DEMONSTRATIO MATHEMATICA
Vol XXII No 1 1989

Vu Van Hung

ON NON-CONDITIONAL STABILITY OF OPEN DIFFERENCE PATTERNS FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction

Even in the simplest cases the choice of the numerically optimal difference pattern for a parabolic equation of second order is not a simple problem. An open pattern is conditionally stable and the number of the operations needed for solving may be large, a not yet open pattern is non-conditionally stable, but it requires inversion of matrices. The Sauliev pattern [3] is applied only in the case of a finite limit problem.

The aim of this paper is to determine the class of patterns which allows the approximation of problems related to parabolic equations of second order in the space L_2, these patterns being open and non-conditionally stable.

We illustrate such patterns for the Cauchy problem

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & \text{for} \quad (x,t) \in R \times (0,T), \\ u(x,0) = u_0(x), & \text{for} \quad x \in R, \quad u_0 \in C(R), \end{cases}$$

on the rectangular net

$$(*)\qquad R^2_{h\tau} := \{(x,t) : m,n \in Z,\ x = mh,\ t = n\tau,\ \tau N = T\},$$

Z - being the set of integers.

The difference pattern for (1.1) has the form

$$
(1.2)\quad
\begin{cases}
\dfrac{2}{\tau}\left[U_{2m+1}^{n+\frac{1}{2}} - U_{2m+1}^{n}\right] = L_h U_{2m+1}^{n}, \\[2mm]
\dfrac{2}{\tau}\left[U_{2m}^{n+\frac{1}{2}} - U_{2m}^{n}\right] = L_h U_{2m}^{n+\frac{1}{2}}, \\[2mm]
\dfrac{2}{\tau}\left[U_{2m}^{n+1} - U_{2m}^{n+\frac{1}{2}}\right] = L_h U_{2m}^{n+\frac{1}{2}}, \\[2mm]
\dfrac{2}{\tau}\left[U_{2m+1}^{n+1} - U_{2m+1}^{n+\frac{1}{2}}\right] = L_h U_{2m+1}^{n+1},
\end{cases}
$$

$$U_m^0 = U(hm), \quad m \in Z, \quad 0 \leqslant n \leqslant N-1,$$

where $L_h U_m^n := \frac{1}{h^2}\left[U_{m+1}^n - 2U_m^n + U_{m-1}^n\right]$, $U_m^n := u(mh, n\tau)$.

The pattern (1.2), as easily seen, approximates the problem (1.1) with the error $O(\tau) + O(h^2)$. Let us observe, moreover, that the pattern (1.2) is open.

In order to examine the stability of this pattern in the norm L_2 we introduce the following functions

$$
(**)\quad
\begin{cases}
V_h^k(h\xi) := \dfrac{h}{\sqrt{2\pi}} \displaystyle\sum_{m\in Z} U_{2m+1}^k e^{-i(2m+1)h\xi}, \\[2mm]
W_h^k(h\xi) := \dfrac{h}{\sqrt{2\pi}} \displaystyle\sum_{m\in Z} U_{2m}^k e^{-i2mh\xi}, \quad 2k \in Z,\ 0 \leqslant 2k \leqslant N, \quad \forall \xi \in R.
\end{cases}
$$

Multiplying the first and the fourth equation (1.2) by $\frac{h}{\sqrt{2\pi}} e^{-i(2m+1)h\xi}$, the second and the third by $\frac{h}{\sqrt{2\pi}} e^{-i2h\xi m}$, and summing with respect to m we get

