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CONVERGENCE OF DIFFERENCE METHODS FOR BOUNDARY VALUE PROBLEMS OF ODE'S WITH DISCONTINUITIES

1. Introduction

Let B be a real Banach space. We consider the boundary value problem on $I = [\alpha, \beta]$, $\alpha < \beta$:

$$y''(t) = f(t, y(t)), \quad t \in I, \tag{1}$$

with the boundary conditions

$$y(\alpha) = y_0, \quad y(\beta) = y_1, \tag{2}$$

where f is a given function $f: I \times B \to B$. We assume that for fixed continuous function $y: I \to B$ the mapping $t \to f(t, y(t))$ is integrable. By a solution of (1)-(2), we mean a function $\varphi: I \to B$ which has an absolutely continuous first derivative on I, and satisfies the boundary conditions (2) and the equation (1) almost everywhere on I i.e. except on a set of Lebesgue measure zero.

Our task is to define a numerical solution of (1)-(2), so due to this fact it will be assumed that the problem (1)-(2) has the bounded solution φ. A fundamental class of numerical methods is based on discrete variables. We consider a discrete set $t_{hi} = \alpha + ih$ for $i \in R_N$ where for some natural number N, $h = (\beta - \alpha)/N$ and $R_N = \{0, 1, \ldots, N\}$. Now we should find a set of corresponding values $y_h(t_{h0}), \ldots, y_h(t_{hN})$ as an approximation to the exact solution $\varphi(t)$ evaluated at $t = t_{hi}$, $i \in R_N$.

The simply numerical method for (1)-(2) is the following

$$(3)\quad \begin{cases} y_h(t+h) - 2y_h(t) + y_h(t-h) = h^2 f(t, y_h(t)), & t \in I_h, \\ y_h(\alpha) = y_0, \qquad y_h(\beta) = y_1, \end{cases}$$

where

$$I_h = \{t_{hi}:\ i=1,2,\dots,N-1\}.$$

Nonstationary linear methods with constant coefficients

$$(4)\quad \begin{cases} y_h(t+h) - 2y_h(t) + y_h(t-h) = \\ = h^2 \sum\limits_{i=1}^{3} b_i\, f(t+2h-ih,\ y_h(t+2h-ih)), & i \in I_h, \\ y_h(\alpha) = y_0, \qquad y_h(\beta) = y_1, \end{cases}$$

or with variable coefficients

$$(5)\quad \begin{cases} y_h(t+h) - 2y_h(t) + y_h(t-h) = \\ = h^2 \sum\limits_{i=1}^{3} b_i(t,h) f(t+2h-ih,\ y_h(t+2h-ih)), & t \in I_h, \\ y_h(\alpha) = y_0, \qquad y_h(\beta) = y_1, \end{cases}$$

are methods of higher order.

To find the numerical solution y_h we want to apply the quasilinear nonstationary method of the form

$$(6)\quad \begin{cases} y_h(t+h) - 2y_h(t) + y_h(t-h) = h^2 \mathcal{F}(t,h,y_h), \\ y_h(\alpha) = y_0, \qquad y_h(\beta) = y_1, \end{cases}$$

where

$$\mathcal{F}(t,h,y_h) = F(t+h,t,t-h,h,y_h(t+h),y_h(t),y_h(t-h)).$$

Indeed, the above mentioned methods are special cases of this general (6).

Linear methods were analysed by several authors under the assumption that f was continuous. The case with discontinuities was discussed in [3] for the method (3). Convergence of onestep and multistep methods, but for initial-value problems with discontinuities, was considered in [3], [4], [6], [9].

The purpose of this paper is to give sufficient conditions for the convergence of the method (6). It will be done under the assumption that F satisfies a Lipschitz condition with a function L and if (6) is consistent.

Obtained the corresponding condition 4^0 of Theorem 2 for convergence is better than it was so far. For linear method (5) will be given conditions when both it is consistent and has convergence of corresponding order.

2. Convergence and consistency

The following definitions are known (see [3], [4], [6], [8], [9]).

