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An overview of some recent developments on the Invariant Subspace Problem

Abstract

This paper presents an account of some recent approaches to the Invariant Subspace Problem. It contains a brief historical account of the problem, and some more detailed discussions of specific topics, namely, universal operators, the Bishop operators, and Read's Banach space counter-example involving a finitely strictly singular operator.

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1. History of the problem

There is an important problem in operator theory called *the invariant subspace problem*. This problem is open for more than half a century, and there are many significant contributions with a huge variety of techniques, making this challenging problem so interesting; however the solution seems to be nowhere in sight. In this paper we first review the history of the problem, and then present an account of some recent developments with which we have been involved. Other recent approaches are discussed in [21].
The invariant subspace problem is the following: let $\mathcal{X}$ be a complex Banach space of dimension at least 2 and let $T \in \mathcal{L}(\mathcal{X})$, i.e., $T : \mathcal{X} \to \mathcal{X}$, linear and bounded. Is there any closed subspace $\mathcal{M} \subset \mathcal{X}$ such that $T(\mathcal{M}) \subset \mathcal{M}$ (i.e., $\mathcal{M}$ is invariant for T) and $\mathcal{M} \neq \{0\}$, $\mathcal{M} \neq \mathcal{X}$ (i.e., $\mathcal{M}$ is not trivial)? In the sequel a "nontrivial invariant subspace" may be abbreviated as "ntis".

- Here is a list of preliminary remarks:

Assume that $\mathcal{X}$ is of finite dimension $n \geq 2$, so that $\mathcal{X}$ is isomorphic to $\mathbb{C}^n$. Then $T \in \mathcal{L}(\mathcal{X})$ is a $n \times n$ matrix with complex entries, and thus T has eigenvectors. Each eigenvector generates a (nontrivial) invariant subspace of dimension 1. The Jordan form of T is of great help in describing the lattice of the invariant subspaces of T.

If $\mathcal{X}$ is not separable, then $T \in \mathcal{L}(\mathcal{X})$ has a nontrivial invariant subspace since for all $x \in \mathcal{X}$, $x \neq 0$, the closure of $\{p(T)x : p \in \mathbb{C}[z]\}$ is invariant for T and nontrivial since it is a separable subspace by construction, containing a nonzero vector.

Note also that for real Banach spaces, there exist operators with no nontrivial invariant subspace: take $\mathcal{X} = \mathbb{R}^2$ and T a rotation of angle $\theta \in (0, 2\pi) \setminus \{\pi\}$.

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- Here is a non-exhaustive list of important results:

In 1950, von Neumann (unpublished) proved that every compact operator T on a Hilbert space has nontrivial hyperinvariant subspaces (i.e., closed subspaces invariant for all operators A such that $AT = TA$). We may write "nthis" for short in what follows.

As a consequence of the spectral theorem, all normal operators also have nontrivial hyperinvariant subspaces unless they are equal to a scalar times the identity map.

As a consequence of the existence of a holomorphic functional calculus, namely the Dunford–Riesz functional calculus, a linear and bounded operator on an arbitrary complex Banach space, whose spectrum is disconnected, has nontrivial hyperinvariant subspaces.

Aronszajn and Smith proved in 1954 [6] that every compact operator on Banach space has a nontrivial hyperinvariant subspace. Later, in 1966, Bernstein and Robinson [11], using nonstandard analysis, proved that if there exists a nontrivial polynomial p such that $p(T)$ is compact, then T has nontrivial hyperinvariant subspace or T is a scalar operator, that is a scalar times the identity map. Halmos, the same year, obtained the same result avoiding the nonstandard analysis. Finally, Lomonosov [40] proved in a very elegant way, namely, using a fixed point theorem due to Schauder, that every operator commuting with a nonzero compact operator has nontrivial hyperinvariant subspaces unless it is of the form λI. More generally, if T commutes with an operator S and S commutes with a nonzero compact operator, then T has a nontrivial invariant subspace. But Troitsky [60] proved that no further generalization holds in an arbitrary Banach space, i.e., there is no such result for longer chains of pairwise commuting operators.
In 1978, Scott Brown [13] proved that a subnormal operator (i.e., a restriction of a normal operator to one of its invariant subspace) has ntis. Later Thomson [59] gave another simpler proof based on an appropriate functional calculus. Nevertheless, S. Brown's paper is important since it is the starting point of a whole theory called the *dual algebra theory* [9], with a spectacular result which we mention below.