$$
(1.3)\quad \begin{cases}
\frac{2}{\tau}\left[V_h^{n+\frac{1}{2}}(h\xi) - V_h^n(h\xi)\right] = \\
\quad = \frac{1}{h^2}\left[(e^{ih\xi}+e^{-ih\xi})W_h^n(h\xi) - 2V_h^n(h\xi)\right]. \\
\frac{2}{\tau}\left[W_h^{n+\frac{1}{2}}(h\xi) - W_h^n(h\xi)\right] = \\
\quad = \frac{1}{h^2}\left[(e^{ih\xi}+e^{-ih\xi})\, V_h^{n+\frac{1}{2}}(h\xi) - 2W_h^{n+\frac{1}{2}}(h\xi)\right], \\
\frac{2}{\tau}\left[W_h^{n+1}(h\xi) - W_h^{n+\frac{1}{2}}(h\xi)\right] = \\
\quad = \frac{1}{h^2}\left[(e^{ih\xi}+e^{-ih\xi})\, V_h^{n+\frac{1}{2}}(h\xi) - 2W_h^{n+\frac{1}{2}}(h\xi)\right], \\
\frac{2}{\tau}\left[V_h^{n+1}(h\xi) - V_h^{n+\frac{1}{2}}(h\xi)\right] = \\
\quad = \frac{1}{h^2}\left[(e^{ih\xi}+e^{-ih\xi})\, W_h^{n+1}(h\xi) - 2V_h^{n+1}(h\xi)\right].
\end{cases}
$$

Hence, setting $\varphi := h\xi$, $\sigma := \frac{\tau}{h^2}$, we obtain the system of two differential equations of order 1

$$
(1.4)\qquad \begin{bmatrix} V_h^{n+1}(\varphi) \\ W_h^{n+1}(\varphi) \end{bmatrix} =
$$

$$
= \begin{bmatrix} \frac{1-\sigma}{1+\sigma} + 2\,\frac{1-\sigma}{(1+\sigma)^2}\,\sigma^2\cos^2\varphi, & \frac{2}{(1+\sigma)^2}\cos\varphi\,(1+\sigma^2\cos^2\varphi) \\ 2\sigma\,\frac{1-\sigma}{1+\sigma}\cos\varphi, & \frac{1-\sigma}{1+\sigma} + 2\,\frac{\sigma^2}{1+\sigma}\cos^2\varphi \end{bmatrix} \cdot \begin{bmatrix} V_h^n(\varphi) \\ W_h^n(\varphi) \end{bmatrix}.
$$

It is easy to see that the eigenvalues (λ_1, λ_2) of the transition matrix of this difference equation are given by the formula

$$\lambda_{1,2}(\varphi) = \frac{1}{(1+\delta)^2}\left[\delta\cos\varphi \pm \sqrt{1-\delta^2\sin^2\varphi}\right]^2,$$

whence we obtain

$$\rho(\varphi) := \max\left[|\lambda_1(\varphi)|,|\lambda_2(\varphi)|\right] =$$

$$= \begin{cases} \dfrac{1}{(1+\delta)^2}\left(\delta|\cos\varphi| + \sqrt{1-\delta^2\sin^2\varphi}\right)^2, & \text{if} \quad |\delta\sin\varphi| \leqslant 1, \\ \dfrac{1}{(1+\delta)^2}\left(\delta^2\cos^2\varphi - \delta^2\sin^2\varphi + 1\right), & \text{if} \quad |\delta\sin\varphi| > 1. \end{cases}$$

Hence $\max\rho(\varphi) = 1$ for $|\varphi| \leqslant \frac{\pi}{2}$, which proves, by von Neumann's conditions (see [1]), the L_2-stability of the pattern (1.2); therefore, the stability is non-conditional.

Furthermore, we will show in this paper that the same result can be extended to the case of a parabolic equation of second order in the normal form in one spatial variable.

2. On the non-conditional stability of the class of open difference patterns which approximate the initial problem for a parabolic equation of order 2 in one spatial variable

Consider the difference patterns for the following problem

$$\text{(2.1)} \quad \begin{cases} \dfrac{\partial u}{\partial t} = a(x,t)\dfrac{\partial^2 u}{\partial x^2} + b(x,t)\dfrac{\partial u}{\partial x} + c(x,t)u + f(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in R\times(0,T), \\ u(x,0) = u_0(x), \quad u_0 \in C(R), \qquad x \in R. \end{cases}$$

Assume that the functions a, b, c, f belong to the space $C(R\times(0,T))$ and $a(x,t) \geqslant a > 0$.

