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ON SOME PROPERTIES OF THE SPHERICAL BUNDLE $T_1Ł_2$

The 2-dimensional Poincaré model of the hiperbolic space is well known. Our considerations are based on the 2-dimensional Riemannian manifold $Ł_2$ with a support $\underline{Ł_2} = \{(x^1, x^2) \in R^2 : x^2 > 0\}$. The differential structure is induced from R^2. The fundamental metric form is given by the formula

$$ds^2 = \left(\frac{k}{x^2}\right)^2 \left[(dx^1)^2 + (dx^2)^2\right],$$

where k is an arbitrary positive constant.

In the paper [3] the geometry of the tangent bundle to $Ł_2$ with the g^c-pseudoriemannian metric which is a complete lift of the metric given on $Ł_2$ to $TŁ_2$ is considered.

It is interesting to give fundamental properties of the spherical bundle $T_1Ł_2$. This paper is concerned in this problem.

We denote by $T_1Ł_2$ the set of all tangent vectors to $Ł_2$ of the length 1. The structure on $T_1Ł_2$ is induced from $TŁ_2$. If we reduce the structural group of the bundle $TŁ_2$ to $O(2)$, we obtain the spherical bundle $T_1Ł_2$.

In the papers [4], [5], [7] the authors investigate the properties of spherical bundle T_1M^2 with the metric

$$d\sigma^2 = g_{ik}dx^i dx^k + g_{ik}\delta y^i \delta y^k,$$

where g_{ik} are local coordinates of the metric tensor on the 2-dimensional Riemannian manifold (M^2,g), x^i are local coordinates on a base, y^i are coordinates in a fibre and

$$\delta y^i = dy^i + \Gamma^i_{jk} y^j dx^k.$$

This Riemannian metric was defined by S. Sasaki [9]. S. Sasaki's paper [10] is concerned with the geometry of (2n-1)-dimensional Riemannian manifold T_1M^n. The aim of this paper is a consideration of the manifold $T_1Ł_2$ with the metric different from Sasaki's metric, namely with the metric induced by the immersion in $TŁ_2$.

We introduce on $T_1Ł_2$ a local coordinate system (x^1,x^2,x^3), where (x^1,x^2) are coordinates of points on $Ł_2$ and x^3 is an angle formed by a tangent vector at this point and a vector $\frac{\partial}{\partial x^1}$ of the natural base. We denote by f the immersion of $T_1Ł_2$ into $TŁ_2$. We have

$$f : T_1Ł_2 \to TŁ_2$$

$$(x^1,x^2,x^3) \mapsto \left(x^1,\ x^2,\ \frac{1}{k}\, x^2 \cos x^3,\ \frac{1}{k}\, x^2 \sin x^3\right).$$

The coordinates $a_{\alpha\beta}$, $\alpha,\beta = 1,2,3$, of the metric tensor a on $T_1Ł_2$ induced by f have the following form

$$a_{\alpha\beta} = \frac{\partial f^A}{\partial x^\alpha}\frac{\partial f^B}{\partial x^\beta}\, g^c_{AB}, \quad A,B = 1,\dots,4.$$

Thus we obtain

$$\left[a_{\alpha\beta}\right] = \begin{bmatrix} -\frac{2k \sin x^3}{(x^2)^2} & \frac{k \cos x^3}{(x^2)^2} & -\frac{k \sin x^3}{x^2} \\ \frac{k \cos x^3}{(x^2)^2} & 0 & \frac{k \cos x^3}{x^2} \\ -\frac{k \sin x^3}{x^2} & \frac{k \cos x^3}{x^2} & 0 \end{bmatrix}.$$

The tensor field a defined by this way on $T_1 Ł_2$ is symmetric of the rank 2 and the signature (-,+,0). The structure given by a on $T_1 Ł_2$ we will call a degenerated Riemannian structure ([2], [8]). Any such structure is called semi-Riemannian manifold ([1]). First we consider the existence of a torsionless connection associated to this degenerated metric. It is well known there exists such unique connection on a Riemannian manifolds ([6], Th. 2.2, p. 158).