On the other hand, there are now several negative results, that is, examples of bounded operators on a Banach space with no ntis. One of them was found by Enflo [25, 26] and simplified by Beauzamy [8]. Another example was produced by Read [52], who later gave the first example on a classical Banach space, namely ℓ_1 [53]. Read was also even able to produce an example of operator with no nontrivial invariant subset [54]. Closely related to our problem, we should also mention that Atzmon [7] was the first to construct an example of operator on a nuclear Fréchet space with no ntis.

In 1988, S. Brown, B. Chevreau and C. Pearcy [14] proved, using dual algebra theory [9] and an approximation scheme linked to the surjectivity of bilinear forms [10], that a contraction on a Hilbert space whose spectrum contains the unit circle has nontrivial invariant subspaces.
In 2004, C. Ambrozie and W. Müller [1] generalized this last impressive result, proving that if a bounded operator T on an arbitrary complex Banach space is polynomially bounded (i.e., there exists $C > 0$ such that $||p(T)|| \leq C \sup\{|p(z)| : |z| \leq 1\}$) and if the spectrum of T contains the unit circle, then T^* has a ntis. One of the extra tools introduced by the authors is the famous Carleson interpolation theorem.
Here are a few comments to explain why the Ambrozie–Müller theorem implies the Brown–Chevreau–Pearcy result. First note that if T has a ntis then T^* has a ntis. Therefore, in a reflexive Banach space the existence of ntis for T or its adjoint are equivalent. Moreover, thanks to the von Neumann inequality, we know that a contraction on a complex Hilbert space is polynomially bounded (with $C = 1$). Finally, Pisier proved in 1997 [48] that there exists a polynomially bounded operator on a Hilbert space which is not similar to any contraction.
One of the most positive results concerning the invariant subspace problem that one could ever imagine is a construction due to Argyros and Haydon in 2009 [5]. They constructed an infinite-dimensional Banach space such that every bounded operator is the sum of a compact operator and a scalar operator. Therefore, in particular, every non-scalar operator on this space has a nthis. Gowers and Maurey [33] had earlier found a space where every bounded operator is the sum of a strictly singular operator and a scalar operator. Nonetheless, since Read [56] gave an example of a strictly singular operator without invariant subspace (detailed in the last section), the Gowers–Maurey example had no direct consequences for the problem under consideration.

In 1998 Ansari and Enflo introduced the notion of minimal vectors in Hilbert space [4, 27, 28]. This is a promising new technique by which it was hoped to show the existence of ntis for quasinilpotent operators on a Hilbert space. However, Troitsky [61] introduced a generalization known as λ-minimal vectors, which are well-defined in an arbitrary Banach space, and for which the ideas developed in the context of a reflexive Banach space apply. This fact considerably reduced the hope since C. Read [55] has also constructed an example of a quasinilpotent operator with no ntis. Nonetheless, the minimal vector technique presented in [4] gave alternative proofs of the existence of invariant subspaces for compact operators and normal operators on a Hilbert space, and has been used for the study of other classes of operators: see, for example, [3, 18, 19, 36].

2. Universal operators on Hilbert space

The subject of universal operators has attracted much interest recently. The basic idea is that there are particular Hilbert space operators such that, if we knew enough information about their lattice of invariant subspaces, we could solve the invariant subspace problem.

Definition 1.

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space. Then an operator $U \in \mathcal{L}(\mathcal{H})$ is said to be *universal* (for Hilbert space) if for each non-zero $T \in \mathcal{L}(\mathcal{H})$ there is a scalar $\lambda \neq 0$ and an invariant subspace $\mathcal{M}$ for U such that the restriction $U_{|\mathcal{M}}$ is similar to λT, i.e., $\lambda J T = UJ$ for some linear isomorphism $J : \mathcal{H} \to \mathcal{M}$.

Assuming the existence of such operators, the following result explains how they can be applied to the invariant subspace problem.

