminus \{x_0\}$.

In the further part of the paper we shall give an additional condition under which the Banach indicatrix of a Darboux function with respect to a closed and locally connected set is a measurable function. So, let L denote the σ-ideal of all sets of the Lebesgue measure zero. Let us adopt the following definition:

D e f i n i t i o n 3. We say that a function $f : R^2 \to R^2$ is L-regular with respect to a set $A \subset R^2$ if, for any set W open in A (as a subspace of R^2) and any component K of the set $f(W)$, $\mathrm{Fr}\, K \in L$.

For a fixed function f and a set D and for a fixed number $m \in N \cup \{0\}$, let $S_m^D(f) = \{\alpha \in R^2 : \mathrm{card}(f^{-1}(\alpha) \cap D) = m\}$ and $S_\infty^D(f) = \{\alpha \in R^2 : \mathrm{card}(f^{-1}(\alpha) \cap D) \geq \aleph_0\}$.

T h e o r e m 4. Let $f : R^2 \to R^2$ be a Darboux function L-regular with respect to a closed and locally connected set D. Then the Banach indicatrix N_f^D of the function f is a measurable function.

P r o o f . To prove the theorem, it suffices to demonstrate that, for any $\alpha \in R$, $E_\alpha(N_f^D)$ is a measurable set, i.e. that

(1) $\bigcup_{m=n}^{\infty} S_m^D(f) \cup S_\infty^D(f)$ is a measurable set for $n = 0, 1, 2, \ldots$.

Let n_0 be a fixed positive integer. Let further $\{V_k\}_{k=1}^{\infty}$ stand for an open base (in D) composed of connected sets of the subspace D (such a base exists - cf. e.g. [8, Remark on p.237]).

We shall show that

(2) $f(V_k)$ is a connected set for $k=1,2,\ldots$.

Let k be a fixed positive integer. Then V_k, as an open subset of the locally connected space D, is a locally connected set and D is a complete subspace. In virtue of the Mazurkiewicz-Moor theorem, V_k is an arcwise connected set in D, thus also in R^2, which proves (2).

Let N_1 denote the set of all $m \in N$ such that f is L-degenerate on V_m (i.e. $f(V_m) \in L$) and let $N_2 = N \setminus N_1$. Let further $A = \bigcup_{n \in N_2} Fr(f(V_n))$ and $B = \bigcup_{n \in N_1} f(V_n)$. From the assumptions we have made it follows, by (2), that $A \in L$ and $B \in L$.

Let us now adopt $P_{n_0} = \left(\bigcup_{m=n_0}^{\infty} S_m^D(f) \cup S_{\infty}^D(f)\right) \setminus (A \cup B)$.

Fix $\alpha^* \in P_{n_0}$. If $\alpha^* \in S_0^D(f)$, then let us adopt as V_{α^*} an arbitrary open set containing α^*. So, further, we shall always assume that $\alpha^* \notin S_0^D(f)$ while constructing V_{α^*}. Then there exist pairwise distinct points $x_1, x_2, \ldots, x_{n_0}$ of the set D, such that $f(x_k) = \alpha^*$ $(k=1,2,\ldots,n_0)$, and there exist positive integers $m_1, m_2, \ldots, m_{n_0} \in N_2$ such that $x_k \in V_{m_k}$ $(k=1,2,\ldots,n_0)$ and the sets $V_{m_1}, \ldots, V_{m_{n_0}}$ are pairwise disjoint.

Since $\alpha^* \notin A$, therefore $\alpha^* \in \operatorname{Int} f(V_{m_k})$ $(k=1,2,\ldots,n_0)$. Let $V_{\alpha^*} = \operatorname{Int}\left(\bigcap_{k=1}^{n_0} f(V_{m_k})\right)$. Then V_{α^*} is a neighbourhood of α^*. We shall show that

(3) $$V_{\alpha^*} \subset \bigcup_{m=n_0}^{\infty} S_m^D(f) \cup S_{\infty}^D(f).$$

Thus, let us suppose that $\beta \in V_{\alpha^*}$. Then $f^{-1}(\beta) \cap V_{m_k} \neq \emptyset$ for $k = 1,\dots,n_0$, which, in view of the disjointness of the sets $V_{m_1},\dots,V_{m_{n_0}}$, proves (3).