On the numerical sequences $\{y_m\}_{m\in Z}$ we define the operations $\{L^n_{h,m}\}_{\substack{m\in Z\\ 2n\in Z}}$, $L^n_{h,m} \in \mathcal{L}(L_\infty)$ as follows

$$(L^n_{h,m}y)_k := L^n_{h,m}y_k := a^n_m \frac{1}{h^2}(y_{k-1} - 2y_k + y_{k+1}) +$$

$$+ b^n_m \frac{1}{2h}(y_{k+1} - y_{k-1}) + c^n_m y_k,$$

$$\forall (k,m,n), \quad 2n \in Z, \quad m,k \in Z.$$

On the rectangular net (*) the difference pattern for (2.1) has the form

$$(2.2)\quad \begin{cases} \frac{2}{\tau}\left[U^{n+\frac{1}{2}}_{2m+1} - U^n_{2m+1}\right] = L^n_{h,2m+1} U^n_{2m+1} + f^n_{2m+1}, \\ \frac{2}{\tau}\left[U^{n+\frac{1}{2}}_{2m} - U^n_{2m}\right] = L^{n+\frac{1}{2}}_{h,2m} U^{n+\frac{1}{2}}_{2m} + f^{n+\frac{1}{2}}_{2m}, \\ \frac{2}{\tau}\left[U^{n+1}_{2m} - U^{n+\frac{1}{2}}_{2m}\right] = L^{n+\frac{1}{2}}_{h,2m} U^{n+\frac{1}{2}}_{2m} + f^{n+\frac{1}{2}}_{2m}, \\ \frac{2}{\tau}\left[U^{n+1}_{2m+1} - U^{n+\frac{1}{2}}_{2m+1}\right] = L^{n+1}_{h,2m+1} U^{n+1}_{2m+1} + f^{n+1}_{2m+1} \end{cases}$$

for $m \in Z$; $0 \leq n \leq N-1$, $U^0_m = u_0(mh)$ for $m \in Z$.

Assume that the functions a, b, c defining the differential operator are constant on the domain $R\times[0,T]$; then the operator $L^n_{h,m}$ has the form $L^n_{h,m}y_k := L_h y_k := a\frac{1}{h^2}\left[y_{k-1}-2y_k+y_{k+1}\right]+$ $+ b\frac{1}{2h}\left[y_{k+1}-y_{k-1}\right] + cy_k$ for $k \in Z$.

In order to perform the Fourier analysis of the pattern (2.2) we shall assume for both complex functions V^k_h and W^k_h the same definition (see (**)). Proceeding as for the system (1.2) we obtain the following relations

(2.3)
$$\begin{bmatrix} 1 & , & 0 \\ -p(\varphi,\tau,\delta), & & 1+\delta-\frac{c}{2}\tau \end{bmatrix} \cdot \begin{bmatrix} V_h^{n+\frac{1}{2}}(\varphi) \\ W_h^{n+1}(\varphi) \end{bmatrix} =$$

$$= \begin{bmatrix} 1-\delta+\frac{c}{2}\tau & , & p(\varphi,\tau,\delta) \\ 0 & , & 1 \end{bmatrix} \cdot \begin{bmatrix} V_h^{n}(\varphi) \\ W_h^{n}(\varphi) \end{bmatrix} + \frac{\tau}{2} \begin{bmatrix} F_h^{n}(\varphi) \\ G_h^{n+\frac{1}{2}}(\varphi) \end{bmatrix},$$

(2.4)
$$\begin{bmatrix} 1+\delta-\frac{c}{2}\tau & , & -p(\varphi,\tau,\delta) \\ 0 & , & 1 \end{bmatrix} \cdot \begin{bmatrix} V_h^{n+1}(\varphi) \\ W_h^{n+1}(\varphi) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & , & 0 \\ p(\varphi,\tau,\delta), & & 1-\delta+\frac{c}{2}\tau \end{bmatrix} \cdot \begin{bmatrix} V_h^{n+\frac{1}{2}}(\varphi) \\ W_h^{n+\frac{1}{2}}(\varphi) \end{bmatrix} + \frac{\tau}{2} \begin{bmatrix} G_h^{n+\frac{1}{2}}(\varphi) \\ F_h^{n+1}(\varphi) \end{bmatrix},$$