There exist manifolds with degenerated metrics for which connections without torsion do not exist. If such connection exists, then respectively to Weyl-Cartan theorem it is not uniquely determined.

Let us define a mapping $b : TT_1Ł_2 \to (TT_1Ł_2)^*$ by the formula $b(X) = a(X,-)$. The kernel of this mapping is a 1-dimensional distribution V. This distribution is spanned by the vector field $E_1 = [-x^2\cos x^3, -x^2\sin x^3, \cos x^3]$. Of course, V is an integrable distribution. The complementary distribution H is a 2-dimensional one. The fields $E_2=[0, x^2\cos x^3, \sin x^3]$ and $E_3 = [1,0,0]$ form a base of H. It is easy to see that H is an integrable distribution, too.

T h e o r e m 1. There exists no degenerated connection without torsion on the manifold $(T_1Ł_2, a)$.

P r o o f . It is sufficient to show (with respect to Th. 3.2 [8]) that the field E_1 does not satisfy the system of Killing differential equations $L_Y a = 0$, e.a.

$$-2Y^1_{|1}\sin x^3 + Y^2_{|1}\cos x^3 - x^2Y^3_{|1}\sin x^3 + \frac{2}{x^2} Y^2\sin x^3 - Y^3\cos x^3 = 0,$$

$$Y^1_{|1}\cos x^3 + x^2Y^3_{|1}\cos x^3 - 2Y^1_{|2}\sin x^3 + Y^2_{|2}\cos x^3 - x^3Y^3_{|2}\sin x^3 - \frac{2}{x^2} Y^2\cos x^3 - Y^3\sin x^3 = 0,$$

$$Y^2_{|1}x^2\cos x^3 - Y^1_{|1}x^2\sin x^3 - 2\sin x^3 Y^1_{|3} + Y^2_{|3}\cos x^3 - Y^3_{|3}x^2\sin x^3 + Y^2\sin x^3 - Y^3x^2\cos x^3 = 0,$$

$$Y^1_{|2} + x^2 Y^3_{|2} = 0,$$

$$Y^2_{|2} x^2 \cos x^3 - Y^1_{|2} x^2 \sin x^3 + Y^1_{|3} \cos x^3 + Y^3_{|3} x^2 \cos x^3 - \\ - Y^2 \cos x^3 - Y^3 x^2 \sin x^3 = 0,$$

$$Y^1_{|3} \sin x^3 - Y^2_{|3} \cos x^3 = 0.$$

It is easy to verify that $L_{E_1} a \neq 0$.

R e m a r k . Let v and h denote the projections of the space $T(T_1Ł_2)$ onto the distributions V and H, respectively. By Q we denote an arbitrary tensor field of the type (1.2) and by ∇ a degenerated Riemannian connection. The family of degenerated Riemannian connections $\tilde{\nabla}$ such that $\tilde{\nabla}_X Y = \nabla_X Y + P(X,Y)$ for $X, Y \in T(T_1Ł_2)$ and $P(X,Y) = \frac{1}{2}\big(\delta^\alpha_\beta \delta^\gamma_\xi + \delta^\alpha_\beta v^\gamma_\xi - a_{\beta\xi} a^{\alpha\gamma} - v^\alpha_\beta h^\gamma_\xi\big) Q^\delta_{\varkappa\alpha} X^\varkappa Y^\beta$, where $\alpha, \beta, \gamma, \delta, \varkappa, \xi = 1,2,3$ and $X^\varkappa$, Y^β denote coordinated of vector fields X, Y respectively, does not depend of a choice of the distribution H. The torsion tensor field T does not vanish and its coordinates are given by $T^\alpha_{\beta\gamma} = P^\alpha_{\beta\gamma} - P^\alpha_{\gamma\beta}$.

