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ON INTEGRAL INEQUALITIES SIMILAR TO OPIAL'S INEQUALITY

1. Introduction

In 1960 Z. Opial [6] proved an inequality which has received considerable attention over the last twenty five years. In [5] C. Olech obtained the following version of the Opial inequality established in [6].

If z is absolutely continuous on $[a,b]$ and $z(a) = z(b) = 0$, then

$$(1)\qquad \int_a^b |z(x)z'(x)|\,dx \leqslant \frac{(b-a)}{4} \int_a^b |z'(x)|^2 dx,$$

where the constant $\frac{(b-a)}{4}$ is the best possible.

After the appearance of the papers [5], [6] in 1960, a number of papers had as their aim to give simpler proofs, various extensions and generalisations of the Opial inequality (1), (see [1], [3]-[12] and the references given therein). The integral inequalities of the form (1) are of considerable interest and also have important applications in the theory of ordinary differential equations and boundary value problems (see [4], [10], [11]). The main aim of this paper is to establish some new integral inequalities involving functions and their derivatives which claim their origin to the Opial inequality given in (1). In fact, our results are motivated by the interesting variants of inequality (1) given by Godunova and Levin in [1] (see, [3], Theorems 12 and 13, p.159). The proofs given here are easy and based on the idea used by Olech [5] to obtain a simple proof of Opial's inequality [6].

2. Statement of results

Our main result in the present paper is established in the following theorem.

Theorem 1. Let u and v be absolutely continuous functions on $[a,b]$ with $u(a) = v(a) = 0$. Let $f(r)$ and $g(r)$ be nonnegative, continuous, nondecreasing functions for $r \geq 0$ and $f(0) = g(0) = 0$ such that $f'(r)$ and $g'(r)$ exist and are nonnegative, continuous and nondecreasing for $r \geq 0$. Then

$$\int_a^b \Big[f(|u(x)|)g'(|v(x)|)|v'(x)| + g(|v(x)|)f'(|u(x)|)|u'(x)|\Big]\,dx \leq f\Big(\int_a^b |u'(x)|\,dx\Big)\, g\Big(\int_a^b |v'(x)|\,dx\Big). \tag{2}$$

The inequality (2) also holds, if we replace the conditions $u(a) = v(a) = 0$ by $u(b) = v(b) = 0$.

Remark 1. (i) If we take $v = u$ and $g = f$ in (2), then becomes

$$\int_a^b f(|u(x)|)f'(|u(x)|)\,|u'(x)|\,dx \leq \frac{1}{2}\left\{f\Big(\int_a^b |u'(x)|\,dx\Big)\right\}^2. \tag{3}$$

We note that the inequality (3) is analogous to Godunova and Levin's inequality (see [3], Theorem 13, p.159) and we believe that the inequality (3) is new to the literature.

(ii) Putting $f(r) = r$ and hence $f'(r) = 1$ in (3) and applying the Schwarz inequality to the resulting integral on the right-hand side of (3), we get the Opial inequality given in ([3], Theorem 2', p.154).

(iii) By taking $f(r) = r^{m+1}$, $g(r) = r^{m+1}$, where $m \geq 0$ is a constant and hence $f'(r) = (m+1)\, r^m$, $g'(r) = (m+1)\, r^m$ in (2), and using the elementary inequality $c_1 c_2 \leq \frac{1}{2}(c_1^2 + c_2^2)$ for $c_1, c_2 \geq 0$ reals and Hölder's inequality with indices $2(m+1)$, $\frac{2(m+1)}{2m+1}$ to the integral on the right-hand side of (2), we have

$$(4)\qquad \int_a^b |u(x)|^m\, |v(x)|^m \left[|u(x)|\,|v'(x)| + |v(x)|\,|u'(x)|\right] dx \leq$$

$$\leq \frac{(b-a)^{2m+1}}{2(m+1)} \int_a^b \left[|u'(x)|^{2(m+1)} + |v'(x)|^{2(m+1)}\right] dx.$$

Here we note that the inequality of the form (4) with additional conditions $u(b) = v(b) = 0$ is established by the author in ([7], Theorem 4).

(iv) Putting $v = u$ and $2m+1 = n$ in (4), we get the inequality

$$\int_a^b |u(x)|^n |u'(x)|\, dx \leq \frac{(b-a)^n}{n+1} \int_a^b |u'(x)|^{n+1}\, dx$$

which in turn is a slight variant of the inequality established by Yang in ([12], Lemma 3) and contains as a special case the Opial inequality of the form given in ([3], Theorem 2', p.154) when $n = 1$.

