DEMONSTRATIO MATHEMATICA
Vol. XXII No 1 1989

Marcin Studniarski

A SIMPLE DERIVATION OF SUFFICIENT CONDITIONS FOR A LOCAL MINIMUM OF A LIPSCHITZIAN FUNCTION

1. Introduction

The aim of this note is to present a simple derivation of sufficient conditions for a strict local minimum of a locally Lipschitzian function defined on a normed space. The conditions obtained include the classical higher order sufficient conditions for Fréchet differentiable functions. We shall formulate our result in two forms. The first of them (Theorem 1) has a straightforward proof (in which only the mean value theorem of Lebourg is used) and enables a comparison with the classical conditions when the function considered is differentiable. The second form (Theorem 2) follows easily from the first one and is similar to the conditions occurring in the recent results of Chaney [2], [3]. The second order sufficient conditions of Chaney concern a more general situation than that considered in this paper. However, our theorems include also conditions of order greater than two. Let us also note that a quite different approach to first and second order conditions for a minimum of a locally Lipschitzian function is presented in ([1], Section 2).

Throughout the paper, X will be a normed space with norm $\|\cdot\|$, W - an open non-empty subset of X, and $f : W \to \mathbb{R}$ - a locally Lipschitzian function (i.e. a function satisfying the Lipschitz condition in a neighbourhood of any point $x \in W$). X^* will denote the topological dual space of X. We recall that the generalized directional derivative of f at x in the direction $v \in X$ is defined by

$$f^{o}(x;v) = \limsup_{\substack{h \to 0 \\ \lambda \downarrow 0}} \lambda^{-1}(f(x+h+\lambda v) - f(x+h)).$$

The function $f^{o}(x;\cdot) : X \to \mathbf{R}$ is convex and positively homogeneous. The generalized gradient of f at x is the set

$$(1) \qquad \partial f(x) = \{x^* \in X^* \mid \forall v \in X,\ f^{o}(x;v) \geq \langle x^*, v\rangle\}$$

(see [4] for more information about these notions).

2. Sufficient optimality conditions

We shall consider the problem of minimization of f over $x \in W$. The following two theorems give conditions sufficient for a point $\bar{x}$ to be a local solution to this problem.

Theorem 1. Let $\bar{x} \in W$. Suppose that there exist a neighbourhood U of $\bar{x}$ ($U \subset W$) and a function $\psi : U \setminus \{\bar{x}\} \to]0,+\infty[$ such that

$$(2) \qquad \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \psi(x) f^{o}(x;\bar{x} - x) < 0.$$

Then f attains a strict local minimum at $\bar{x}$ (i.e. there exists a neighbourhood V of $\bar{x}$ such that $f(x) > f(\bar{x})$ for all $x \in V$, $x \neq \bar{x}$).

This theorem is a particular case of ([8], Theorem 5.1). However, the proof given below is simpler since it does not require the knowledge of the theory presented in [8].

Proof. It follows from (2) that there exists a convex neighbourhood V of $\bar{x}$ such that

$$(3) \qquad \sup_{x \in V \setminus \{\bar{x}\}} \psi(x) f^{o}(x;\bar{x} - x) < 0.$$

Suppose that the desired conclusion is false; then there exists $y \in V$ such that $y \neq \bar{x}$ and $f(y) \leq f(\bar{x})$. By Lebourg's mean value theorem [5,6], we have

$$f(\bar{x}) - f(y) = \langle x^*, \bar{x} - y\rangle \quad \text{where} \quad x^* \in \partial f(z),$$

$z = \bar{x} + \theta(y - \bar{x})$ for some $\theta \in \,]0,1[$. Hence from (1) we obtain $0 \leqslant f(\bar{x}) - f(y) \leqslant f^o(z;\bar{x} - y)$. Since $\bar{x} - z = \theta(\bar{x} - y)$ and $f^o(z;\cdot)$ is positively homogeneous, we have

$$\psi(z)f^o(z;\bar{x} - z) = \theta\psi(z)f^o(z;\bar{x} - y) \geqslant 0,$$

which contradicts (3) because $z \in V$ and $z \neq \bar{x}$.

Theorem 2. Let $\bar{x} \in W$. Suppose that there exist a neighbourhood U of $\bar{x}$ ($U \subset W$) and a function $\psi : U\setminus\{\bar{x}\} \to \,]0,+\infty[$, such that

$$\limsup_{n\to\infty} \psi(x_n)\langle x_n^*, x_n - \bar{x}\rangle > 0$$

whenever $\{x_n\}$ and $\{x_n^*\}$ are sequences in U and X^*, respectively, such that $\{x_n\}$ converges to $\bar{x}$, $x_n \neq \bar{x}$ for every n, and $x_n^* \in \partial f(x_n)$ for every n. Then f attains a strict local minimum at $\bar{x}$.

Proof. It suffices to show that the assumptions of Theorem 2 imply condition (2). Suppose that (2) is false; then there exists a sequence $\{x_n\}$ in U such that $\{x_n\}$ converges to $\bar{x}$, $x_n \neq \bar{x}$ for every n, and

$$\lim_{n\to\infty} \psi(x_n)f^o(x_n;\bar{x} - x_n) \geqslant 0$$

(where the limit may be finite or equal to $+\infty$).

