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ON A GENERALIZATION OF THE HILBERT SPACE

Introduction

In this paper we have shown a certain generalization of the Hilbert space using for this purpose the properties of the Mikusiński space and the space normed by elements belonging to the cone of the Mikusiński space. We have given some examples of generalized unitary spaces and Hilbert spaces and applications of these spaces in the Bittner operational calculus.

1. The Hilbert space of the type (X,Y,K)

Let X be a linear space over the field Γ of real or complex numbers. Let Y be a Mikusiński space (see [3]) with a distinguished cone $K \subset Y$. Assume that in the latter space also a commutative multiplication

(Y.1) $$y_1y_2 = y_2y_1 \in Y, \quad y_1,y_2 \in Y$$

is defined.

Assume also that the multiplication is distributive with respect to addition, i.e. that

(Y.2) $$y_1(y_2+y_3) = y_1y_2 + y_1y_3, \quad y_1,y_2,y_3 \in Y$$

and

(K.1) $$(y_1,y_2 \in K) \Longrightarrow (y_1y_2 \in K),$$

(Y.3) $$(y \in Y) \Longrightarrow (y^2 := yy \in K),$$

(Y.4) $$[(y \in Y) \wedge (y^2 = 0)] \Longrightarrow [y = 0],$$

(K.2) $$[(y_1, y_2 \in K) \wedge (y_2^2 - y_1^2 \in K)] \Longrightarrow [y_2 - y_1 \in K],$$

(K.3) $$[(y_1, y_2 \in K) \wedge (y_1^2 = y_2^2)] \Longrightarrow [y_1 = y_2],$$

(K.4) $$\bigwedge_{y_1 \in K} \bigvee_{y_2 \in K} y_1 = y_2^2.$$

It is obvious that

(K.5) $$[(y_1, y_2 \in K) \wedge (y_1 + y_2 = 0)] \Rightarrow [y_1 = y_2 = 0].$$

Let Z be a linear complex space Y+iY (see [1]) with the multiplication defined by the formula

$$z_1 z_2 := (y_1 y_3 - y_2 y_4) + i(y_1 y_4 + y_2 y_3),$$

where $z_1 = y_1 + iy_2$, $z_2 = y_3 + iy_4 \in Z$. Then

(Z.1) $$z_1 z_2 = z_2 z_1 \in Z, \quad z_1, z_2 \in Z,$$

(Z.2) $$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3, \quad z_1, z_2, z_3 \in Z$$

by (Y.1) and (Y.2).

Put

$$\bar{z} := y_1 - iy_2,$$

where $z = y_1 + iy_2 \in Z$. It can easily be seen that

(Z.3) $$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad z_1, z_2 \in Z.$$

Let $|z|$ be such an element of the cone K that

(1) $$|z|^2 = z\bar{z} = y_1^2 + y_2^2,$$

where $z = y_1+iy_2 \in Z$. Assume also that for any $z \in Y$ the element $|z|$ is the module defined in K. Moreover, let

$$(Z.4) \qquad |z_1| + |z_2| - |z_1+z_2| \in K, \quad z_1,z_2 \in Z.$$

D e f i n i t i o n 1. We shall say that in the space X a scalar product

$$\langle x_1|x_2\rangle \in Z, \quad x_1,x_2 \in X$$

is defined, i.e. a scalar product with values taken from Z, if the following conditions hold*):

$$(i.1) \qquad \langle x_1+x_2|x_3\rangle = \langle x_1|x_3\rangle + \langle x_3|x_3\rangle,$$

$$(i.2) \qquad \langle \gamma x_1|x_2\rangle = \gamma\langle x_1|x_2\rangle,$$

$$(i.3) \qquad \langle x_1|x_2\rangle = \overline{\langle x_2|x_1\rangle},$$

$$(i.4) \qquad \langle x|x\rangle \in K, \quad (\langle x|x\rangle = 0) \Longrightarrow (x = 0),$$

$$(i.5) \qquad (\langle x_1|x_1\rangle\langle x_2|x_2\rangle)^2 - (\langle\overline{x_1|x_2}\rangle\langle x_1|x_2\rangle)^2 \in K$$

(the Cauchy-Buniakowski-Schwarz condition), $x,x_1,x_2,x_3 \in X$, $\gamma \in \Gamma$.

D e f i n i t i o n 2. The couple $(X,\langle\cdot|\cdot\rangle)$ is called unitary space of the type (X,Y,K).

D e f i n i t i o n 3 (cf. [3]). We say that in the space X a norm

$$|x| \in K, \quad x \in X$$

is defined, i.e. a norm with values taken from K, if the following conditions hold:

$$(n.1) \qquad (|x| = 0) \Longrightarrow (x = 0),$$

$$(n.2) \qquad |x_1| + |x_2| - |x_1+x_2| \in K,$$

$$(n.3) \qquad |\gamma x| = |\gamma|\,|x|, \quad x,x_1,x_2 \in X, \quad \gamma \in \Gamma.$$

*) $Z = Y+iY$ if X is a complex space and $Z = Y$ if X is real.