where

$$F_h^k(h\xi) := \frac{h}{\sqrt{2\pi}} \sum_{m\in Z} f_{2m+1}^k e^{-i(2m+1)h\xi},$$

$$G_h^k(h\xi) := \frac{h}{\sqrt{2\pi}} \sum_{m\in Z} f_{2m}^k e^{-i2mh\xi}$$

for $k \in Z$, $0 \leqslant 2k \leqslant N$,

(2.5)
$$\delta := \frac{a\tau}{h^2},$$

(2.6)
$$p(\varphi,\tau,\delta) := \delta \cos\varphi + i\frac{b}{\sqrt{2a}}\sqrt{\delta\tau}\,\sin\varphi, \qquad \varphi := h\xi.$$

Denoting

(2.7)
$$\tau_0 = \begin{cases} +\infty, & \text{if } \quad c \leqslant 0 \\ \frac{2}{c}, & \text{if } \quad c > 0 \end{cases}$$

we see that the system of difference equations is solvable for $\tau \in (0,\tau_0)$.

From (2.3), (2.4) we shall deduce that

$$\begin{bmatrix} V_h^{n+1}(\varphi) \\ W_h^{n+1}(\varphi) \end{bmatrix} = C(\varphi,\tau,\delta) \begin{bmatrix} V_h^{n}(\varphi) \\ W_h^{n}(\varphi) \end{bmatrix} +$$

$$+ \frac{\tau}{2} B(\varphi,\tau,\delta) \begin{bmatrix} F_h^{n}(\varphi) \\ G_h^{n+\frac{1}{2}}(\varphi) \end{bmatrix} + \frac{\tau}{2} A(\varphi,\tau,\delta) \begin{bmatrix} G_h^{n+\frac{1}{2}}(\varphi) \\ F_h^{n+1}(\varphi) \end{bmatrix},$$

where the matrices A, B, C are given by the formulas

$$A = \begin{bmatrix} \frac{1}{1+\rho} , & \psi \\ 0 , & 1 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{1+\rho} + 2\psi^2, & \frac{1-\rho}{1+\rho}\psi \\ 2\psi , & \frac{1-\rho}{1+\rho} \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{1-\rho}{1+\rho} + 2(1-\rho)\psi^2, & \frac{2}{1+\rho}\psi + 2(1+\rho)\psi^3 \\ 2(1-\rho)\psi , & \frac{1-\rho}{1+\rho} + 2(1+\rho)\psi^2 \end{bmatrix} \tag{2.8}$$

with

$$\begin{cases} \rho(\tau,\delta) = \rho := \delta - \frac{c}{2}\tau , \\ \psi := \psi(\varphi,\tau,\delta) := \frac{1}{1+\rho(\tau,\delta)}\, p(\varphi,\tau,\delta). \end{cases} \tag{2.9}$$

The matrix C, defined by formula (2.8), is called the transition matrix of the pattern (2.2) (see [1]) and plays a fundamental role in the study of the L_2-stability of this pattern.

In order to apply von Neumann s theorems on stability, we have to examine the eigenvalues of the matrix C. In the considered case the eigenvalues of the matrix C are

$$\lambda_{1,2} = \left[\psi \pm \sqrt{\frac{1-\rho}{1+\rho} + \psi^2}\right]^2. \tag{2.10}$$

Hence, we obtain the following lemmas [4].

L e m m a 2.1. Let $\tau \in (0, \tau_1)$, where

$$\tau_1 := \begin{cases} \infty, & \text{if} \quad c \leqslant 0, \\ 2\sqrt{\dfrac{a}{c[ac + b^2]}}, & \text{if} \quad c > 0. \end{cases} \tag{2.11}$$

If λ is the double eigenvalue of the matrix given by formula (2.10), then

$$|\lambda(\varphi, \tau, \delta)| < 1, \quad \text{for} \quad \tau \in (0, \tau_1), \ \delta \in (0, \infty), \ |\varphi| \leqslant \frac{\pi}{2}.$$

L e m m a 2.2. Let a_0, a_1 be arbitrary real numbers, non-negative, and let Ω denote the ellipse defined in the complex plane $\mathbf{C}$ as follows

$$\Omega := \left\{z \in \mathbf{C} : z = a_1 \cos\varphi + i\, a_0 \sin\varphi, \ |\varphi| \leqslant \pi\right\}.$$