A slightly different version of inequality (2) analogous to the inequality given in ([3], Theorem 12, p. 159) is embodied in the following theorem.

T h e o r e m 2. Let u, v, f, g, f', g' be as in Theorem 1. Let $p(x) > 0$ be defined on $[a,b]$ and $\int_a^b p(x)dx = 1$. If $h(r)$ is a positive, convex and increasing function for $r > 0$, then

$$(5)\qquad \int_a^b \Big[f(|u(x)|)g'(|v(x)|)|v'(x)| + g(|v(x)|)f'(|u(x)|)|u'(x)|\Big]\,dx \leq$$
$$\leq f\left(h^{-1}\left(\int_a^b p(x)h\left(\frac{|u'(x)|}{p(x)}\right)dx\right)\right)g\left(h^{-1}\left(\int_a^b p(x)h\left(\frac{|v'(x)|}{p(x)}\right)dx\right)\right).$$

The inequality (5) also holds, if we replace the conditions $u(a) = v(a) = 0$ by $u(b) = v(b) = 0$.

Remark 2. If we take $v = u$ and $g = f$, then (5) reduces to the following inequality analogous to that of ([3], Theorem 12, p.159),

$$(6)\qquad \int_a^b f(|u(x)|)f'(|u(x)|)|u'(x)|\,dx \leq \frac{1}{2}\left\{f\left(h^{-1}\left(\int_a^b p(x)h\left(\frac{|u'(x)|}{p(x)}\right)dx\right)\right)\right\}^2.$$

We also note that in the special cases the inequality (5) yields the various inequalities as discussed in Remark 1.

3. Proofs of Theorems 1 and 2

Let $x \in [a,b]$ and define

$$(7)\qquad y(x) = \int_a^x |u'(t)|\,dt, \qquad z(x) = \int_a^x |v'(t)|\,dt.$$

From (7) we have

$$(8)\qquad y'(x) = |u'(x)|, \quad z'(x) = |v'(x)| \quad \text{for } x \in [a,b].$$

On the other hand

$$(9)\qquad u(x) = \int_a^x u'(t)\,dt, \qquad v(x) = \int_a^x v'(t)\,dt \quad \text{for } x \in [a,b],$$

since $u(a) = v(a) = 0$. From (9) and (7) we get

$$(10) \qquad |u(x)| \leqslant y(x), \quad |v(x)| \leqslant z(x) \quad \text{for} \quad x \in [a,b].$$

Using (10), (8), (7) we obtain

$$(11) \qquad \int_a^b \Big[f(|u(x)|)g'(|v(x)|)|v'(x)| + $$

$$+ g(|v(x)|)f'(|u(x)|)|u'(x)|\Big] \, dx \leqslant$$

$$\leqslant \int_a^b [f(y(x))g'(z(x))z'(x) + g(z(x))f'(y(x))y'(x)]dx =$$

$$= \int_a^b \frac{d}{dx} [f(y(x))g(z(x))]dx = f(y(b))g(z(b)) =$$

$$= f\Big(\int_a^b |u'(x)|dx\Big) g\Big(\int_a^b |v'(x)|dx\Big)$$

which is the required inequality (2). The proof of (2) is similar in the case of $u(b) = v(b) = 0$. This completes the proof of Theorem 1.

In order to prove Theorem 2, we first observe that having $\int_a^b p(x)dx = 1$, by assumption, we can write

$$(12) \qquad \int_a^b |u'(x)| dx = \frac{\int_a^b p(x) \frac{|u'(x)|}{p(x)} dx}{\int_a^b p(x)dx}.$$

Since h is convex, from (12) and using Jansen's inequality (see [2], p. 133) we obtain

$$(13) \qquad h\Big(\int_a^b |u'(x)| \, dx\Big) \leqslant \int_a^b p(x)h\Big(\frac{|u'(x)|}{p(x)}\Big) dx$$

which implies

$$(14)\qquad \int_a^b |u'(x)|dx \leq h^{-1}\Big(\int_a^b p(x)h'\Big(\frac{|u'(x)|}{p(x)}\Big)dx\Big).$$

Similarly we have

$$(15)\qquad \int_a^b |v'(x)|dx \leq h^{-1}\Big(\int_a^b p(x)h'\Big(\frac{|v'(x)|}{p(x)}\Big)dx\Big).$$

Since all hypotheses of Theorem 1 are among these of Theorem 2, we can write the inequality (2) which, together with (14), (15), gives the required inequality (5). Similarly we arrive to the inequality (5) in the case of $u(b) = v(b) = 0$. This completes the proof of Theorem 2.

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