D e f i n i t i o n 4. The couple $(X, |\cdot|)$ will be referred to as the normed space of the type (X,Y,K).

For a given cone K a partial order in the Mikusiński space Y is defined by the relation

$$(y_1 \leq y_2) \overset{\text{def}}{\Longleftrightarrow} (y_2 - y_1 \in K), \quad y_1, y_2 \in Y.$$

We shall also use the symbol $y_2 \geq y_1$.

Let $|x|$ be an element of K such that

$$|x|^2 = \langle x|x\rangle, \quad x \in X. \tag{2}$$

Then

$$|x|^2 = |\langle x|x\rangle|, \quad x \in X. \tag{2'}$$

It is obvious that $|x| \geq 0$ for $x \in X$. If $|x| = 0$ then $|x|^2 = \langle x|x\rangle = 0$. Herefrom and from (i.4) we obtain $x = 0$. Using (i.2) and (i.3) we can easily verify that

$$\langle x_1|\gamma x_2\rangle = \bar{\gamma}\langle x_1|x_2\rangle, \quad x_1, x_2 \in X, \quad \gamma \in \Gamma .$$

Hence

$$|\gamma x|^2 = \langle \gamma x|\gamma x\rangle = \gamma\bar{\gamma}\langle x|x\rangle = (|\gamma|\,|x|)^2, \quad x \in X, \quad \gamma \in \Gamma.$$

Herefrom and from (K.3) we obtain

$$|\gamma x| = |\gamma|\,|x|, \quad x \in X, \quad \gamma \in \Gamma .$$

From conditions (i.5) and (K.2) as well as from (1) and (2) we obtain the Cauchy-Buniakowski-Schwarz inequality

$$|\langle x_1|x_2\rangle| \leq |x_1|\,|x_2|, \quad x_1, x_2 \in X. \tag{3}$$

By (i.1), (i.3) and (Z.3) we can easily verify that

$$\langle x_1|x_2+x_3\rangle = \langle x_1|x_2\rangle + \langle x_1|x_3\rangle, \quad x_1, x_2, x_3 \in X. \tag{4}$$

As

$$|x_1+x_2|^2 = |\langle x_1+x_2|x_1+x_2\rangle|, \quad x_1, x_2 \in X,$$

what can be obtained from (2′), therefore from (i.1) and (4) we get

$$|x_1+x_2|^2 = |\langle x_1|x_1\rangle+\langle x_2|x_1\rangle+\langle x_1|x_2\rangle+\langle x_2|x_2\rangle, \quad x_1,x_2 \in X.$$

Furthermore, by (Z.4) and inequality (3) we have

$$|x_1+x_2|^2 \leqslant \langle x_1|x_1\rangle+2|\langle x_1|x_2\rangle|+\langle x_2|x_2\rangle \leqslant |x_1|^2+2|x_1||x_2|+|x_2|^2 =$$
$$= (|x_1|+|x_2|)^2, \quad x_1,x_2 \in X.$$

This last equality follows from conditions (Y.1) and (Y.2). Herefrom and from (K.2) we obtain

$$|x_1+x_2| \leqslant |x_1| + |x_2|, \quad x_1,x_2 \in X.$$

C o r o l l a r y 1. The unitary space $(X,\langle\cdot|\cdot\rangle)$ of the type (X,Y,K) is a normed space $(X,|\cdot|)$ of the type (X,Y,K) where the norm satisfies (2).

The proof of the following theorem can be easily obtained (cf. Th.2.1, 2.2, XX [1]).

T h e o r e m 1. The norm $|\cdot|$ in the unitary space $(X,\langle\cdot|\cdot\rangle)$ of the type (X,Y,K) satisfies the following identity of parallelogram

$$(5) \qquad |x_1+x_2|^2 + |x_1-x_2|^2 = 2(|x_1|^2 + |x_2|^2), \quad x_1,x_2 \in X.$$

Moreover

$$(6) \qquad \langle x_1|x_2\rangle = \frac{1}{4}(|x_1+x_2|^2 - |x_1-x_2|^2), \quad x_1,x_2 \in X$$

if X is a real space and

$$(7) \quad \langle x_1|x_2\rangle = \frac{1}{4}(|x_1+x_2|^2-|x_1-x_2|^2) - \frac{1}{4}i(|ix_1+x_2|^2-|ix_1-x_2|^2), \quad x_1,x_2 \in X$$

if X is complex.

In a normed space $(X,|\cdot|)$ of the type (X,Y,K) a convergence is defined by the convergence with the regulator $f \in K$ (regular convergence) in the Mikusiński space Y (see [2,3]).

