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CONVERGENCE WITH RESPECT TO THE MYCIELSKI σ-IDEAL

In this paper we shall prove that convergence with respect to the Mycielski σ-ideal $\mathcal{J}_M$ in the Cantor set C does not yield a topology in the space of measurable functions. The σ-ideals $\mathcal{J}_M \cap \mathcal{K}$, $\mathcal{J}_M \cap \mathcal{L}$, $\mathcal{J}_M \cap \mathcal{K} \cap \mathcal{L}$ have the same property, where $\mathcal{K}$ and $\mathcal{L}$ are respectively the σ-ideals of sets of the first category and measure zero in C.

Let $(X, \mathcal{S})$ be a measurable space. Let $\mathcal{J}$ be a proper σ-ideal in a σ-field $\mathcal{S}$.

D e f i n i t i o n 1.1. We shall say that a property holds $\mathcal{J}$-almost everywhere on a set X (abbr. $\mathcal{J}$-a.e.) if and only if the set of points of X which do not have this property belongs to $\mathcal{J}$.

D e f i n i t i o n 1.2. We shall say that a sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable functions defined on X convergence with respect to $\mathcal{J}$ to an $\mathcal{S}$-measurable function f defined on X if and only if every subsequence $\{f_{m_n}\}_{n\in N}$ of $\{f_n\}_{n\in N}$ contains a subsequence $\{f_{p_{m_n}}\}_{n\in N}$ convergent to f $\mathcal{J}$-a.e. on X. We shall use the notation $f_n \xrightarrow[n\to\infty]{\mathcal{J}} f$.

The set of $\mathcal{S}$-measurable functions defined on X equipped with convergence with respect to $\mathcal{J}$ is an $\mathcal{L}^*$ space. We can define the closure of the set A: $f \in \bar{A}$ iff there exists a sequence $\{f_n\}_{n\in N}$ in A such that $f_n \xrightarrow[n\to\infty]{\mathcal{J}} f$. This operation has the properties:

$$\bar{\emptyset} = \emptyset, \quad A \subset \bar{A}, \quad \overline{A \cup B} = \bar{A} \cup \bar{B} \quad \text{for every } A \text{ and } B.$$

The condition $\bar{A} = \bar{\bar{A}}$ is fulfilled if and only if the convergence has the following property:

(*) If $f_j \xrightarrow[j\to\infty]{\mathfrak{J}} f$, and $f_{j,n} \xrightarrow[n\to\infty]{\mathfrak{J}} f_j$

for $j \in N$, then there exist two sequences of natural numbers $\{j_p\}_{p\in N}$, $\{n_p\}_{p\in N}$ such that $f_{j_p,n_p} \xrightarrow[p\to\infty]{\mathfrak{J}} f$.

If the space $\mathcal{L}^*$ fulfils (*) then the topology introduced by the closure operation described above is often called a Frechet topology.

D e f i n i t i o n 1.3. We shall say that a pair $(\mathfrak{S},\mathfrak{J})$ fulfils the condition (E) if and only if for every set $B \in \mathfrak{S} - \mathfrak{J}$ and for every double sequence $\{B_{j,n}\}_{j,n\in N}$ of $\mathfrak{S}$-measurable sets such that

1^0 $B_{j,n} \subset B_{j,n+1}$ for any $j,n \in N$

2^0 $\bigcup_{n=1}^{\infty} B_{j,n} = B$ for every $j \in N$

there exist an increasing sequence $\{j_p\}_{p\in N}$ of natural numbers and a sequence $\{n_p\}_{p\in N}$ of natural numbers such that $\bigcap_{p=1}^{\infty} B_{j_p,n_p} \notin \mathfrak{J}$.