Then we have, for any real number x,

$$\sup_{z\in\Omega} \left|z + \sqrt{x+z^2}\right| = \begin{cases} a_0 + \sqrt{a_0^2 - x}, & \text{if} \quad x \leqslant -a_1^2, \\ \max\left[a_0 + \sqrt{a_0^2 - x}, \ a_1 + \sqrt{a_1^2 + x}\right], & \text{if} \quad -a_1^2 \leqslant x \leqslant a_0^2, \\ a_1 + \sqrt{a_1^2 + x}, & \text{if} \quad a_0^2 \leqslant x. \end{cases}$$

R e m a r k 2.1. The assertion of the Lemma 2.2 can be written in the still simpler form

$$\sup_{z\in\Omega} \left|z + \sqrt{x+z^2}\right| = \max\left\{a_0 + \sqrt{[a_0^2 - x]_+}, \ a_1 \mp \sqrt{[a_1^2 + x]_+}\right\}, \tag{2.12}$$

where the operation $[\ ]_+$ is defined on R as follows:

$$[\xi]_+ = \max[0, \xi], \quad \forall \xi \in R.$$

T h e o r e m 2.1. If $\tau \in (0,\tau_0)$ (τ_0 being given by formula (2.7)), then

$$(2.13)\qquad \sup_{\substack{\lambda \in \operatorname{spectr} C \\ |\varphi| \leq \pi}} |\lambda(\varphi,\delta,\tau)| = \frac{1}{\left(1+\delta-\frac{c}{2}\tau\right)\sqrt{2a}}$$

$$\cdot \max\left\{\left(|b|\sqrt{\delta\tau}+\sqrt{\left[2a\delta^2+\delta\tau(b^2-2ac)-2a\left(1-\frac{c^2}{4}\tau^2\right)\right]_+}\right)^2,\right.$$

$$\left.\left(\delta\sqrt{2a}+\sqrt{2a\left[c\,\delta\tau+1-\frac{c^2}{4}\tau^2\right]_+}\right)^2\right\}.$$

P r o o f . From formulae (2.6) and (2.9) we get

$$\psi(\varphi,\delta,\tau) = \frac{\delta}{1+\rho}\cos\varphi + i\,\frac{b}{1+\rho}\sqrt{\frac{\delta\tau}{2a}}\sin\varphi, \quad \text{for } |\varphi| \leq \pi .$$

Hence, setting

$$\Omega := \left\{z : z = \frac{\delta}{1+\rho}\cos\varphi + i\,\frac{|b|}{1+\rho}\sqrt{\frac{\delta\tau}{2a}}\sin\varphi\right\},$$

we find that

1° $\{z : z = \psi(\varphi,\delta,\tau),\ |\varphi| \leq \pi\} = \Omega$;

2° if $\tau \in (0,\tau_0)$, then (see (2.7), (2.9))

$1+\rho = 1+\delta-\frac{c}{2}\tau > \delta > 0$, that is $a_1 = \frac{\delta}{1+\rho} > 0$,

$a_0 = \frac{|b|}{1+\rho}\sqrt{\frac{\delta\tau}{2a}} \geq 0$;

3° if $z \in \Omega$, then $-z \in \Omega$.

Since the eigenvalues of the matrix C (formula (2.8)) are given by (2.10), applying Lemma 2.2 and Remark 2.1 we have

$$\max\left[\sup_{|\varphi|\leq\pi} |\lambda_1(\varphi,\delta,\tau)|,\quad \sup_{|\varphi|\leq\pi} |\lambda_2(\varphi,\delta,\tau)|\right] =$$

$$= \max\left[\sup_{\psi\in\Omega}\left|\psi+\sqrt{\frac{1-\rho}{1+\rho}+\psi^2}\right|^2,\right.$$

$$\sup_{\psi\in\Omega}\left|\psi-\sqrt{\frac{1-\rho}{1+\rho}+\psi^2}\right|^2\Bigg] = \sup_{\psi\in\Omega}\left|\psi+\sqrt{\frac{1-\rho}{1+\rho}+\psi^2}\right|^2 = \max\left\{\left(\frac{|b|}{1+\rho}\sqrt{\frac{\delta\tau}{2a}}+\right.\right.$$

$$\left.\left.+\sqrt{\left[\frac{b^2\delta\tau}{2a(1+\rho)^2}-\frac{1-\rho}{1+\rho}\right]_+}\right)^2,\ \left(\frac{\delta}{1+\rho}+\sqrt{\left[\frac{\delta^2}{(1+\rho)^2}+\frac{1-\rho}{1+\rho}\right]_+}\right)^2\right\}.$$