Let $\{x_\nu\}_{\nu\in N}$ and $\{y_\nu\}_{\nu\in N}$ be sequences of elements from spaces X and Y respectively.

D e f i n i t i o n 5 (see [2,3])

$$\left(\lim_{\nu\to\infty} y_\nu\right) = y \overset{\text{def}}{\iff} \left(\bigvee_{f\in K} \bigwedge_{\varepsilon>0} \bigvee_{M} \bigwedge_{\nu\geq M} |y_\nu - y| \leq \varepsilon f\right),$$

$$\left(\lim_{\nu\to\infty} x_\nu = x\right) \overset{\text{def}}{\iff} \left(\lim_{\nu\to\infty} |x_\nu - x| = 0\right).$$

Therefore

$$\left(\lim_{\nu\to\infty} x_\nu = x\right) \iff \left(\bigvee_{f\in K} \bigwedge_{\varepsilon>0} \bigvee_{M} \bigwedge_{\nu\geq M} |x_\nu - x| \leq \varepsilon f\right).$$

D e f i n i t i o n 6 (see [3]). We say that a sequence $\{x_\nu\}_{\nu\in N}$ satisfies the Cauchy condition if

$$\bigvee_{f\in K} \bigwedge_{\varepsilon>0} \bigvee_{M} \bigwedge_{\mu,\nu\geq M} |x_\mu - x_\nu| \leq \varepsilon f.$$

D e f i n i t i o n 7. If any arbitrary sequence $\{x_\nu\}_{\nu\in N}$ of elements of X satisfying the Cauchy condition converges to some element of X, then the normed space $(X, |\cdot|)$ of the type (X,Y,K) will be called a space of the type $X_{Y,K}$ (in [3] $X_{Y,K}$ is referred to as the space of the type X_Y), whereas the unitary space $(X, \langle\cdot|\cdot\rangle)$ of the type (X,Y,K) will be called a Hilbert space of the type (X,Y,K).

2. Examples

A. Let $X = Y := C(N)$ be a real linear space of real sequences $x = \{x_\nu\}_{\nu\in N}$ with the common addition of sequences and multiplication of a sequence by a real number.

If

$$K = C_+(N) := \left\{x \in C(N) : \bigwedge_{\nu\in N} x_\nu \geq 0\right\}$$

and

$$|x| := \left\{|x_\nu|\right\}_{\nu\in N}, \quad x \in C(N)$$

then C(N) is a Mikusiński space. The space defined in this way is a normed space of the type $(C(N), C(N), C_+(N))$ if to any sequence x correspond a sequence of real quasi-norms $\{\|x\|_j\}_{j\in N}$ and the norm

$$|x| := \{\|x\|_j\}_{j\in N}, \quad x \in C(N),$$

where

$$\|x\|_j := \sup\{|x_\nu| : \nu \leqslant j\}, \quad j \in N \qquad \text{(see [3])}.$$

The multiplication of sequences belonging to C(N) we define by the formula

$$xy := \{x_\nu y_\nu\}_{\nu\in N}, \quad x,y \in C(N).$$

C(N) is not an unitary space. E.g. for the sequences

$$x = \{0,1,2,0,0,0,0,\dots\},$$
$$y = \{0,0,0,3,4,5,6,\dots\}$$

we have

$$|x+y|^2 + |x-y|^2 = \{0,2,8,18,32,50,72,\dots\}$$

and

$$2(|x|^2 + |y|^2) = \{0,2,8,26,40,58,80,\dots\}.$$

Thus the parallelogram identity condition (5) does not hold in this case.

B. For $Y := R^1$, $K = R^1_+ := \{y \in R^1 : y \geqslant 0\}$, $f := 1$ the Hilbert space $H = (X, \langle\cdot|\cdot\rangle)$ of the type (X, R^1, R^1_+) is a Hilbert space in the classical sense.

C. Let $C([\alpha,\beta], \mathbb{C})$ be a complex linear space of continuous functions defined on the interval $[\alpha,\beta] \subset R^1$ with complex values. Similarily, let $C([\alpha,\beta], R^1)$ be a real linear space of continuous functions defined on the interval $[\alpha,\beta]$ with real values. Define

$$X := \bigoplus_{j=1}^{n} C([\alpha,\beta],\mathbb{C}), \qquad Y := C([\alpha,\beta],R^1).$$

Then Z is a complex linear space of continuous functions defined on $[\alpha,\beta]$ with complex values.

Let

$$K := \left\{ \{y(t)\} \in Y : \bigwedge_{t\in[\alpha,\beta]} y(t) \geq 0 \right\}$$

and

$$|y| := \{|y(t)|\}, \quad \{(y_1y_2)(t)\} := \{y_1(t)y_2(t)\},$$

where $y = \{y(t)\}$, $y_1 = \{y_1(t)\}$, $y_2 = \{y_2(t)\} \in Y$.