Definition 1.3 is equivalent to the following one (see p.101 in [3]):

D e f i n i t i o n 1.4. We shall say that a pair $(\mathfrak{S},\mathfrak{J})$ fulfils the condition (E′) if and only if for every set $B \in \mathfrak{S} - \mathfrak{J}$ and for every double sequence $\{B_{j,n}\}_{j,n\in N}$ of $\mathfrak{S}$-measurable sets fulfilling the above conditions 1^0 and 2^0 there exists a sequence $\{n_j\}_{j\in N}$ of natural numbers such that $\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathfrak{J}$.

Conditions (E), (E′) have been defined for the pair $(\mathfrak{S},\mathfrak{J})$ but we can define them also for a pair $(\mathfrak{R},\mathfrak{J})$ where $\mathfrak{R}$ is a family of sets stable under countable intersections and unions and the σ-ideal $\mathfrak{J}$ need not be included in $\mathfrak{R}$.

As the consequence of the theorem proved by Wagner (see th. 1 in [2]) we have

T h e o r e m 1.1. If the set of all $\mathfrak{S}$-measurable real functions is equipped with the Fréchet topology generated by the convergence with respect to $\mathfrak{J}$ then the pair $(\mathfrak{S},\mathfrak{J})$ fulfils the condition ($\mathfrak{E}$).

2. Preliminaries

It has been proved that convergence with respect to the σ-ideal of sets of the first category in the real line does not yield a topology for the space of Baire functions but convergence with respect to the σ-ideal of sets of measure zero yields a topology for the space of Lebesgue functions. This last assertion is true in a more general case (see [2]).

Now, we consider convergence with respect to the σ-ideal introduced by Mycielski (see [1]).

Let C be a countable product of the cyclic two element group $\langle\{0,1\},+\rangle$ with the discrete topology. Thus C is homeomorphic to and can be identified with the set of Cantor $\left\{\sum_{i=0}^{\infty}\frac{2x_i}{3^{i+1}} : x_i\in\{0,1\}\right\}$, and the Haar measure in C is introduced by the Lebesgue measure over the unit interval I and the Cantor mapping of C into I. For $S\subset C$ and a set K of the natural numbers in [1] defined a positional game $\Gamma(S,K)$ with perfect information between two players I and II. The players choose a sequence $(x_0,x_1,x_2,\ldots)\in C$ in such a way that the choice x_i is done by Player I if $i\notin K$ and by Player II if $i\in K$. The player choosing x_i knows S,K and $x_0,\ldots,x_{i-1}$. Player I wins if $(x_0,x_1,\ldots)\in S$ and Player II wins in the other case.

D e f i n i t i o n 2.1. A strategy of Player I is any function $\sigma : \chi_{N-K}\to\{0,1\}$, where χ_{N-K} is the set of sequences whose length is equal to $l\in N-K$ together with empty sequence.

D e f i n i t i o n 2.2. A strategy of Player II is any function $\sigma : \chi_K \to \{0,1\}$, where χ_K is the set of sequences whose length is equal to $1 \in K$.

Thus, if $i \in K$ then $x_i = \sigma(x_0, x_1, \dots, x_{i-1})$ and $x_i = \tau(x_0, \dots, x_{i-1})$ if $i \notin K$. Applying the strategies σ and τ the result of this game is an element $x \in C$ and we denote this element by $\langle \tau, \sigma \rangle$.

D e f i n i t i o n 2.3. A strategy τ is a winning strategy of the Player I in the game $\Gamma(S,K)$ if and only if for every strategy σ Player II $\langle \tau, \sigma \rangle \in S$. A winning strategy of Player II is defined analogously.

Let $V_{II}(K)$ denote the class of sets $S \subset C$ for which Player II has a winning strategy in the game $\Gamma(S,K)$.