Definition 1. We say that the method y_h is convergent to the exact solution φ of (1)-(2) if

$$\lim_{N\to\infty} \max_{i\in R_N} \|\varphi(t_{hi}) - y_h(t_{hi})\| = 0.$$

The order of convergence is p if

$$\max_{i\in R_N} \|\varphi(t_{hi}) - y_h(t_{hi})\| = O(h^p).$$

Definition 2. We say that the numerical method y_h is consistent with the boundary-value problem (1)-(2) on the solution φ if there exists a function $\varepsilon : J_h\times H \to R_+ = [0,\infty)$, $J_h = [\alpha+h, \beta-h]$, $H = [0,h_0]$, $h_0 \in (0,\infty)$ such that the following two conditions are satisfied

$$\|\varphi(t+h) - 2\varphi(t) + \varphi(t-h) - h^2 \mathcal{F}(t,h,\varphi)\| \leqslant \varepsilon(t,h),$$

$$\lim_{N\to\infty} h^{-1} \sum_{i=1}^{N-1} \varepsilon(t_{hi},h) = 0.$$

The order of consistency is p if

$$h^{-1}\sum_{i=1}^{N-1}\varepsilon(t_{hi},h) = O(h^p).$$

R e m a r k 1. The first condition in this definition may be written in the following way

$$\left\| \int_{t-h}^{t+h} K_1(t,\tau)\ f(\tau,\varphi(\tau))d\tau - h^2\mathcal{F}(t,h,\varphi)\right\| \leqslant \varepsilon(t,h),$$

where

$$K_1(t,\tau) = \begin{cases} \tau + h - t & \text{if} \quad t-h \leqslant \tau \leqslant t, \\ t + h - \tau & \text{if} \quad t \leqslant \tau \leqslant t+h. \end{cases}$$

Indeed, the boundary value problem (1)-(2) may be converted to the integral equation

$$y(t) = y_0 + \frac{t-\alpha}{\beta-\alpha}(y_1 - y_0) + \int_\alpha^\beta K(t,\tau)\ f(\tau,y(\tau))d\tau$$

where

$$K(t,\tau) = \begin{cases} \frac{t-\beta}{\beta-\alpha}(\tau - \alpha) & \text{if} \quad \alpha \leqslant \tau \leqslant t, \\ \frac{\tau-\beta}{\beta-\alpha}(t - \alpha) & \text{if} \quad t \leqslant \tau \leqslant \beta. \end{cases}$$

Now it is very easy to get our result.

Consistency is a necessary condition for the discrete convergence of the method (6). Now we want to get a general condition for this fact. The main result of this section is the following consistency theorem.

T h e o r e m 1. If

1° $f: I\times B \to B$, $F: I^3\times H\times B^3 \to B$, and f is bounded,

2° there exists the exact solution φ of (1)-(2),

3° φ'' is a Riemann integrable function, then the method (6) is consistent with the boundary value problem (1),(2) on φ if

(7) $$\lim_{N\to\infty} h \sum_{i=1}^{N-1} \| f(t_{hi}, \varphi(t_{hi})) - \mathfrak{F}(t_{hi}, h, \varphi) \| = 0.$$

P r o o f . Applying the Taylor formula

$$\varphi(t+u) = \sum_{i=0}^{p} \frac{u^i}{i!} \varphi^{(i)}(t) +$$

$$+ \frac{1}{(p-1)!} \int_t^{t+u} (t+u-s)^{p-1} \left[\varphi^{(p)}(s) - \varphi^{(p)}(t)\right] ds$$

for p = 2 gives

$$\varphi(t+h) - 2\varphi(t) + \varphi(t-h) =$$

$$= h^2\varphi''(t) + \int_t^{t+h} (t+h-s) \left[\varphi''(s) - \varphi''(t)\right] ds +$$

$$+ \int_t^{t-h} (t-h-s) \left[\varphi''(s) - \varphi''(t)\right] ds.$$

Now changing the intervals of integration for our integrals we are able to get

$$\varphi(t+h) - 2\varphi(t) + \varphi(t-h) =$$

$$= h^2\varphi''(t) + \int_0^h (h-\theta) \left[\varphi''(t+\theta) + \varphi''(t-\theta) - 2\varphi''(t)\right] d\theta.$$

Integration by parts gives the same result (see [3]).