The expression

$$\langle \vec{x}_1|\vec{x}_2\rangle := \left\{ \sum_{j=1}^{n} x_j^1(t)\overline{x_j^2(t)} \right\}, \quad \vec{x}_1 = \{\vec{x}_1(t)\}, \ \vec{x}_2 = \{\vec{x}_2(t)\} \in X$$

defines a scalar product in X, as for any $t \in [\alpha,\beta]$ $\langle \vec{x}_1|\vec{x}_2\rangle(t)$ is a scalar product in l_n^2.

We shall demonstrate that the considered unitary space $(X,\langle\cdot|\cdot\rangle)$ of the type (X,Y,K) is a Hilbert space.

We have here

$$|\vec{x}| = \left\{ \sqrt{\sum_{j=1}^{n} |x_j(t)|^2} \right\}, \quad \vec{x} = \{\vec{x}(t)\} \in X.$$

Let $\{\vec{x}_\nu\}_{\nu\in N}$ be a sequence of vector functions belonging to X satisfying the Cauchy condition

$$\bigvee_{\{f(t)\}\in K} \ \bigwedge_{\varepsilon>0} \ \bigvee_{M} \ \bigwedge_{\mu,\nu\geq M} \left\{ \sqrt{\sum_{j=1}^{n} \left|x_j^\mu(t) - x_j^\nu(t)\right|^2} \right\} \leq \{\varepsilon f(t)\} .$$

Hence for any arbitrary $\varepsilon > 0$ there exists a number M such that for any $\mu,\nu \geq M$ and for any $t\in[\alpha,\beta]$ we have

$$|x_j^\mu(t) - x_j^\nu(t)| \leq \varepsilon f(t) \leq \varepsilon \max\{f(t) : t\in[\alpha,\beta]\}, \quad j = 1,\dots,n.$$

It follows from this condition, that the sequence $\{x_j^\nu(t)\}$, $j=1,\dots,n$ is uniformly convergent to some continuous function

$\{x_j(t)\}$, $j=1,\dots,n$. Therefore for any arbitrary $\varepsilon > 0$ there exists a number M such that for any $\nu \geq M$ and for any $t \in [\alpha,\beta]$ we have

$$\left|x_j^\nu(t) - x_j(t)\right| \leqslant \varepsilon, \quad j = 1,\dots,n.$$

Taking $f := \{\sqrt{n}\}$ we infer that $\{\vec{x}_\nu\} \Rightarrow \vec{x} \in X$, where " $\Rightarrow$ " stands for the regular convergence.

D. Let $(X,\langle\cdot|\cdot\rangle_1)$ be an unitary space of the type (X,Y,K) and let $V \in L(Z,Z)$ and $V|_Y \in L(Y,Y)$ be a non-negative operation, i.e.

$$(8) \qquad V(K) \subset K,$$

satisfying the condition

$$(9) \quad (V\langle x_1|x_1\rangle_1 V\langle x_2|x_2\rangle_1)^2 - (V\langle \overline{x_1|x_2}\rangle_1 V\langle x_1|x_2\rangle_1)^2 \in K, \; x_1,x_2 \in X.$$

Moreover, let $V|_K$ be an injection, i.e.

$$(10) \qquad (Vy=0) \Longrightarrow (y=0), \quad y \in K.$$

Then

$$(11) \qquad \langle x_1|x_2\rangle_2 := V\langle x_1|x_2\rangle_1, \quad x_1,x_2 \in X$$

is a scalar product in X.

As the operation V is additive and homogeneous the conditions (i.1) and (i.2) for the product (11) are obvious. As

$$(12) \qquad V\bar{z} = (\overline{Vz}), \quad z \in Z$$

therefore

$$\langle x_1|x_2\rangle_2 = V\langle x_1|x_2\rangle_1 = V\langle\overline{x_2|x_1}\rangle_1 = (\overline{V\langle x_2|x_1\rangle_1}) = \langle\overline{x_2|x_1}\rangle_2, \quad x_1,x_2 \in X,$$

i.e. condition (i.3) holds.

From properties (8) and (10) we obtain conditions (i.4):

$$\langle x|x\rangle_2 = V\langle x|x\rangle_1 \in K, \quad \text{as} \quad \langle x|x\rangle_1 \in K, \quad x \in X,$$

$$(\langle x|x\rangle_2 = V\langle x|x\rangle_1 = 0) \Rightarrow (\langle x|x\rangle_1 = 0) \Rightarrow (x=0), \qquad x \in X.$$

From (9) and definition (11) as well as from property (12) of operation V we get the Cauchy-Buniakowski-Schwarz condition (i.5) for the scalar product $\langle\cdot|\cdot\rangle_2$.