Proposition 2.

Let $U \in \mathcal{L}(\mathcal{H})$ be a universal operator for Hilbert space. Then the following conditions are equivalent:
(i) Every non-zero $T \in \mathcal{L}(\mathcal{H})$ has a non-trivial closed invariant subspace.
(ii) Every minimal non-trivial invariant subspace for U is one-dimensional.

Caradus [15] gave a simple sufficient condition for an operator to be universal.

Theorem 3.

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and $U \in \mathcal{L}(H)$. Suppose that
(i) Ker U is infinite-dimensional, and
(ii) Im $U = \mathcal{H}$.
Then U is universal.

This result is extremely useful for constructing universal operators. Recently, Pozzi [51, Thm. 3.8] gave an extension of the Caradus result as follows:

Theorem 4.

Suppose that $U \in \mathcal{L}(\mathcal{H})$ satisfies
(i) Ker U is infinite-dimensional, and
(ii) Im U has finite codimension.
Then U is universal.

The condition (i) is necessary, since U needs to have restrictions similar to operators with infinite-dimensional kernel. Condition (ii) is not necessary (e.g. if U is universal, then so is $U \oplus O$ in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, where O is the zero operator).

An easy example of a universal operator is the backward shift B of infinite multiplicity, which can be realised on $L^2(0, \infty)$ by writing $(Bf)(t) = f(t + 1)$ for $t > 0$; we give some further examples below.

2.1. Adjoints of composition operators

For $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic we define the composition operator $C_\varphi : H^2 \to H^2$ on the Hardy space $H^2 = H^2(\mathbb{D})$ by $C_\varphi f = f \circ \varphi$ for $f \in H^2$. It is well-known that C_φ is always a bounded operator on H^2 (Littlewood's subordination theorem [39]).

Now define a Borel measure μ_φ on $\mathbb{T}$ by $\mu_\varphi(E) = m(\varphi^{-1}(E))$, where m is Lebesgue measure; then μ_φ is absolutely continuous with respect to m and we write g_φ for the Radon–Nikodym derivative $d\mu_\varphi/dm$. The following result is apparently new.

Theorem 5.
Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and non-constant, satisfying:
(i) the function g_φ defined above is essentially bounded away from zero, and
(ii) φ is not univalent.
Then the adjoint composition operator C_φ^* is universal.

Proof. The operator C_φ^* is surjective if and only if C_φ is bounded below (since by Banach's closed range theorem C_φ^* will have closed range precisely when C_φ does, and C_φ is injective when φ is non-constant). This happens precisely when g_φ is essentially bounded away from zero, as was shown in [22]: see also [37] for a more general result on weighted composition operators. Also if φ is not univalent then, by the open mapping theorem for holomorphic functions, for each N there exist $w_1, \dots, w_n$ and $w_1', \dots, w_N'$ such that $\varphi(w_j) = \varphi(w_j')$ for $j = 1, 2, \dots, N$. Then $C_\varphi^*(k_{w_j} - k_{w_j'}) = k_{\varphi(w_j)} - k_{\varphi(w_j')} = 0$ for each j; we conclude that the kernel of C_φ^* is infinite-dimensional, and so C_φ^* is universal, by Theorem 3. □

The result clearly applies to any finite Blaschke product φ of degree at least two.

2.2. Weighted shifts and related operators

Let $a_0 < a_1$ be real numbers and consider the space $\mathcal{H} = L^2(\mathbb{Z}, L^2(a_0, a_1))$. Let $(k_n)_{n\in\mathbb{Z}}$ be a sequence of positive continuous functions on $[a_0, a_1]$ such that (k_n) converges uniformly to constants a and b as $n \to \infty$ and $n \to -\infty$ respectively, with $a < b$. Consider now the bilateral weighted shift on $\mathcal{H}$ defined by $(Tx)_n = k_{n-1}x_{n-1}$ for $x \in \mathcal{H}$ and $n \in \mathbb{Z}$. The following result can be found in [47].

Theorem 6.
Let T be the weighted bilateral shift defined above. Then $\sigma(T) = \{\lambda \in \mathbb{C} : a \leqslant |\lambda| \leqslant b\}$. If $a < |\lambda| < b$ then λ is an eigenvalue of T of infinite multiplicity, whereas $T^* - \lambda I$ is bounded below. It follows that $T - \lambda I$ is a universal operator.