We note that

$$\varphi(t+h) - 2\varphi(t) + \varphi(t-h) - h^2 \mathfrak{F}(t,h,\varphi) =$$

$$= h^2 P_1(t,h,\varphi) + P_2(t,h,\varphi),$$

where

$$P_1(t,h,\varphi) = \varphi''(t) - \mathcal{F}(t,h,\varphi),$$

$$P_2(t,h,\varphi) = \int_0^h (h-\theta)\,[\varphi''(t+\theta) - \varphi''(t)]\,d\theta +$$

$$+ \int_0^h (h-\theta)\,[\varphi''(t-\theta) - \varphi''(t)]\,d\theta.$$

Now

$$\lim_{N\to\infty} h^{-1} \sum_{i=1}^{N-1} \|P_2(t_{hi},h,\varphi)\| =$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N-1} \left\{ \left\| \int_0^h \left(1 - \frac{\theta}{h}\right)\left[\varphi''(t_{hi}+\theta) - \varphi''(t_{hi})\right] d\theta \right\| + \right.$$

$$\left. + \left\| \int_0^h \left(1 - \frac{\theta}{h}\right) \left[\varphi''(t_{hi}-\theta) - \varphi''(t_{hi})\right] d\theta \right\| \right\} \leqslant$$

$$\leqslant \lim_{N\to\infty} \left\{ h \sum_{i=1}^{N-1} (M_i - m_i) + h \sum_{i=1}^{N-1} (M_{i-1} - m_{i-1}) \right\},$$

where

$$M_i = \sup_{s\in[0,h]} f(t_{hi} + s, \varphi\,(t_{hi} + s)), \quad i=0,1,\ldots,N-1,$$

$$m_i = \inf_{s\in[0,h]} f(t_{hi} + s, \varphi\,(t_{hi} + s)), \quad i=0,1,\ldots,N-1.$$

So if φ'' is a Riemann integrable function then

$$\lim_{N\to\infty} h^{-1} \sum_{i=1}^{N-1} \|P_2(t_{hi},h,\varphi)\| = 0,$$

(see also [3]). This completes the proof.

R e m a r k 2. Theorem 1 remains true if the condition 3^0 is replaced by the following: φ'' is of bounded variation or φ'' satisfies the Hölder condition of order $\gamma \in (0,1]$.

Now the method (6) is consistent of order $\min(1,r)$ or $\min(\gamma,r)$ if φ'' is of bounded variation or φ'' satisfies the Hölder condition of order $\gamma \in (0,1]$, respectively, provided that

$$h \sum_{i=1}^{N-1} \| f(t_{hi}, \varphi(t_{hi})) - \mathcal{F}(t_{hi}, h, \varphi) \| = O(h^r) \quad \text{for every } N.$$

Indeed, this follows from the proof of Theorem 1 and the fact that

$$h^{-1} \sum_{i=1}^{N-1} \| P_2(t_{hi}, h, \varphi) \| = O(h^s),$$

where $s = 1$ if φ'' is of bounded variation, and $s = \gamma$ if φ'' satisfies the Hölder condition of order $\gamma \in (0,1]$.

R e m a r k 3. For the nonstationary linear method (5) the condition (7) has the form

$$1 = b_1(t,h) + b_2(t,h) + b_3(t,h) \quad \text{for} \quad (t,h) \in I \times H,$$

if $\varphi''(t+h) = \varphi''(t) + O(h)$.