E. Given the Bittner operational calculus

$$CO(L^0, L^1, S, T_q, s_q, q, Q) \qquad \text{(see [2,3])},$$

where L^0 and L^1 are real linear spaces such that $L^1 \subset L^0$; the operation $S \in L(L^1, L^0)$ called the derivative is a surjection, i.e. $S(L^1) = L^0$. The elements of the kernel of S, i.e. the elements of

$$\operatorname{Ker}S := \{c \in L^1 : Sc=0\}$$

will be called the constants of the derivative S.

Q is an arbitrary set of indexes q for the operations $T_q \in L(L^0, L^1)$ such that $ST_q w = w$, $w \in L^0$ (called integrals) and for the operations $s_q \in L(L^1, L^1)$ such that $T_q Sx = x - s_q x$, $x \in L^1$ (called limit conditions). By induction we define a sequence of spaces L^ν, $\nu \in N$ such that

$$L^\nu := \{x \in L^{\nu-1} : Sx \in L^{\nu-1}\}, \qquad \nu \in N.$$

Then

$$\ldots \subset L^\nu \subset L^{\nu-1} \subset \ldots \subset L^1 \subset L^0, \qquad \nu \in N$$

and

$$S^\nu(L^\nu) = L^0,$$

where

$$S^\nu := \underbrace{S \circ S \circ \ldots \circ S}_{\nu\text{-times}} \in L(L^\nu, L^0), \quad \nu \in N.$$

Put also

$$L_n^\nu := \bigoplus_{j=1}^{n} L^\nu, \quad (\operatorname{Ker}S)_n := \bigoplus_{j=1}^{n} \operatorname{Ker}S, \quad n, \nu \in N.$$

E.1. Let $X := L^o_n$, $Z = Y := L^o$ possess properties of the spaces X and Y quoted in §1 and let

$$\langle \vec{x}_1 | \vec{x}_2 \rangle_1 := \sum_{j=1}^{n} x_j^1 x_j^2, \quad \vec{x}_1, \vec{x}_2 \in L^o_n.$$

Suppose also that condition (i.5) holds.

From (Y.3) and (K.5) we get

$$(\langle \vec{x} | \vec{x} \rangle_1 = 0) \Rightarrow (x_1^2 = \dots = x_n^2 = 0), \quad \vec{x} \in L^o_n.$$

Herefrom and from (Y.4) we obtain $x_1 = \dots = x_n = 0$, i.e. $\vec{x} = \vec{0}$. It is easy to verify that also the remaining properties of the scalar product are satisfied.

Let

$$V := T_{q_0} - T_{q_1} = s_{q_1} T_{q_0} \in L(L^o, L^o), \quad q_0, q_1 \in Q$$

be an operation satisfying the same assumptions as in Example D. It follows from Example D that

$$(13) \qquad \langle \vec{x}_1 | \vec{x}_2 \rangle_2 = \sum_{j=1}^{n} \left(T_{q_0} - T_{q_1}\right) x_j^1 x_j^2, \quad \vec{x}_1, \vec{x}_2 \in L^o_n$$

is also a scalar product in L^o_n. As

$$s_q(L^1) = \mathrm{Ker} S, \quad q \in Q \quad (\text{see } [3])$$

therefore

$$V(L^o) \subset \mathrm{Ker} S$$

and for any $\vec{x}_1, \vec{x}_2 \in L^o_n$ we have $\langle \vec{x}_1 | \vec{x}_2 \rangle_2 \in \mathrm{Ker} S \subset L^o$.

Mieloszyk has defined in his paper [6] the scalar product (13) for $n = 1$ quoting its forms in various models of the operational calculus.

E.2. Let $X := L^n$, $Z = Y := L^o$ possess properties of the spaces X and Y as in §1 and let the operation

$$V := T_{q_0} - T_{q_1}, \quad q_0, q_1 \in Q$$

satisfy the assumptions (8) and (10).

Define

(14) $\langle x_1|x_2\rangle_1 := \left(T_{q_0}-T_{q_1}\right)(x_1x_2+Sx_1Sx_2+\ldots+S^nx_1S^nx_2),\quad x_1,x_2\in L^n$

and let (i.5) be satisfied.

As for any $x\in L^n\subset L^0$ we have

$$x^2,(Sx)^2,\ldots,(S^nx)^2\in K,$$

what follows from (Y.3), and because operation V is non-negative, therefore

$$\langle x|x\rangle_1\in K,\qquad x\in L^n.$$

As $V_{|K}$ is an injection, therefore from (K.5) we infer

$$[\langle x|x\rangle_1=0]\Rightarrow[x^2=(Sx)^2=\ldots=(S^nx)^2=0],\qquad x\in L^n.$$

Therefrom and from (Y.4) we get $x=0$.

It is easy to verify that also the remaining properties of the scalar product are satisfied. Therefore $(L^n,\langle\cdot|\cdot\rangle_1)$ is an unitary space of the type (L^n,L^0,K). And also $\langle x_1|x_2\rangle_1\in$ $\in\operatorname{Ker}S\subset L^0$, $x_1,x_2\in L^n$.