Substituting ρ from formula (2.9) and performing the necessary algebraic transformations we obtain the assertion of Theorem 2.1.

We now return to the proof of the stability. Let us first prove the following lemma.

L e m m a 2.3. Let α, β be real numbers such that $\frac{1}{2}<\alpha$, $0\leqslant\beta\leqslant\frac{1}{2}$ and let f_1, g_1, f_2 be functions defined on R as follows

$$f_1(x) := \beta x(1-\alpha x),$$

$$f_2(x) := (1-\alpha x)^2 + 2x - 1,$$

$$g_1(x) := f_1(x) - 2x + 1,$$

Let

$$\Omega_1 := \left\{x : x\in\Omega_2,\ f_1(x)\geqslant 0,\ g_1(x)\geqslant 0\right\},$$

$$\Omega_2 := \left\{x\ :\ 0<x<\frac{1}{\alpha}\right\}.$$

Then

$$\max\left\{\sup_{\Omega_1}\left[\sqrt{f_1(x)}+\sqrt{g_1(x)}\right]^2,\sup_{\Omega_2}\left[1-\alpha x+\sqrt{f_2(x)}\right]^2\right\}\leqslant\max\left[1+2\beta,\ \frac{1}{\alpha}\ (2-\alpha)\right].$$

P r o o f . Let

$$F_1(x) := \sqrt{f_1(x)}+\sqrt{g_1(x)},\quad \text{for}\quad x\in\Omega_1,$$

$$F_2(x) := 1-\alpha x+\sqrt{f_2(x)},\quad \text{for}\quad x\in\Omega_2.$$

We shall first find an estimate for F_1^2. Under the assumptions made on α and β we have for any $x \in R$ the inequality

$$x^2 \left(1 + \frac{\alpha\beta}{1-\beta}\right) + \frac{\beta^2}{4(1-\beta)^2} \geqslant 0.$$

Setting $d := \frac{\beta}{2(1-\beta)}$ we can write this inequality in the form

$$x^2[1+\alpha\beta(1+2d)] + x[2d-\beta(1+2d)] + d^2 \geqslant 0.$$

Hence, by the definition of f_1, we obtain

$$f_1(x)(1+2d) = -\alpha\beta(1+2d)x^2 + \beta(1+2d)x \leqslant$$
$$\leqslant x^2 + 2dx + d^2 = (x+d)^2,$$

and, furthermore,

$$f_1(x)g_1(x) = f_1(x)\left[f_1(x) - 2x+1\right] =$$
$$= f_1(x)\left[f_1(x) + (1+2d) - 2(x+d)\right] \leqslant$$
$$\leqslant f_1^2(x) + (x+d)^2 - 2f_1(x)(x+d) = \left[x+d-f_1(x)\right]^2,$$

that is,

(i) $$f_1(x)g_1(x) \leqslant \left[x + d - f_1(x)\right]^2, \qquad \forall x \in R.$$

For $x \in \Omega_1$ we have

$$d + x - f_1(x) > d - f_1(x) \geqslant \frac{\beta}{2(1-\beta)} - \frac{\beta}{4\alpha} =$$

$$= \frac{\beta}{4\alpha(1-\beta)}\left[2\alpha - (1-\beta)\right] \geqslant \frac{\beta}{4\alpha(1-\beta)}(2\alpha-1) \geqslant 0,$$

since $\sup\limits_{\Omega_1} f_1(x) = \frac{\beta}{4\alpha}$.