Indeed, it is easy to see that

$$\| \varphi''(t) - b_1(t,h)\varphi''(t+h) - b_2(t,h)\varphi''(t) - b_3(t,h)\varphi''(t-h) \| =$$

$$= (1 - b_1(t,h) - b_2(t,h) - b_3(t,h)) \|\varphi''(t)\| + O(h).$$

3. Convergence of the method (6)

In this section we wish to examine the convergence behaviour as $N \to \infty$ (or $h \to 0$) of the approximate solution y_h given by (6). We can prove the following main theorem:

T h e o r e m 2. Suppose that

1^0 the problem (1)-(2) has the exact solution φ,

2^0 $F: I^3 \times H \times B^3 \to B$, and there exist functions $L: I \times H \to R_+$ and $\eta: I \times H \to R_+$ such that for $(s_0, s_1, s_2, h) \in I^3 \times H$, $z_j, \bar{z}_j \in B$, $j = 1,2,3$, we have

$$\|F(s_0,s_1,s_2,h,z_1,z_2,z_3) - F(s_0,s_1,s_2,h,\bar{z}_1,\bar{z}_2,\bar{z}_3)\| \leq$$

$$\leq L(s_1,h)\sum_{i=1}^{3}\|z_i - \bar{z}_i\| + \eta(s_1,h),$$

$$\lim_{N\to\infty} h\sum_{i=1}^{N-1}\eta(t_{hi},h) = 0,$$

3^0 the method (6) is consistent with (1)-(2) on the exact solution φ,

4^0 $s_h = 3h^2 \max_{i=1,2,\dots,N-1} \sum_{j=1}^{N-1} r_{ij} L(t_{hj},h) < 1,$

where

$$r_{ij} = \begin{cases} i - \frac{ij}{N} & \text{if } i \leq j, \\ j - \frac{ij}{N} & \text{if } i > j, \end{cases}$$

then the method (6) is convergent to the exact solution φ of (1)-(2).

P r o o f . Put

$$v_h(t) = \varphi(t) - y_h(t),$$

$$g(t,h) = h^2 \mathcal{F}(t,h,y_h) - \varphi(t+h) + 2\varphi(t) - \varphi(t-h).$$

Using (6) it follows at once that

$$v_h(t+h) - 2v_h(t) + v_h(t-h) = g(t,h).$$

It is known that the solution of this difference equation is given by the formula

$$v_h(t_{hi}) = -\sum_{j=1}^{N-1} r_{ij} g(t_{hj},t), \quad i \in R_n. \tag{8}$$

Observe, from the assumptions of our theorem it follows that

$$\|g(t,h)\| \leqslant h^2 \|\mathcal{F}(t,h,y_h) - \mathcal{F}(t,h,\varphi)\| +$$

$$+ \|\varphi(t+h) - 2\varphi(t) + \varphi(t-h) - h^2 \mathcal{F}(t,h,\varphi)\| \leqslant$$

$$\leqslant h^2 \{L(t,h) \; [\|v_h(t-h)\| + \|v_h(t)\| + \|v_h(t+h)\|] + \eta(t,h)\} +$$

$$+ \varepsilon(t,h).$$

Now, using this and (8) we obtain

$$\|v_h\|_\infty = \max_{i=1,2,\dots,N-1} \|v_h(t_{hi})\| \leqslant$$

$$\leqslant \max_{i=1,2,\dots,N-1} \sum_{j=1}^{N-1} r_{ij} \left\{h^2 \left[L(t_{hj},h) 3 \|v_h\|_\infty + \eta(t_{hj},h)\right] + \varepsilon(t_{hj},h)\right\}.$$

Hence

$$\|v_h\|_\infty \leqslant \frac{\beta - \alpha}{4(1-s_h)} \sum_{j=1}^{N-1} \left[h\eta(t_{hj},h) + h^{-1}\varepsilon(t_{hj},h)\right].$$

To complete this proof it is sufficient to use assumptions 2^0-4^0 of this theorem.

R e m a r k 4. Let

$$L(t,h) = L > 0.$$

Indeed, now we have

$$\max_{i=1,2,\dots,N-1} \sum_{j=1}^{N-1} r_{ij} = \max_i .5(iN - i^2) \leqslant \frac{N^2}{8},$$

and the condition 4^0 will be satisfied if

$$L < \frac{8}{3(\beta-\alpha)^2}. \tag{9}$$

Moreover for

$$\mathcal{F}(t,h,\varphi) = f(t,\varphi)$$

the condition (9) has the form

$$L < \frac{8}{(\beta-\alpha)^2}$$

(see [3]).