Let

(15) $\langle x_1|x_2\rangle_2 := s_{q_0}x_1s_{q_0}x_2+\ldots+s_{q_0}S^{n-1}x_1s_{q_0}S^{n-1}x_2+$
$$+\left(T_{q_0}-T_{q_1}\right)S^nx_1S^nx_2,\qquad x_1,x_2\in L^n.$$

Assume also that (i.5) is satisfied. As we have for any $x\in L^n$

$$\left(s_{q_0}x\right)^2,\ldots,\left(s_{q_0}S^{n-1}x\right)^2\in K$$

and

$$\left(T_{q_0}-T_{q_1}\right)(S^nx)^2\in K$$

therefore

$$\langle x|x\rangle_2\in K,\quad x\in L^n.$$

From (K.5) we obtain

$$[\langle x|x\rangle_2=0]\Longrightarrow\left[\left(s_{q_0}x\right)^2=\ldots=\left(s_{q_0}S^{n-1}x\right)^2=\left(T_{q_0}-T_{q_1}\right)(S^nx)^2=0\right],\quad x\in L^n.$$

Therefrom and from (Y.4) we get

$$s_{q_0}x = \dots = s_{q_0}S^{n-1}x = 0$$

and

$$\left[\left(T_{q_0}-T_{q_1}\right)(S^n x)^2=0\right] \Rightarrow \left[(S^n x)^2=0\right] \Rightarrow \left[S^n x=0\right], \quad x \in L^n.$$

Therefore $x = 0$, as any element $x \in L^n$ possesses a Taylor development

$$x = s_q x + T_q s_q Sx + \dots + T_q^{n-1} s_q S^{n-1} x + T_q^n S^n x, \qquad q \in Q \quad (\text{see } [2,3]).$$

It is easy to prove that also the remaining properties of the scalar product are satisfied. Therefore $(L^n, \langle\cdot|\cdot\rangle_2)$ is an unitary space. Moreover, if KerS is an algebra, then $\langle x_1|x_2\rangle_2 \in \mathrm{Ker}S \subset L^0$, $x_1, x_2 \in L^n$.

If $L^0 := L^2([0,1],R^1)$ is a real linear space of functions

$$x : [0,1] \rightarrow R^1$$

such that the functions $\{[x(t)]^2\}$ are Lebesgue-integrable and

$$S := \frac{d}{dt}, \quad s_{q_0} := \Big|_{t=0}, \quad T_{q_0} := \int_0^t, \quad T_{q_1} := \int_1^t, \quad q_0=0, q_1=1 \in Q := [0,1]$$

then the norms $|\cdot|_1$, $|\cdot|_2 : L^n \rightarrow \left\{\{c\} \in \mathrm{Ker}\,\frac{d}{dt} : c \geqslant 0\right\} \simeq R^1_+$ determined by the scalar products (14) and (15) are equivalent in the Hilbert space

$$L^n \simeq W_n^2([0,1],R^1)$$

being a real Sobolev space of functions

$$x : [0,1] \rightarrow R^1$$

such that the derivative of order n-1 is absolutely continuous, whereas the derivative of order n belongs to $L^2([0,1],R^1)$ (see [4]).

E.3. Let $(X,\langle\cdot|\cdot\rangle)$ be an unitary space of the type (X,Y,K) and let $X^o \subset X$.

D e f i n i t i o n 8 (cf. [1,5]). The operation $\Phi \in L(X^o,X)$ is called quasi-unitary (quasi-orthogonal) if

$$\langle \Phi x_1 | \Phi x_2 \rangle = \langle x_1 | x_2 \rangle, \quad x_1, x_2 \in X^o.$$

D e f i n i t i o n 9 (see [5]). The endomorphism $A \in L(X,X)$ is called antisymmetric (antihermitian) if

$$\langle Ax_1 | x_2 \rangle = -\langle x_1 | Ax_2 \rangle, \quad x_1, x_2 \in X.$$

Consider the abstract differential equation

$$S\vec{x} = \hat{A}\vec{x} \tag{16}$$

with the limit condition

$$s_{q_o}\vec{x} = \vec{x}_o, \tag{17}$$

where

$$q_o \in Q, \quad \vec{x} \in L_n^1, \quad \vec{x}_o \in (\mathrm{Ker}S)_n, \quad \hat{A} \in L(L_n^o, L_n^o)$$

and

$$S\vec{x} := \begin{bmatrix} Sx_1 \\ \vdots \\ Sx_n \end{bmatrix}, \qquad s_{q_o}\vec{x} := \begin{bmatrix} s_{q_o}x_1 \\ \vdots \\ s_{q_o}x_n \end{bmatrix}, \qquad \hat{A}\vec{x} := \begin{bmatrix} \sum_{j=1}^{n} A_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} A_{nj}x_j \end{bmatrix}.$$