Note that T and $T - \lambda I$ have the same lattice of invariant subspaces.

We have seen some examples of the universality of adjoints of composition operators on the Hardy space H^2. In [46], the following result is proved.

Theorem 7.
Let C_φ be the composition operator on H^2 induced by the hyperbolic automorphism $\varphi(z) = \frac{z+r}{1+rz}$, where $0 < r < 1$, and write $a = (1-r)/(1+r)$. Then for every complex number λ with $\sqrt{a} < |\lambda| < 1/\sqrt{a}$ the operator $C_\varphi - \lambda I$ is universal.

It can be shown that C_φ is similar to a bilateral weighted shift of infinite multiplicity. One way to see this is by writing

$$H = K_B \oplus BK_B \oplus B^2 K_B \oplus \dots,$$

where K_B is the model space $H^2 \ominus BH^2$ corresponding to the Blaschke product B whose zeroes are the iterates (positive and negative) of 0 under φ. Each of the spaces $B^j K_B$ is invariant under C_φ and on these spaces it acts as a shift.

This particular universal operator has attracted a certain amount of interest, as people have considered cyclic subspaces generated by functions in H^2 and attempted to decide whether they were minimal invariant subspaces for C_φ. See, for example, [32, 43, 45].
A more recent example is the following, proved in [47]:

Theorem 8.
For $\varphi(x) = x^s$ with $0 < s < 1$ the composition operator C_φ on $L^2(0,1)$ satisfies $\sigma(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1/\sqrt{s}\}$, and for $0 < |\lambda| < 1/\sqrt{s}$ the operator $C_\varphi - \lambda I$ is universal.

Note that for $s > 1$ these operators are not bounded.
The proof of this result is achieved by representing C_φ as a weighted shift, using the decomposition of $(0,1)$ into a countable union of intervals; these can be taken to have the form $\left[\frac{1}{2^{s^n}}, \frac{1}{2^{s^{n+1}}}\right)$, for $n \in \mathbb{Z}$.

In [51] these techniques are taken further and examples of universal weighted composition operators are given, both on $L^2(0,1)$ and on the Sobolev space $W_0^1(0,1)$ of absolutely continuous functions f such that $f(0) = 0$ and $f' \in L^2(0,1)$.

3. An open problem involving Liouville numbers

Definition 9.
For $\alpha \in (0,1)$ the *Bishop operator* $T_\alpha : L^2(0,1) \to L^2(0,1)$ is defined by

$$(T_\alpha f)(t) = tf(\{t+\alpha\}), \qquad (f \in L^2(0,1)),$$

where the curly brackets denote the fractional part of a number, i.e., $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x.

These weighted composition operators were originally introduced as candidates for operators that might not have non-trivial invariant subspaces, at least for α irrational.

Davie [23] showed that, for every α that is not a Liouville number, T_α has a nontrivial closed hyperinvariant subspace. Recall that a Liouville number is an irrational that is very well approximable by rational numbers, in the sense that for each $n \geqslant 2$ there is a number K_n such that

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{K_n}{q^n}$$

for infinitely many $p, q \in \mathbb{Z}$. Such numbers are necessarily transcendental. Flattot [30] extended Davie's methods to a larger class of α, but it is clear that the approach cannot work for all α. See also [12, 20, 41] for other generalizations of Davie's results.

Davie's method consists in defining a functional calculus from a particular Beurling algebra

$$\mathcal{A}_\rho := \left\{ f \in C(\mathbb{T}) : ||f||_\rho := \sum_{n=-\infty}^{\infty} \rho_n |\hat{f}(n)| < \infty \right\},$$

into the space of linear mappings from $\mathcal{S}$ into $L^2(0,1)$), where $\mathcal{S}$ is an appropriately-chosen dense subspace of $L^2(0,1)$. Thus we have a multiplicative homomorphism Φ such that $\Phi(I) = I$ and $\Phi(Z) = T$, where I and Z are the functions $z \mapsto 1$ and $z \mapsto z$ respectively.
The hyperinvariant subspaces for T are then constructed using the fact that for a suitable choice of weights $(\rho_n)_{n\in\mathbb{Z}}$ there exist non-negative functions $\varphi, \psi \in \mathcal{A}_\rho$ which do not vanish identically but whose pointwise product is zero. The required hyperinvariant subspace is then

$$\mathcal{M} := \{f \in L^2(0,1) : \psi(T)Vf = 0 \text{ for all } V \text{ with } TV = VT\}.$$