Remark 5. Let

$$\sum_{i=1}^{N-1} L(t_{hi},h) \leq M, \quad M > 0.$$

Now we obtain the following inequality

$$s_h \leq 3h^2 \frac{N}{4} M = \frac{3}{4} hM(\beta - \alpha),$$

and the condition 4^o is satisfied if the stepsize h is sufficiently small such that

$$h < \frac{4}{3} \frac{1}{M(\beta - \alpha)} .$$

It is interesting to note that if $\psi:I \to R_+$ is a Lebesgue integrable function such that

$$L(t,h) = \int_t^{t+h} \psi(s)ds,$$

then

$$M = \int_\alpha^\beta \psi(s)ds.$$

4. Consistency of order q of the method (5)

Take the following

Definition 3 (see [4], [9]). We say $\varphi:I \to B$ is in class $S_p^R(I)$, $p \geq 1$, if φ is $p-1$ times differentiable on I and there exists a bounded function we will denote by $\varphi^{(p)}: I \rightarrow B$ such that (p-1)th derivative $\varphi^{(p-1)}$ is the Riemann integral of $\varphi^{(p)}$. We say $\varphi \in S_p^B(I)$ if $\varphi \in S_p^R(I)$ and $\varphi^{(p)}$ is of bounded variation i.e. there exists a constant V such that for any partition $\alpha \leqslant t_0 < t_1 < \ldots < t_r \leqslant \beta$ we have

$$\sum_{j=1}^{r} \left\| \varphi^{(p)}(t_j) - \varphi^{(p)}(t_{j-1}) \right\| \leqslant V.$$

We say $\varphi \in S_p^H(I)$ if $\varphi \in S_p^R(I)$ and $\varphi^{(p)}$ satisfies the Hölder condition of order $\gamma \in (0,1]$.

Let b_1, b_2, b_3 are bounded and

$$C_2(t,h) = 1 - b_1(t,h) - b_2(t,h) - b_3(t,h),$$

$$C_i(t,h) = \frac{1}{i!}\left\{1 + (-1)^i - i(i-1)\left[b_1(t,h) + (-1)^{i-2} b_3(t,h)\right]\right\}$$

for $i=3,4,\ldots$

Now the nonstationary linear method (5) has the following property

Lemma 1. If

1^0 there exists the exact solution of the problem (1)-(2),

2^0 $C_j(t,h) = 0$ for $t \in I$, $h \in H$, $j=2,3,\ldots,p-1$,
$C_p(t,h) \not\equiv 0$,

then the method (5) is consistent of order p-3 if $\varphi \in S_p^R(I)$, and of order p-2 if $\varphi \in S_p^B(I)$ and of order $p-3+\gamma$ if $\varphi \in S_p^H(I)$.

Proof. Using the Taylor formula for $\varphi \in S_p^R(I)$ (see the proof of Th.1) and combining the same terms we have

$$\varphi(t+h) - 2\varphi(t) + \varphi(t-h) -$$

$$- h^2\left[b_1(t,h)\varphi''(t+h) + b_2(t,h)\varphi''(t) + b_3(t,h)\varphi''(t-h)\right] =$$

$$= \sum_{i=2}^{p} h^i \varphi^{(i)}(t)\, C_i(t,h) + T(t,h) = O(h^p) + T(t,h),$$

where

$$T(t,h) = \frac{1}{(p-1)!}\int_t^{t+h}(t+h-s)^{p-1}\left[\varphi^{(p)}(s) - \varphi^{(p)}(t)\right]ds +$$
$$+ \frac{1}{(p-1)!}\int_t^{t-h}(t-h-s)^{p-1}\left[\varphi^{(p)}(s) - \varphi^{(p)}(t)\right]ds -$$
$$- \frac{1}{(p-3)!}h^2 b_1(t,h)\int_t^{t+h}(t+h-s)^{p-3}\left[\varphi^{(p)}(s) - \varphi^{(p)}(t)\right]ds -$$
$$- \frac{1}{(p-3)!}h^2 b_3(t,h)\int_t^{t-h}(t-h-s)^{p-3}\left[\varphi^{(p)}(s) - \varphi^{(p)}(t)\right]ds.$$