Let us now observe that

$$(4) \qquad \bigcup_{m=n_0}^{\infty} S_m^D(f) \cup S_\infty^D(f) = \bigcup_{\alpha^* \in P_{n_0}} V_{\alpha^*} \cup C,$$

where C is some subset of the set $A \cup B$.

On the ground of (4), seeing that $C \subset A \cup B \in L$, we may infer that the set $\bigcup_{m=n_0}^{\infty} S_m^D(f) \cup S_\infty^D(f)$ is Lebesgue-measurable, which, in view of the free choice of n_0, proves (1).

It is not difficult to notice that, under some modified assumptions concerning the transformations and the sets under consideration, one can obtain successive versions of the above theorem. Also, the following evident proposition holds.

P r o p o s i t i o n 5. Let $f : R^2 \to R^2$ be a Darboux function such that $m_2(f(R^2)) = 0$. Then, for any set $D \subset R^2$, N_f^D is a measurable function.

Knowing that, for certain Darboux functions, their Banach indicatrices with respect to some sets are measurable functions, one can define (in a similar way as, for instance, in [18, Definition 2, p.217]) the variation of the function and, thereby, answer the question what it means that such a function possesses a bounded variation.

D e f i n i t i o n 6. Let $f : E \to R^2$ $(E \subset R^2)$ be a Darboux function whose Banach indicatrix N_f^D $(D \subset E)$ is a measurable function. We then say that the function f has a bounded variation in the sense of Banach on the set $D \in E$ if $\int_{R^2} N_f^D(p)\,dp < +\infty$. The value of the integral $\int_{R^2} N_f^D(p)\,dp$ will be called a variation of the function f and denoted by $\bigvee_D f$.

R e m a r k . If we do not state on what set the function possesses a bounded variation, then we understand that this fact takes place on the whole domain of the transformation considered.

It is evident that, in the case of real functions defined on $[a,b]$, if f is a discontinuous (even if only at one point) Darboux function, then $\bigvee_a^b (f) = +\infty$ (cf. [17]). However, it is not difficult to observe that, in the case of functions defined and taking their values in certain subsets of the plane, such a situation need not take place. For it is easily noticed that if f satisfies the assumptions of Proposition 5, then, for any set $D \subset R^2$, $\bigvee_D (f) = 0$. And consequently, one may put the question: How often do discontinuous functions occur in the class of Darboux functions with bounded variation? The answer to this question is included in the following theorem:

T h e o r e m 7. In the space of bounded Darboux functions $f : K \to R^2$ (where $K = [0,1]\times[0,1]$) with bounded variation, with the metric of uniform convergence, discontinuous functions constitute a dense set of cardinality $2^{\mathfrak{c}}$.

P r o o f . We shall first prove the density of this set. Let $g : K \to R^2$ be a continuous function with bounded variation and let $\varepsilon > 0$.

Consider the following cases:

1^o $g(K) = \{y_o\}$. Let K_o be a closed cube with centre at some point x^o, such that $K_o \subset K(y_o, \varepsilon)$. Let further, for any $r \in [0, \frac{1}{3}]$, $K^r = \{r\}\times[r,1-r] \cup [r,1-r]\times\{1-r\} \cup \{1-r\}\times[r,1-r] \cup$ $\cup [r,1-r]\times\{r\}$. Let K^* denote the closed cube which is determined by $K^{\frac{1}{3}}$. Let h be a continuous function mapping $[0, \frac{1}{3})$ into $[0,1]$, such that, for any $x_o' \in (0, \frac{1}{3})$, $h((x_o', \frac{1}{3})) =$ $= [0,1]$. Let further $h_* : [0,1] \xrightarrow{onto} Fr\, K_o$ be a continuous function such that $h_*(0) = h_*(1)$. Finally, let $h^* : K^* \xrightarrow{onto} K_o$ be a homeomorphism such that $h^*(Fr\, K^*) = Fr\, K_o$. Let us then adopt:

$$f(p) = \begin{cases} h_*(h(r)) & \text{when} \quad p \in K^r, \quad \text{where} \quad r \in \left[0, \frac{1}{3}\right), \\ h^*(p) & \text{when} \quad p \in K^*. \end{cases}$$

Note that $f : K \to K_0$ is a Darboux function. Indeed, let Ł be any arc contained in K. If $Ł \subset K^*$, then $f(Ł) = h^*(Ł)$ is a connected set. In the case when $Ł \cap (K \setminus K^*) \neq \emptyset$, it can easily be seen that $f(Ł \cap (K \setminus K^*))$ is a connected set; if we assume additionally that $Ł \cap \mathrm{Fr}\, K^* \neq \emptyset$, then $f(Ł \cap (K \setminus K^*)) = \mathrm{Fr}\, K_0$.