Note that Bishop operators are order-preserving; that is, if $f \geqslant 0$ a.e., then $T_\alpha f \geqslant 0$ a.e. Indeed, the invariant subspace problem is still unsolved for order-preserving operators on L^2 spaces.

4. Finitely strictly singular operators

It is a well-known result, due originally to von Neumann (1950) but unpublished by him, that any compact operator on a Hilbert space of dimension at least 2 has a nontrivial invariant subspace. Stronger results are known, for example the Lomonosov theorem [40] that any operator commuting with a compact operator has a non-trivial hyperinvariant subspace, unless it is a multiple of the identity. Two related properties are as follows.

Definition 10.
Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces.
(i) An operator $T : \mathcal{X} \to \mathcal{Y}$ is *strictly singular (SS)* if for each $\varepsilon > 0$ and for each infinite-dimensional subspace $\mathcal{Z} \subset \mathcal{X}$ there is a vector $z \in \mathcal{Z}$ with $||z|| = 1$ and $||Tz|| < \varepsilon$.
(ii) An operator $T : \mathcal{X} \to \mathcal{Y}$ is *finitely strictly singular (FSS)* if for each $\varepsilon > 0$ there is a positive integer N such that for every subspace $\mathcal{Z} \subset \mathcal{X}$ with $\dim \mathcal{Z} \geqslant N$ there is an element $z \in \mathcal{Z}$ with $||z|| = 1$ and $||Tz|| < \varepsilon$.

The notion of strictly singular operators goes back to Kato in the late sixties, and is equivalent to the fact that the restriction of T to an infinite-dimensional subspace can never be an isomorphism.
The next proposition explains their link with compact operators.

Proposition 11.
Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ be linear and bounded. The following implications hold:

$$T \text{ is compact } \Rightarrow T \text{ is finitely strictly singular } \Rightarrow T \text{ is strictly singular.}$$

Proof. The last implication follows from the definitions. The first implication is not so obvious since a compact operator is not always in the closure of finite rank operators. Let $\varepsilon > 0$ and $B = \{x \in \mathcal{X} : ||x||_{\mathcal{X}} \leq 1\}$. Then, obviously,

$$T(B) \subset \overline{T(B)} \subset \bigcup_{x \in B} B_{\mathcal{Y}}(Tx, \varepsilon/2),$$

where $B_{\mathcal{Y}}(Tx, \varepsilon/2) := \{y \in \mathcal{Y} : ||Tx - y|| < \varepsilon/2\}$. If T is compact, there exist $x_1, \cdots, x_N \in B$ such that

$$T(B) \subset \bigcup_{i=1}^{N} B_{\mathcal{Y}}(Tx_i, \varepsilon/2).$$

By the Hahn–Banach theorem, there exist $\varphi_1, \cdots, \varphi_N \in Y^*$ such that $||\varphi_i|| \leq 1$ and $\varphi_i(Tx_i) = ||Tx_i||$ for $i \in \{1, \cdots, N\}$. Then, note that for all $x \in B$, there exists $k \in \{1, \cdots, N\}$ such that $||Tx - Tx_k|| < \varepsilon/2$. Since $||\varphi_k|| \leq 1$, it follows that

$$|\varphi_k(Tx)| > ||Tx_k|| - \varepsilon/2 > ||Tx|| - \varepsilon. \tag{1}$$

Let E be a closed subspace of $\mathcal{X}$ with dimension greater or equal than $N + 1$. If $T(E)$ is of dimension at most N, then there exists a unit vector $x \in E$ such that $Tx = 0$. Otherwise, there always exists a unit vector $x \in E$ with $Tx \in \bigcap_{i=1}^{N} \operatorname{Ker} \varphi_k$. Then, by (1), there exists an index $k_0 \in \{1, \cdots, N\}$ with $|\varphi_{k_0}(Tx)| > ||Tx|| - \varepsilon$. Since $Tx \in \operatorname{Ker} \varphi_{k_0}$, it follows that $||Tx|| < \varepsilon$. □