Let $\mathcal{M} = \langle K_{s_1,\dots,s_n} : s_i = 0,1;\ n = 1,2,\dots \rangle$ be a system of sets of natural numbers such that $K_{s_1,\dots,s_n,s_{n+1}} \subset K_{s_1,\dots,s_n}$ and $K_{s_1,\dots,s_{n-1},0} \cap K_{s_1,\dots,s_{n-1},1} = \emptyset$ for every $n \in N$. Let $\mathcal{J}_{\mathcal{M}} = \bigcap_{K_{s_1,\dots,s_n} \in \mathcal{M}} V_{II}(K_{s_1,\dots,s_n})$. Mycielski proved that $\mathcal{J}_{\mathcal{M}}$ is a proper σ-ideal of the subsets C not containing nonempty open sets and $\mathcal{J}_{\mathcal{M}} \neq \emptyset$ if and only if each subset $K_{s_1,\dots,s_n}$ is nonempty.

For an arbitrary σ-field $\mathcal{S}$ and σ-ideal $\mathcal{J}$ the smallest σ-field containing $\mathcal{S}$ and $\mathcal{J}$ will be denoted by $\mathcal{S} \Delta \mathcal{J}$.

3. The main results

L e m m a 3.1. If $K \subset N$ is infinite, then there exists an increasing sequence $\{B_n\}_{n \in N}$ of Borel sets such that $C = \bigcup_{n=1}^{\infty} B_n$ and $B_n \in V_{II}(K)$ for every $n \in N$.

P r o o f . Let $K = \{k_i\}_{i \in N}$ and let for every $i \in N$, $B_i^* = \prod_{t \in N} X_t$, where $X_t = \{0,1\}$ for $t \neq k_i$ and $X_t = \{0\}$ for $t = k_i$. Then putting for every $n \in N$, $B_n = \left(C - \bigcup_{i=1}^{\infty} B_i^*\right) \cup \bigcup_{i=1}^{n} B_i^*$, we have obtained the increasing sequence of Borel sets such that $\bigcup_{n=1}^{\infty} B_n = C$ and $B_n \in V_{II}(K)$ for every $n \in N$. Indeed, it suffices for the Player II to use for every $n \in N$ a strategy σ_n such that $x_{k_1} = x_{k_2} = \dots = x_{k_n} = 1$ and $x_{k_{n+1}} = 0$. Then for an arbitrary strategy τ of the Player I $<\tau,\sigma_n> \in V_{II}(K)$.

Let $K_{s_1,s_2,\dots,s_n} \neq \emptyset$ for every $n \in N$. Let $\mathcal{B}^* = \mathcal{B} \Delta \mathcal{J}_{\mathcal{M}}$ where $\mathcal{B}$ is the family of Borel sets in C.

T h e o r e m 3.1. Convergence with respect to the σ-ideal $\mathcal{J}_{\mathcal{M}}$ does not yield a topology in the space of all B^*-measurable real functions.

P r o o f . We shall prove that condition (E′) is not fulfilled. For this purpose we find a set $B \in \mathcal{B}^* - \mathcal{J}_{\mathcal{M}}$ and a sequence $\{B_{j,n}\}_{j,n \in N}$ of $\mathcal{B}^*$-measurable sets such that conditions 1^0 and 2^0 will be satisfied and for each sequence $\{n_j\}_{j \in N}$ of natural numbers $\bigcap_{j=1}^{\infty} B_{j,n_j} \in \mathcal{J}_{\mathcal{M}}$. Let $\mathcal{M} = \{K_j\}_{j \in N}$ for every $j \in N$, K_j is infinite, then by previous lemma there exists a sequence $\{B_{j,n}\}_{n \in N}$ of Borel sets such that

1^0 $B_{j,n} \subset B_{j,n+1}$ for any $n \in N$,

2^0 $\bigcup_{j=1}^{\infty} B_{j,n} = C$ for every $j \in N$

and $B_{j,n} \in V_{II}(K_j)$ for any $j,n \in N$. Hence for each sequence $\{n_j\}_{j \in N}$ of natural numbers $\bigcap_{j=1}^{\infty} B_{j,n_j} \in \bigcap_{K_j \in \mathcal{M}} V_{II}(K_j) = \mathcal{J}_{\mathcal{M}}$.