Now changing the intervals of integration it is possible to get

$$h^{-1}\sum_{i=1}^{N-1} T(t_{hi},h) = h^{p-2}\sum_{i=1}^{N-1} W(t_{hi},h),$$

where

$$W(t,h) = \frac{1}{(p-1)!}\int_0^h\left(1 - \frac{\theta}{h}\right)^{p-1}\left[\varphi^{(p)}(t+\theta) - \varphi^{(p)}(t)\right]d\theta +$$
$$+ \frac{(-1)^p}{(p-1)!}\int_0^h\left(1 - \frac{\theta}{h}\right)^{p-1}\left[\varphi^{(p)}(t-\theta) - \varphi^{(p)}(t)\right]d\theta -$$
$$- \frac{1}{(p-3)!}b_1(t,h)\int_0^h\left(1 - \frac{\theta}{h}\right)^{p-3}\left[\varphi^{(p)}(t+\theta)-\varphi^{(p)}(t)\right]d\theta -$$
$$- \frac{(-1)^p b_3(t,h)}{(p-3)!}\int_0^h\left(1 - \frac{\theta}{h}\right)^{p-3}\left[\varphi^{(p)}(t-\theta) - \varphi^{(p)}(t)\right]d\theta.$$

Hence

$$h^{-1}\sum_{i=1}^{N-1}\left\|T(t_{hi},h)\right\| = O(h^{p-3+m}),$$

where $m = 0$ if $\varphi \in S_p^R(I)$, and $m = 1$ if $\varphi \in S_p^B(I)$, and $m = \gamma$ if $\varphi \in S_p^H(I)$.

It says that the method (5) is consistent of corresponding order.

R e m a r k 6. We consider the nonstationary method (4) with

$$b_1 = 1/12, \quad b_2 = 10/12, \quad b_3 = 1/12.$$

It is widely used (see for example [5]). The coefficients C_i are

$$C_2 = C_3 = C_4 = C_5 = 0, \qquad C_6 = -1/240.$$

We see that $p = 6$. Now this method is consistent of order 3 or 4 or $3-\gamma$ if $\varphi \in S_6^R(I)$ or $\varphi \in S_6^B(I)$ or $\varphi \in S_6^H(I)$, respectively.

Now taking the stationary method (3) we see

$$C_2 = C_3 = 0, \quad C_4 = 1/12,$$

and $p = 4$.

For the nonstationary method (5) with variable coefficients

$$b_1(t,h) = b_3(t,h) = (\sqrt{t+h} - \sqrt{t})/12,$$

$$b_2(t,h) = 1 - (\sqrt{t+h} - \sqrt{t})/6,$$

we have the similar result, namely

$$C_2 = C_3 = 0, \quad C_4 = (1 - \sqrt{t+h} + \sqrt{t})/12,$$

and hence $p = 4$.

Now the last two methods are consistent of order 1 or 2 or $1-\gamma$ if $\varphi \in S_4^R(I)$ or $\varphi \in S_4^B(I)$ or $\varphi \in S_4^H(I)$, respectively.

Now we are in a position to establish the convergence theorem for the nonstationary linear method (5). It follows directly from Theorem 2 and Lemma 1.

T h e o r e m 3. Assume that

1^0 the assumptions of Lemma 1 are satisfied,

2^0 the assumptions 1^0, 2^0, 4^0 of Theorem 2 are satisfied with the condition

$$h \sum_{i=1}^{N-1} \eta(t_{hi},h) = O(h^{\rho}),$$

then the method (5) has convergence of order $\min(\rho,p-3)$ if $\varphi \in S_p^R(I)$, and of order $\min(\rho,p-2)$ if $\varphi \in S_p^B(I)$, and of order $\min(\rho,p+\gamma-3)$ if $\varphi \in S_p^H(I)$.

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