Taking the square root of inequality (i) we get

$$\sqrt{f_1(x)}\,\sqrt{g_1(x)} \leqslant x + d - f_1(x) \quad \text{for} \quad x \in \Omega_1.$$

This inequality yields the estimate

$$F_1^2(x) = 2\sqrt{f_1(x)}\,\sqrt{g_1(x)} + 2f_1(x) + 1 - 2x \leqslant 1 + 2d.$$

Hence, making use of the assumption made on β and of the definition of d, we have

$$\text{(ii)} \qquad \sup_{x \in \Omega_1} F_1^2(x) \leqslant 1 + \frac{\beta}{1-\beta} \leqslant 1 + 2\beta.$$

Let us now find an estimate for $F_2(x)$ on Ω_2. From the evident inequality

$$\alpha^2(2x - 1) \leqslant 1 - 2\alpha(1 - \alpha x), \quad \forall x \in R,$$

we have

$$\alpha^2 f_2(x) = \alpha^2\left[(1-\alpha x)^2 + 2x - 1\right] \leqslant \left[1-\alpha(1-\alpha x)\right]^2 = \frac{1}{4}\left[f_2'(x)\right]^2$$

that is

$$\text{(iii)} \qquad \alpha^2 f_2(x) \leqslant \frac{1}{4}\left[f_2'(x)\right]^2 \quad \text{for} \quad x \in R.$$

If $x \in \Omega_2$, then $f_2(x) > 0$ and

$$\frac{1}{2} f_2'(x) = 1 - \alpha(1-\alpha x) \geqslant 1 - \alpha > \frac{1}{2}.$$

From inequality (iii) we get

$$\alpha \leqslant \frac{f_2'(x)}{2\sqrt{f_2(x)}},$$

whence

$$F_2'(x) = -\alpha + \frac{f_2'(x)}{2\sqrt{f_2(x)}} \geqslant 0 \quad \text{for} \quad x \in \Omega_2,$$

that is F_2 is a non-decreasing function on Ω_2. Therefore

$$\sup_{\Omega_2} F_2(x) = F_2\left(\frac{1}{\alpha}\right) = \sqrt{\frac{2-\alpha}{\alpha}},$$

since $F_2 > 0$ on Ω_2, and we finally have

$$\sup_{\Omega_2} F_2^2(x) \leqslant \frac{1}{\alpha}(2-\alpha).$$

This estimate, together with the estimate (ii), gives the assertion of Lemma 2.3.

T h e o r e m 2.2 (on the non-conditional stability). If we apply the difference pattern (2.2) to the initial problem (2.1) with constant coefficients a, b, c ($a > 0$), then this pattern is non-conditionally L_2-stable for $\tau \in (0, \tau_*)$, where

$$\tau_* := \sup\left\{\tau : c\tau < 1,\ b^2\tau \leqslant a,\ 0 < \tau < \tau_1\right\},$$

$$\tau_1 := \begin{cases} +\infty & \text{for} \quad c \leqslant 0, \\ 2\sqrt{\dfrac{a}{c(a+b^2)}} & \text{for} \quad c > 0. \end{cases}$$

P r o o f . The proof of the theorem will be based on von Neumann's theorem on the sufficient condition for the L_2-stability of a difference pattern.

Assuming that Theorem 2.2 holds, we state, by Theorem 2.1, that

$$\sup_{\lambda \in \operatorname{spectr} C} \lambda(\varphi, \delta, \tau) = \max\left\{\left(\frac{|b|}{1+\delta-\frac{c\tau}{2}}\sqrt{\frac{\delta\tau}{2a}} + \sqrt{\left[\frac{b^2\delta\tau}{2a\left(1+\delta-\frac{c\tau}{2}\right)^2} - \frac{1-\delta+\frac{c}{2}\tau}{1+\delta-\frac{c}{2}\tau}\right]_+}\right)^2,\right. \tag{2.14}$$

$$\left(\frac{\delta}{1+\delta-\frac{c\tau}{2}}+\sqrt{\left[\frac{\delta^2}{\left(1+\delta-\frac{c\tau}{2}\right)^2}+\frac{1-\delta+\frac{c}{2}\tau}{1+\delta-\frac{c\tau}{2}}\right]_+}\right)^2\Bigg\} \text{ for } \delta\in(0,\infty).$$

Let us now set $\alpha := 1-\frac{c}{2}\tau$, $\beta := \frac{b^2\tau}{2a}$. If $\tau\in(0,\tau_*)$, then $\alpha<\frac{1}{2}$, $0\leqslant\beta\leqslant\frac{1}{2}$. The non-singular change of variables $(\delta\to x)$, given by $x=\frac{1}{1+\delta-\frac{c\tau}{2}}$ maps $(0,\infty)$ onto the interval $(0,\frac{1}{\alpha})$.