D e f i n i t i o n 10 (cf. [3]). For a fixed value of $q \in Q$, the endomorphism $\hat{A} \in L(L_n^o, L_n^o)$ is called a quasi-logarithm if

$$[(\hat{I} - T_q\hat{A})\vec{w} = \vec{0}] \Rightarrow [\vec{w} = \vec{0}],$$

where

$$\hat{I} := id_{L_n^o}, \quad \vec{w} \in L_n^o, \quad T_q\hat{A} = T_q\left[A_{ij}\right]_{n\times n} := \left[T_qA_{ij}\right]_{n\times n}.$$

It is easy to prove that $\hat{A}$ is a quasi-logarithm if and only if

$$\left[(S\vec{w} = \hat{A}\vec{w}) \wedge (s_q\vec{w} = \vec{0})\right] \Rightarrow [\vec{w} = \vec{0}], \qquad \vec{w} \in L_n^1.$$

Let

$$M_{(n)} := \left\{\vec{x} \in L_n^1 : S\vec{x} = \hat{A}\vec{x}\right\}.$$

D e f i n i t i o n 11. An operation $\hat{\Phi}_{q_o}(\hat{A}) \in L((\mathrm{KerS})_n, M_{(n)})$ such that $s_{q_o}\hat{\Phi}_{q_o}(\hat{A}) = id_{(\mathrm{KerS})_n}$ is called the resolvent of the equation (16).

Assume that for arbitrary fixed $q_o \in Q$, $\vec{x}_o \in (\mathrm{KerS})_n$ the solution of the problem (16), (17) exists.

If $\hat{A}$ is a quasi-logarithm then the unique solution of that problem has the form

$$\vec{x} = \hat{\Phi}_{q_o}(\hat{A})\vec{x}_o,$$

where $\hat{\Phi}_{q_o}(\hat{A})$ is the resolvent of the equation (16).

Using the notion of result and operator (see [2,3]) we can give the form of the operation $\hat{\Phi}_{q_o}(\hat{A})$. Namely, it is easy to prove that the problem (16), (17) is equivalent to the abstract integral equation

$$(\hat{I} - T_{q_o}\hat{A})\vec{x} = \vec{x}_o.$$

If $\hat{A}$ is a quasi-logarithm then the solution of the equation is the result

$$\vec{x} = \frac{\vec{x}_o}{\hat{I} - T_{q_o}\hat{A}}.$$

If that result is an element of the space L_n^o then it is also an element of L_n^1 and the operator

$$\hat{\Phi}_{q_o}(\hat{A}) = \frac{\hat{I}}{\hat{I} - T_{q_o}\hat{A}}$$

is the resolvent of the equation (16).

Let $X := L_n^o$, $Z = Y := L^o$ possess the properties of the spaces X and Y as in §1 and let $(L_n^o, \langle\cdot|\cdot\rangle)$ be an unitary space of the type (L_n^o, L^o, K) with the scalar product $\langle\cdot|\cdot\rangle$ possessing the following property

(18) $$(\vec{x}_1, \vec{x}_2 \in L_n^1) \Rightarrow (\langle\vec{x}_1|\vec{x}_2\rangle \in L^1).$$

D e f i n i t i o n 12. We shall say that the derivative S satisfies the Leibniz condition on the scalar product $\langle\cdot|\cdot\rangle$ if

(19) $$S\langle\vec{x}_1|\vec{x}_2\rangle = \langle S\vec{x}_1|\vec{x}_2\rangle + \langle\vec{x}_1|S\vec{x}_2\rangle, \quad \vec{x}_1, \vec{x}_2 \in L_n^1.$$

We shall say that the limit condition $s_q, q \in Q$ is multiplicative on the scalar product $\langle\cdot|\cdot\rangle$ if

(20) $$s_q\langle\vec{x}_1|\vec{x}_2\rangle = \langle s_q\vec{x}_1|s_q\vec{x}_2\rangle, \quad q \in Q, \quad \vec{x}_1, \vec{x}_2 \in L_n^1.$$

T h e o r e m 2 (cf. Lemma IX.9 [5]). If the solution of the problem (16), (17) exists, the derivative S satisfies the Leibniz condition, the limit condition s_{q_o} is multiplicative, the quasi-logarithm $\hat{A}$ is antisymmetric, then the resolvent $\hat{\Phi}_{q_o}(\hat{A})$ of equation (16) is a quasi-unitary operation.