Consequently, assume that $Ł \cap \mathrm{Int}\, K^* \neq \emptyset \neq Ł \cap (K \setminus K^*)$ and suppose that $f(Ł)$ is not a connected set; i.e. $f(Ł) = A \cup B$ where A and B are separated sets. Denote $A_1 = Ł \cap f^{-1}(A)$, $B_1 = Ł \cap f^{-1}(B)$. Then $A_1 \cap B_1 = \emptyset$ but these sets are not separated. Without loss of generality let us assume that $\bar{A}_1 \cap B_1 \neq \emptyset$. Let $x_0 \in \bar{A}_1 \cap B_1$. Then there exists a sequence $\{x_n\} \subset A_1$ such that $x_0 = \lim_{n \to \infty} x_n$.

At present, we shall show that

(1) there exists a sequence $\{y_n\} \subset A_1 \cap \mathrm{Fr}\, K^*$ such that $x_0 = \lim_{n \to \infty} y_n$.

Let $\varphi : [0,1] \xrightarrow{\text{onto}} Ł$ be a homeomorphism. Let $z_n = \varphi^{-1}(x_n)$ $(n=0,1,2,\ldots)$. Then $\lim_{n \to \infty} z_n = z_0$ (of course, $z_n \neq z_0$). With no loss of generality we may assume that all elements of the sequence $\{z_n\}$ lie on the same side of the point z_0; moreover, assume that $z_n < z_0$ $(n=1,2,\ldots)$.

So, let n stand for a fixed positive integer. If $x_n \in \mathrm{Fr}\, K^*$, then let us put $y_n = x_n$. In the contrary case we notice that

$$\{t \in [z_n, z_0) : \varphi(t) \in \mathrm{Fr}\, K^*\} \neq \emptyset.$$

Let us then adopt

$$s_n = \inf\{t \in [z_n, z_0) : \varphi(t) \in \mathrm{Fr}\, K^*\} \quad \text{and} \quad y_n = \varphi(s_n).$$

It can be demonstrated that $y_n \in \mathrm{Fr}\, K^*$ and $y_n \in A_1$. Besides, note that $z_n < s_n < z_0$.

From the reasoning we have carried out it follows that relationship (1) does take place.

In virtue of (1), we may infer that $x_0 \in \mathrm{Fr}\, K^*$.

Since $f_{|K^*} = h^*$ is a continuous function, therefore, in view of (1), we may deduce that $\lim_{n \to \infty} f(y_n) = f(x_0) \in B$ and, as $\{f(y_n)\} \subset A$, we obtain a contradiction with the supposition that A and B are separated sets. The contradiction obtained proves that f is really a Darboux function.

It is not difficult to notice that f is a discontinuous function.

Obviously, $\bigvee_K (f) = m_2(K_0) < +\infty$, and so, f is a function with bounded variation.

Moreover, it is easy to verify that $\rho_0(f,g) < \varepsilon$ where ρ_0 denotes the metric of uniform convergence.

2^0 Assume now that g is a non-constant continuous function with bounded variation. Let $y_0 \in g(K)$ and $K_0 = K(y_0, \frac{\varepsilon}{2})$. Let n_0 be a positive integer such that $\frac{\varepsilon}{2} > \frac{\varepsilon}{n_0}$ and $\overline{g(K) \setminus K(y_0, \frac{\varepsilon}{n_0})} \neq \emptyset$. Then, denote $K^0 = K(y_0, \frac{\varepsilon}{n_0})$. Let T be a segment joining $\mathrm{Fr}\, K_0$ and $\mathrm{Fr}\, K^0$ such that either of the intersections $T \cap \mathrm{Fr}\, K_0$ and $T \cap \mathrm{Fr}\, K^0$ is a one-element set. Further, for any $r \in \left(\frac{\varepsilon}{n_0}, \frac{\varepsilon}{2}\right)$, let $K_r = \mathrm{Fr}\, K(y_0, r)$. Let $h : \left(\frac{\varepsilon}{n_0}, \frac{\varepsilon}{2}\right) \to [0,1]$ be a continuous function such that, for any $x_0 \in \left(\frac{\varepsilon}{n_0}, \frac{\varepsilon}{2}\right)$, $h\left(\left(\frac{\varepsilon}{n_0}, x_0\right)\right) = [0,1] = h\left(\left(x_0, \frac{\varepsilon}{2}\right)\right)$. Let further $h_* : [0,1] \xrightarrow{\text{onto}} \mathrm{Fr}\, K_0 \cup \mathrm{Fr}\, K^0 \cup T$ be a continuous function. Define a function $f_1 : R^2 \to R^2$ in the following manner:

$$f_1(p) = \begin{cases} h_*(h(r)) & \text{when } p \in K_r, \text{ where } r \in \left(\frac{\varepsilon}{n_0}, \frac{\varepsilon}{2}\right), \\ p & \text{when } p \in (R^2 \setminus K_0) \cup \overline{K^0}. \end{cases}$$

Similarly as in 1° we can show that f_1 is a Darboux function.

So, let $f = f_1 \circ g$. Then, of course, f is a discontinuous Darboux function.

Note now that

$$N_f^K(p) = \begin{cases} N_g^K(p) & \text{when} \quad p \notin \overline{K_0}, \\ +\infty & \text{when} \quad p \in \mathrm{Fr}\, K_0 \cup \mathrm{Fr}\, K^0 \cup T, \\ 0 & \text{when} \quad p \in K_0 \setminus (\overline{K^0} \cup T), \\ N_g^K(p) & \text{when} \quad p \in K^0. \end{cases}$$

Then $\bigvee_K (f) \leqslant \int_{R^2} N_g^K(p)\, dp < +\infty$ and, moreover, $f(p) = g(p)$ for $p \in g^{-1}(R^2 \setminus K_0)$ and $f(p) \in \overline{K_0}$ for $p \in g^{-1}(K_0)$, thus $\rho_0(f,g) \leqslant \leqslant \operatorname{dia} \overline{K_0} = \varepsilon$.

The first part of this theorem has been proved. We shall now prove the veracity of the second part of the theorem.

Let us adopt the following notations: $K_1 = \left[\frac{1}{2}, 1\right] \times [0,1]$, $L_1 = \left\{\frac{1}{2}\right\} \times [0,1]$, $L_2 = \left[\frac{1}{2}, 1\right] \times \{0\}$. Let $k : K_1 \to L_1 \cup L_2$ be a function defined as follows: let us assign to each point $x \in K_1$ a point x' being the point of intersection of a line L_x passing through the point x and parallel to the vector with coordinates $\left[\frac{1}{2}, 1\right]$, and the set $L_1 \cup L_2$. Then k is a continuous function.

Let C be a nowhere dense perfect set contained in $(\frac{1}{2}, 1) \times \{0\}$. Now, we shall assign to each set $C_1 \subset C$, in a one-to-one way, some discontinuous Darboux function h_{C_1} with bounded variation.

So, let $C_1 \subset C$ and let a and b denote any two distinct elements belonging to Int K. Let further $\hat{L} = L((1,0), c)$ stand for an arc such that $a, b \in \hat{L} \setminus \{(1,0), c\}$ and $m_2(\hat{L}) = 0$. Let $\varphi : [-1,1] \to \hat{L} \cup [(1,0), (1,1)]$ be a homeomorphism such that $\varphi(0) = (1,0)$, $\varphi(1) = c$, and $\varphi(\frac{1}{3}) = a$ and $\varphi(\frac{2}{3}) = b$ (or $\varphi(\frac{1}{3}) = b$ and $\varphi(\frac{2}{3}) = a$, but then one may interchange the denotations of the points a and b in order to avoid the consideration of two cases); moreover, $\varphi(x) = (1,-x)$ for $x \in [-1,0]$. Besides, for any interval $((p,0),\ (q,0)) \subset L_2$, denote by the symbol h_p^q a continuous function mapping this interval onto $[0,1]$, such that, for any $(x,0) \in ((p,0),\ (q,0))$, $h_p^q(((x,0),\ (q,0))) = [0,1] = h_p^q(((p,0),\ (x,0)))$. Define a transformation $\psi_{C_1} : L_1 \cup L_2 \xrightarrow{\text{onto}} [-1,1]$ in the following manner:

$$\psi_{C_1}((x,y)) = \begin{cases} 0 & \text{when } (x,y) = (\frac{1}{2}, 0), \\ 1 & \text{when } (x,y) = (1,0), \\ \frac{1}{3} & \text{when } (x,y) \in C_1, \\ \frac{2}{3} & \text{when } (x,y) \in C \setminus C_1, \\ h_p^q((x,y)) & \text{when } (x,y) \in ((p,0),\ (q,0)) \text{ where } ((p,0),\ (q,0)) \text{ is a component of the set } L_2 \setminus (C \cup \{(\frac{1}{2}, 0),\ (1,0)\}), \\ -y & \text{when } (x,y) \in L_1. \end{cases}$$

Let us then define the function $h_{C_1} : K \to K$ by the formula

$$h_{C_1}((x,y)) = \begin{cases} (2x,y) & \text{when } (x,y) \in \overline{K \setminus K_1}, \\ \varphi(\psi_{C_1}(k(x,y))) & \text{when } (x,y) \in K_1. \end{cases}$$

It is not hard to check that h_{C_1} is a Darboux function and, moreover, $\bigvee_K (h_{C_1}) = 1$. Of course, h_{C_1} is not a continuous function. To finish with, let us notice that if $C_1 \neq C_2$, then $h_{C_1} \neq h_{C_2}$, which finally completes the proof of the theorem.

REFERENCES

[1] A.M. Bruckner: Differentiation of real functions, Springer-Verlag (1978).

[2] A.M. Bruckner, J.B. Bruckner: Darboux transformations, Trans. Amer. Math. Soc. 128 (1967) 103-111.

[3] G. Darboux: Memoire sur les fonctions discontinues, Ann. Sci. Scuola Norm. Sup. 4 (1875) 57-112.

[4] K.M. Garg: Properties of connected functions in terms of their levels, Fund. Math. 47 (1977) 17-36.

[5] C. Goffman, C. Neugebauer: An approximate derivatives, Proc. Amer. Math. Soc. 11 (1960) 962-966.

[6] J.M. Jędrzejewski: Własności funkcji związane z pojęciem spójności, Acta Univ. Lodz. (1984) 1-84.

[7] M. Kulbacka: Sur certaines properties des derivées approximatives, Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys. 12, No. 1 (1964) 17-20.

[8] K. Kuratowski: Wstęp do teorii mnogości i topologii, Ed. 5, Warszawa (1972).

[9] L. Mišik: Uber die Funtionen der ersten Baireschen Klasse mit der Eigenschaft von Darboux, Mat. Fyz. Časopis SAV 14 (1964) 44-48.

[10] I.P. Natanson: Theorie der Funktionen einer reelen Veränderlichen, Berlin (1961).

[11] C.J. Neugebauer: Darboux property for functions of several variables, Trans. Amer. Math. Soc. 107 (1963), 30-37.

[12] J.C. Oxtoby: Measure and category (in Russian) Moskwa (1974).

[13] H. Pawlak, R. Pawlak: On continuity and limit points of w-Darboux function, Demonstratio Math. 16, No. 3 (1983), 771-775.

[14] H. Pawlak, W. Wilczyński: On the condition of Darboux and Swiątkowski for functions of two variables, Zeszyty Nauk. Politech. Łódz. Mat. z. 15 (1982) 31-35.

[15] R.J. Pawlak: On Zahorski classes of functions of two variables, Revue Roum. de Math. Pures et Appl. (in print).

[16] R.J. Pawlak: On local characterization of Darboux functions, Comment. Math. (in print).

[17] T. Šalat: Generalization of the notions of the Banach indicatric, Fund. Math. 73 (1971) 29-36.

[18] W. Wilczyński: Superpositions of transformations of bounded variation, Fund. Math. 40 (1976) 211-231.

[19] Z. Zahorski: Sur la classe de Baire des dérivées approximatives d'une fonctions quelconque, Ann. Soc. Pol. Math. 21 (1948) 306-323.

[20] Z. Zahorski: Sur la première dérivée, Trans. Amer. Math. Soc. 69 (1950) 1-54.

INSTITUTE OF MATHEMATICS, ŁÓDŹ UNIVERSITY, 90-238 ŁÓDŹ, POLAND
Received March 25, 1987.