Let $(X,\mathcal{S})$ be a measurable space and $\mathcal{J}$ be proper σ-ideal subsets of X. Let $\mathcal{S}^* = \mathcal{S} \Delta \mathcal{J}$ and $\mathcal{R}^* = \{A \in X : A = B \cup C;\ B \in \mathcal{S},\ C \in \mathcal{J}\}$. It is easy to check that $\mathcal{R}^*$ is stable under countable unions and intersections.

L e m m a 3.1. The following conditions are equivalent:

(i) a pair $(\mathcal{S}^*,\mathcal{J})$ fulfils (E′)

(ii) a pair $(\mathcal{R}^*,\mathcal{J})$ fulfils (E′)

(iii) a pair $(\mathcal{S},\mathcal{J})$ fulfils (E′).

P r o o f . Since $\mathcal{S} \subset \mathcal{R}^* \subset \mathcal{S}^*$, then the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. We shall prove that (iii) implies (ii). Let an arbitrary set $D \in \mathcal{R}^* - \mathcal{J}$ and an arbitrary double sequence $\{B_{j,n}\}_{j,n \in N}$ of $\mathcal{R}^*$-measurable sets fulfil 1^o and 2^o of condition (E'). For any $j,n \in N$, $B_{j,n} = C_{j,n} \cup E_{j,n}$, where $C_{j,n} \in \mathcal{S}$, $E_{j,n} \in \mathcal{J}$. We may require that, for any $j,n \in N$, $C_{j,n} \subset C_{j,n+1}$. Let $E = \bigcup_{\substack{j=1\\n=1}}^{\infty} E_{j,n}$. Then $E \in \mathcal{J}$ and it is easy to check that, for each $j \in N$, $D = \bigcup_{n=1}^{\infty} C_{j,n} \cup E$. Let $D' = \bigcap_{j=1}^{\infty} \bigcup_{n=1}^{\infty} C_{j,n}$. Then D and the sets $B'_{j,n} = C_{j,n} \cap D'$ fulfil 1^o and 2^o for the pair $(\mathcal{S},\mathcal{J})$. Since the pair $(\mathcal{S},\mathcal{J})$ fulfils condition (E'), there exists a sequence of natural numbers such that $\bigcap_{j=1}^{\infty} B'_{j,n_j} \notin \mathcal{J}$. Hence $\bigcap_{j=1}^{\infty} B_{j,n_j} \notin \mathcal{J}$. This proves that the pair $(\mathcal{R}^*,\mathcal{J})$ fulfils condition (E'). Now, we shall prove that (ii) implies (i). Let an arbitrary set $D \in \mathcal{S}^* - \mathcal{J}$ and an arbitrary double sequence $\{B_{j,n}\}_{j,n \in N}$ of $\mathcal{S}^*$-measurable sets fulfil conditions 1^o and 2^o of condition (E'). For any $j,n \in N$, $B_{j,n} = (A_{j,n} - C_{j,n}) \cup E_{j,n}$, $D = (A_o - C_o) \cup E_o$, where $A_o, A_{j,n} \in \mathcal{S}$, $C_{j,n}, E_{j,n}, C_o, E_o \in \mathcal{J}$. Let $E = \bigcup_{\substack{j=1\\n=1}}^{\infty} C_{j,n} \cup C_o$. The set E belongs to $\mathcal{J}$. For any $j,n \in N$ put $B'_{j,n} = B_{j,n} \cup E$ and $D' = D \cup E$. Then the sets $B'_{j,n}$ fulfil 1^o and 2^o for the pair $(\mathcal{R}^*,\mathcal{J})$. Since the pair $(\mathcal{R}^*,\mathcal{J})$ fulfils condition (E'), there exists a sequence $\{n_j\}_{j \in N}$ of natural numbers such that $\bigcap_{j=1}^{\infty} B'_{j,n_j} \in \mathcal{R}^* - \mathcal{J}$. Hence $\bigcap_{j=1}^{\infty} B'_{j,n_j} \notin \mathcal{J}$. This proves that the pair $(\mathcal{S}^*,\mathcal{J})$ fulfils condition (E).