P r o o f . For any $\vec{x}_{o1}, \vec{x}_{o2} \in (\mathrm{Ker}S)_n$ the elements

$$\vec{x}_1 = \hat{\Phi}_{q_o}(\hat{A})\vec{x}_{o1}, \quad \vec{x}_2 = \hat{\Phi}_{q_o}(\hat{A})\vec{x}_{o2}$$

are solutions of equation (16) with the limit conditions

$$s_{q_o}\vec{x}_1 = \vec{x}_{o1}, \quad s_{q_o}\vec{x}_2 = \vec{x}_{o2}$$

respectively. Hence we have

$$S\langle\vec{x}_1|\vec{x}_2\rangle = \langle S\vec{x}_1|\vec{x}_2\rangle + \langle\vec{x}_1|S\vec{x}_2\rangle = \langle\hat{A}\vec{x}_1|\vec{x}_2\rangle + \langle\vec{x}_1|\hat{A}\vec{x}_2\rangle = -\langle\vec{x}_1|\hat{A}\vec{x}_2\rangle + \langle\vec{x}_1|\hat{A}\vec{x}_2\rangle = 0.$$

Therefore

$$\langle \vec{x}_1 | \vec{x}_2 \rangle = c, \qquad c \in \mathrm{KerS}.$$

As

$$s_{q_o} c = c, \qquad c \in \mathrm{KerS}$$

therefore

$$c = s_{q_o} \langle \vec{x}_1 | \vec{x}_2 \rangle = \langle s_{q_o} \vec{x}_1 | s_{q_o} \vec{x}_2 \rangle = \langle \vec{x}_{o1} | \vec{x}_{o2} \rangle$$

and finally

$$\langle \hat{\Phi}_{q_o}(\hat{A}) \vec{x}_{o1} | \hat{\Phi}_{q_o}(\hat{A}) \vec{x}_{o2} \rangle = \langle \vec{x}_{o1} | \vec{x}_{o2} \rangle.$$

C o r o l l a r y 2. If the assumptions of Theorem 2 are satisfied, then the resolvent $\hat{\Phi}_{q_o}(\hat{A})$ of equation (16) is a continuous bijection.

P r o o f . As the operation $\hat{\Phi}_{q_o}(\hat{A})$ is quasi-unitary, therefore for any $\vec{x}_o \in (\mathrm{KerS})_n$ we have

$$| \hat{\Phi}_{q_o}(\hat{A}) \vec{x}_o | = | \vec{x}_o |.$$

It follows therefrom, that the resolvent $\hat{\Phi}_{q_o}(\hat{A})$ is a bounded injection. On the other hand, for any $\vec{x} \in M_{(n)}$ there exists $\vec{x}_o = s_{q_o} \vec{x} \in (\mathrm{KerS})_n$ such that $\vec{x} = \hat{\Phi}_{q_o}(\hat{A}) \vec{x}_o$, i.e. $\hat{\Phi}_{q_o}(\hat{A})$ maps $(\mathrm{KerS})_n$ onto $M_{(n)}$.

The system of equations

$$\begin{cases} x_1'(t) = g(t) x_2(t) \\ x_2'(t) = -g(t) x_1(t) \end{cases}$$

with the initial conditions

$$\{x_1(t)|_{t=0}\} = \{x_{1o}\}, \ \{x_2(t)|_{t=0}\} = \{x_{2o}\}, \ \{x_{1o}\}, \{x_{2o}\} \in \mathrm{Ker}\, \tfrac{d}{dt} \simeq R^1$$

has the form

$$S\vec{x} = \hat{A}\vec{x}, \qquad s_{q_o} \vec{x} = \vec{x}_o$$

if we introduce an operational calculus in which

$$L^{\vartheta} := C^{\vartheta}([\alpha,\beta],R^1),\quad \vartheta\in N\cup\{0\},\quad q_0 = 0\in Q := [\alpha,\beta]\subset R^1,$$

$$S := \frac{d}{dt},\quad T_0 := \int_0^t,\quad s_0 := \Big|_{t=0}$$

and

$$\hat{A} := \left\{\begin{bmatrix} 0 & g(t) \\ -g(t) & 0 \end{bmatrix}\right\},\quad \{g(t)\}\in L^0.$$

The resolvent of the considered system has the form

$$\hat{\Phi}_0(\hat{A}) = \left\{\begin{bmatrix} \cos\int_0^t g(\tau)d\tau & \sin\int_0^t g(\tau)d\tau \\ -\sin\int_0^t g(\tau)d\tau & \cos\int_0^t g(\tau)d\tau \end{bmatrix}\right\}.$$

The quasi-logarithm $\hat{A}$ is an antisymmetric operation in the scalar product

$$\langle\vec{x}_1|\vec{x}_2\rangle := \{x_1^1(t)x_1^2(t)+x_2^1(t)x_2^2(t)\},\quad \vec{x}_1=\{\vec{x}_1(t)\},\quad \vec{x}_2=\{\vec{x}_2(t)\}\in L_2^0$$

possessing property (18).

The derivative $S = \frac{d}{dt}$ and the limit condition $s_0 = \Big|_{t=0}$ possess properties (19) and (20) respectively. It follows therefore from the last theorem that the resolvent $\hat{\Phi}_0(\hat{A})$ is a quasi-unitary operation.

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