By [4, Proposition 1], we have, for every n,

$$f^o(x_n;\bar{x} - x_n) = \max\{\langle x^*, \bar{x} - x_n\rangle \mid x^* \in \partial f(x_n)\}.$$

Therefore, we can choose a sequence $\{x_n^*\}$ in X^* such that $x_n^* \in \partial f(x_n)$ for every n, and

$$-\lim_{n\to\infty} \psi(x_n)\langle x_n^*, x_n - \bar{x}\rangle = \lim_{n\to\infty} \psi(x_n^*)\langle x_n, \bar{x} - x_n\rangle \geqslant 0,$$

which contradicts our assumptions.

Remark. In particular, we can assume that the function ψ occurring in Theorems 1 and 2 is defined by $\psi(x) =$

$= \|x - \bar{x}\|^{-k}$ where k is any positive integer. In this way, we obtain a sequence of sufficient conditions which, as is shown below, generalize the classical k-th order conditions for Fréchet differentiable functions. In order to compare our results with those of Chaney [2], let us observe that the assumptions of Theorem 2 for $\psi(x) = \|x - \bar{x}\|^{-2}$ are somewhat stronger than those of ([2], Theorem 3.1). On the other hand, Theorem 2 has a simpler proof and includes also higher order conditions.

3. Comparison with the classical higher order sufficient conditions

Let us now assume that f is k times $(k > 1)$ Frechet differentiable in W. The derivatives $f^{(m)}(x)$, $m = 1,2,\dots,k$, for $x \in W$, will be interpreted as m-linear forms on X^m. We shall show that if f satisfies the classical k-th order sufficient conditions for a local minimum (in such a form as in ([7], Theorem 24), then it satisfies the assumptions of Theorem 1, as well.

Theorem 3. Let $\bar{x} \in W$. Suppose that the derivatives of f at $\bar{x}$, of order $1,2,\dots,k-1$, are equal to zero, and that there exists $\delta > 0$ such that $f^{(k)}(\bar{x})\cdot h^k \geq \delta$ for all unit vectors h in X (where $h^k = (h,\dots,h) \in X^k$). Then k is the first positive integer for which the inequality

$$\limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \langle f'(x), \bar{x} - x\rangle / \|x - \bar{x}\|^k < 0 \tag{4}$$

is true.

Remark. Condition (4) implies that the assumptions of Theorem 1 are satisfied for $\psi(x) = \|x - \bar{x}\|^{-k}$ since, for a continuously differentiable f, we have $f^{o}(x;v) = \langle f'(x), v\rangle$ (by [4], Proposition 4).

Proof. Applying Taylor's formula ([7], Theorem 21) to the function $f' : W \to X^*$, we obtain that, for each x in some neighbourhood of $\bar{x}$,

$$f'(x) = (f')^{(k-1)}(\bar{x})\cdot(x-\bar{x})^{k-1}/(k-1)! + \|x-\bar{x}\|^{k-1}\alpha(x)$$

where $\lim_{x\to\bar{x}} \alpha(x) = 0$ (with respect to the standard norm in X^*). Hence

$$\langle f'(x),\bar{x} - x\rangle/\|x - \bar{x}\|^k =$$

$$= -\frac{1}{(k-1)!} f^{(k)}(\bar{x}) \cdot \left(\frac{x - \bar{x}}{\|x-\bar{x}\|}\right)^k + \|x - \bar{x}\|^{k-1}\left\langle\alpha(x), \frac{\bar{x} - x}{\|x-\bar{x}\|^k}\right\rangle \leqslant$$

$$\leqslant -\frac{\delta}{(k-1)!} + \left\langle\alpha(x), \frac{\bar{x} - x}{\|\bar{x}-x\|}\right\rangle \leqslant -\frac{\delta}{(k-1)!} + \|\alpha(x)\|,$$

which implies (4).

Since $f'(\bar{x}) = 0$ and f' is continuous, it is obvious that (4) does not hold for $k = 1$. Now, let $m \in \{2,\dots,k-1\}$. Proceeding similarly as before and using the fact that $f'(\bar{x}),\dots$ $\dots,f^{(m)}(\bar{x})$ are equal to zero, we obtain

$$\langle f'(x),\bar{x} - x\rangle/\|x - \bar{x}\|^m = \langle\alpha(x),\bar{x} - x\rangle/\|x - \bar{x}\|$$

where $\lim_{x\to\bar{x}} \alpha(x) = 0$, and so, condition (4) does not hold with k replaced by m.

E x a m p l e s . We shall show that, even for differentiable functions, our sufficient conditions (that is, condition (4) for $k = 2,3,\dots$) can give more information than the classical ones.

1) Let $f : \mathbb{R}\to\mathbb{R}$ be given by $f(x) = |x|^3$ and let $\bar{x} = 0$. Then f satisfies (4) for $k = 3$. It is twice differentiable, but $f'''(0)$ does not exist, and so, the classical conditions cannot be used.

2) Let $f : \mathbb{R}^2\to\mathbb{R}$ be given by $f(x_1,x_2) = x_1^2 + x_2^4$. Then (4) holds for $\bar{x} = (0,0)$ and $k = 4$. However, the assumptions of Theorem 3 are not satisfied for any k since $f''(0,0)$ is neither positive nor identically zero.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ,
90-238 ŁÓDŹ, POLAND
Received March 6, 1987.