Let S^1 denote the 1-dimensional sphere. Then we have the following theorem.

T h e o r e m 2. The spaces $H_{(x^1,x^2,x^3)}$, $V_{(x^1,x^2,x^3)}$ are diffeomorphic to $T_{(x^1,x^2)}Ł_2$ and $T_{x^3}S^1$, respectively.

P r o o f . It is known that $T_1Ł_2$ is diffeomorphic to $Ł_2 \times S^1$ ([11], p.242). Thus we have $T_{(x^1,x^2,x^3)}T_1Ł_2 = T_{(x^1,x^2)}Ł_2 \oplus T_{x^3}S^1$. On the other hand we have $T_{(x^1,x^2,x^3)}T_1Ł_2 = H_{(x^1,x^2,x^3)} \oplus V_{(x^1,x^2,x^3)}$. The mapping $\Phi : T_{(x^1,x^2)}Ł_2 \oplus T_{x^3}S^1 \to H_{(x^1,x^2,x^3)} \oplus V_{(x^1,x^2,x^3)}$ defined by $\Phi\left(\frac{\partial}{\partial x^1}\right) = E_3$, $\Phi\left(\frac{\partial}{\partial x^2}\right) = E_2$, $\Phi\left(\frac{\partial}{\partial x^3}\right) = E_1$ is desired diffeomorphism.

We will show that

T h e o r e m 3. The isometry group of the manifold $(T_1 Ł_2, a)$ is 3-dimensional.

P r o o f . To find the isometry group of the manifold $T_1 Ł_2$ we have to determine $I = \{ i \in K : i(f(T_1 Ł_2)) \subset f(T_1 Ł_2) \}$, where K denotes the isometry group of the manifold $TŁ_2$ ([3]). Some calculations show that I is a subgroup of K generated by 1-parameter transformation groups determined by complete lifts to $TŁ_2$ ([3]) of Killing vector fields of the manifold $Ł_2$. The immersion f is isometric thus an arbitrary isometry of $T_1 Ł_2$ can be locally represented by composition of isometries of the form $f^{-1} \circ i \circ f$, where $i \in I$.

REFERENCES

[1] М.А. А к и в и с , Б.П. Ч е б ы ш е в а : Об инвариантном оснащении полуриманова многообразия, Sibirsk. Mat. Ž., 22, N° 6 (1981), 7-14.

[2] И.В. Б е л ь к о : О вырожденных римановых метриках. Mat. Zametki 18, N° 5 (1975), 767-774.

[3] H. F e l i ń s k a : Geometry of the tangent bundle over 2-dimensional hiperbolic space, submitted to Serdica,

[4] А.Л. Я м п о л ь с к и й : К геометрии сферических касательных расслоений римановых многообразий, Ukrain. Geometr. Sb., 24 (1981), 129-132.

[5] W. K l i n g e n b e r g , S. S a s a k i : On the tangent bundle of 2-sphere, Tôhoku Math. Journ. 27 (1975), 49-56.

[6] S. K o b a y a s h i , K. N o m i z u : Fondations of differential geometry, Interscience publishers, New York-London, 1963.

[7] P.T. N a g y : On the tangent sphere bundle of a Riemannian 2-manifold, Tôhoku Math. Journ. 29 (1977), 203--208.

[8] V. O p r o i u : Degenerate Riemannian and degenerate conformal connexions, Analele Ştüntifice ale Universitătii "Al. I. Cuza", Dui Iaşi, sect. I, 16 (1970), 357-376.

[9] S. S a s a k i : On the differential geometry of tangent boundles of Riemannian manifolds, Tôhoku Math. Jour. 10 (1958), 338-354.

[10] S. S a s a k i : On the differential geometry of tangent bundles of Riemannian manifolds II, Tôhoku Math. Jour. 14 (1962), 146-155.

[11] И. Т а м у р а : Топология слоений. Москва 1979.

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