L e m m a 3.2. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the proper σ-ideals subsets of X. If there exist $\mathcal{S}$-measurable sets $X_1 \in \mathcal{J}_1$, $X_2 \in \mathcal{J}_2$, such that $X = X_1 \cup X_2$, then the pair $(\mathcal{S} \Delta (\mathcal{J}_1 \cap \mathcal{J}_2), \mathcal{J}_1 \cap \mathcal{J}_2)$

fulfils condition (E′) if and only if both pairs $(\mathcal{S}\Delta\mathcal{I}_1,\mathcal{I}_1)$ and $(\mathcal{S}\Delta\mathcal{I}_2,\mathcal{I}_2)$ fulfil condition (E′).

P r o o f . Sufficiency. Suppose that the pair $(\mathcal{S}\Delta\mathcal{I}_1,\mathcal{I}_1)$ does not fulfil condition (E′). According to Lemma 3.1 it is equivalent that the pair $(\mathcal{S},\mathcal{I}_1)$ does not fulfil condition (E′). Thus there exist a $\mathcal{S}$-measurable set $B\in\mathcal{I}_1$ and a double sequence $\{B_{j,n}\}_{j,n\in N}$ of $\mathcal{S}$-measurable sets fulfilling 1^0 and 2^0 and such that for an arbitrary sequence $\{n_j\}_{j\in N}$ of natural numbers $\bigcap_{j=1}^{\infty} B_{j,n_j}\in\mathcal{I}_1$. Since $B=(B\cap X_1)\cup(B\cap X_2)$, then $B\cap X_2\notin\mathcal{I}_1$. Putting $A_{j,n}=B_{j,n}\cap X_2$ for any $j,n\in N$ we claim that for an arbitrary sequence $\{n_j\}_{j\in N}$ of natural numbers $\bigcap_{j=1}^{\infty} A_{j,n_j}\in\mathcal{I}_1\cap\mathcal{I}_2$, but this contradicts that the pair $(\mathcal{S}\Delta(\mathcal{I}_1\cap\mathcal{I}_2),\mathcal{I}_1\cap\mathcal{I}_2)$ does not fulfil condition (E′).

Necessity. Suppose that the pair $(\mathcal{S}\Delta(\mathcal{I}_1\cap\mathcal{I}_2),\mathcal{I}_1\cap\mathcal{I}_2)$ does not fulfil condition (E′). Then it is easy to conclude that the pair $(\mathcal{S}\Delta\mathcal{I}_1,\mathcal{I}_1)$ or the pair $(\mathcal{S}\Delta\mathcal{I}_2,\mathcal{I}_2)$ does not have this property.

Let H denote Haar measure in C, $\mathcal{L}$ - the σ-ideal of all sets of H measure zero in C, $\mathcal{K}$ - the σ-ideal of all sets of the first category in C and B - the family of all Borel sets in C.

Let $\mathcal{I}_1=\mathcal{I}_{\mathcal{M}}\cap\mathcal{L}$, $\mathcal{I}_2=\mathcal{I}_{\mathcal{M}}\cap\mathcal{K}$, $\mathcal{I}_3=\mathcal{I}_{\mathcal{M}}\cap\mathcal{K}\cap\mathcal{L}$ and $\mathcal{B}_i^*=\mathcal{B}\Delta\mathcal{I}_i$ for $i=1,2,3$. Mycielski proved (see th. 4 and property 5 in [1]) that there exist Borel sets A and B such that $C=A\cup B$ and $A\in\mathcal{K}\cap\mathcal{L}$, $B\in\mathcal{I}_{\mathcal{M}}$.

By Theorem 1.1, 3.1 and Lemma 3.2, we have the following theorem.

T h e o r e m 3.2. Convergence with respect to the σ-ideal $\mathcal{I}_i$ does not yield a topology in the space of all B_i^*-measurable real functions for $i=1,2,3$.

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