Then we have

$$\frac{b^2\delta\tau}{2a\left(1+\delta^2-\frac{c}{2}\tau\right)^2}=\beta x(1-\alpha x) =: f_1(x),$$

$$\frac{1-\delta+\frac{c}{2}\tau}{1+\delta-\frac{c}{2}\tau}=2x-1,$$

$$\frac{\delta^2}{\left(1+\delta-\frac{c}{2}\tau\right)^2}=(1-\alpha x)^2 =: f_2(x)+1-2x.$$

Setting additionally $g_1(x) := f_1(x)-2x+1$,

$$\Omega_2 := \left(0,\frac{1}{\alpha}\right),$$

$$\Omega_1 := \left\{x : x\in\Omega_2,\ f_1(x)\geqslant 0,\ g_1(x)\geqslant 0\right\},$$

we obtain from formula (2.14)

$$\sup_{\delta\in(0,\infty)}\ \sup_{\lambda\in\operatorname{spectr} C}|\lambda(\varphi,\delta,\tau)| =$$

$$=\sup_{x\in\Omega_1}\max\left\{\left(\sqrt{f_1(x)}+\sqrt{[g_1(x)]_+}\right)^2,\left(1-\alpha x+\sqrt{[f_2(x)]_+}\right)^2\right\}=$$

$$=\max\left\{\sup_{\Omega_1}\left(\sqrt{f_1(x)}+\sqrt{g_1(x)}\right)^2,\ \sup_{\Omega_2}\left(1-\alpha x+\sqrt{f_2(x)}\right)^2\right\}.$$

From Lemma 2.3 it follows that for $\tau \in (0, \tau_*)$ and for any eigenvalue λ of the transition matrix C given by formula (2.8) there holds the inequality

$$|\lambda| \leqslant \max\left[1+2\beta, \frac{1}{\alpha}(2-\alpha)\right] =$$

$$= \max\left[1 + \frac{b^2}{a}\tau, \frac{1+\frac{c}{2}\tau}{1-\frac{c}{2}\tau}\right] \leqslant 1 + \tau \max\left(\frac{b^2}{a}, 2c\right),$$

since the condition $c\tau < 1$ implies that

$$\frac{1+\frac{c}{2}\tau}{1-\frac{c}{2}\tau} = 1 + \frac{c\tau}{1-\frac{c}{2}\tau} \leqslant 1 + 2c\tau;$$

that is

$$(2.15) \qquad |\lambda| \leqslant 1 + M\tau, \quad M = \max\left(\frac{b^2}{a}, 2c\right).$$

Moreover, it follows from Lemma 2.1 that under the assumptions of Theorem 2.2 the modules of the double eigenvalues of the matrix C are less than 1.

Thus, by von Neumann's theorem on the sufficient condition for L_2-stability, the proof of Theorem 2.2 is completed.

The results related to the case of variable coefficients will be presented in a paper prepared for publication, where the case of several spatial variables will also be investigated.

REFERENCES

[1] R. Richtmyer, K. Morton: Raznastnye metody rešeniya krayevykh zadač, Moskva 1977.

[2] V.K. Sauliev: Ob odnom sposobe čislennogo integrirovaniya uravneniy diffuzii. DAN SSSR 115 (1957) 1077-1079.

[3] V.K. S a u l i e v : Integrirovanije uravneniy paraboličeskogo tipa metodom setok, F.M. Moskva 1960.
[4] V u V a n H u n g : Doctoral dissertation. Warszawa 1986.

30-HANG DA, HANOI, VIETNAM
Received March 13, 1987.