In the particular case where $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, an operator $T : \mathcal{X} \to \mathcal{Y}$ is compact if and only if T is strictly singular. In general it is not the case but nevertheless the gap between those classes is somehow rather small. Indeed, if $T : \mathcal{X} \to \mathcal{X}$ with $\mathcal{X}$ of infinite dimension, then T is strictly singular if and only if each infinite-dimensional subspace of $\mathcal{X}$ contains an infinite-dimensional subspace $\mathcal{Z}$ such that the restriction of T to $\mathcal{Z}$ is compact [38]. It is precisely this property which can be used to derive the following structural property.

Proposition 12.
The set of strictly singular and finitely strictly singular operators are bilateral closed ideals.

Here are some general results for specific Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. Recall that Pitt (see for example [29], p. 175 and [24]) proved that if $1 \leq p < q \leq \infty$, any linear and bounded operator from ℓ_q into ℓ_p, as well as from c_0 into ℓ_p is compact. Moreover, any operator $T : \ell_p \to \ell_q$ is strictly singular, but not necessarily finitely strictly singular. Another interesting result is due to Milman: the canonical injection from $\ell_p \to \ell_q$ is clearly not compact, but is finitely strictly singular.
Milman's proof is based on the following lemma for which we include his clever and elegant proof.

Lemma 13.
Every k-dimensional subspace E of c_0 contains a vector with a "flat", namely, a vector x with sup-norm one with (at least) k coordinates equal in modulus to 1.

Proof. The proof follows by induction on k. For $k = 1$, it is obvious since $E \subset c_0$. Assume that it is true for a fixed $k \geq 1$ and consider E a subspace of size $k+1$. By hypothesis, there exists $x \in E$ with at least k coordinates, say $x_{j_1}, \cdots, x_{j_k}$ of modulus one and with a supremum norm equal to 1. Without loss of generality we may assume that $|x_j| < 1$ for all $j \notin J$, where $J = \{j_1, \cdots, j_k\}$. Now consider the function $f : [0, +\infty) \to [0, +\infty)$ defined by

$$f(t) = \max_{j \notin J} |x_j + t y_j|,$$

where $y = (y_j)_j$ is a nonzero vector in both E and the subspace F of codimension n defined by $F = \{z = (z_j)_j \in c_0 : z_j = 0 \text{ if } j \in J\}$.
Since f is continuous with $f(0) < 1$ and $\lim_{t \to +\infty} f(t) = \infty$, the intermediate value theorem implies the existence of $t_0 > 0$ such that $f(t_0) = 1$. Then $x + t_0 y \in E$ has $k+1$ coordinates of modulus one, which ends the proof. □

For a vector x as in Milman's lemma, one has $||x||_{\ell_q} \ll ||x||_{\ell_p}$.
In fact the following refinement of this observation is true: every k-dimensional subspace E of c_0 contains a vector x so that these k coordinates have alternating signs [17]. More precisely, a finite or infinite sequence of real numbers in $[-1, 1]$ will be called a *zigzag* of order k if it has a subsequence of the form $(-1, 1, -1, 1, \ldots)$ of length k.

Theorem 14.
Let $k \in \mathbb{N}$, then every k-dimensional subspace of c_0 contains a zigzag of order k.

There are two proofs of Theorem 14, presented in [17], one based on combinatorial properties of polytopes (see [62]) and the other one based on the geometry of the set of all zigzags and algebraic topology (see [31, 34]). Such a result can be used in order to provide a counterexample for the invariant subspace problem in the theory of finitely strictly singular operators.
Recall that James' p-space J_p is a sequence space consisting of all sequences $x = (x_n)_{n=1}^{\infty}$ in c_0 satisfying $||x||_{J_p} < \infty$ where

$$||x||_{J_p} = \left(\sup\left\{\sum_{i=1}^{n-1} |x_{k_{i+1}} - x_{k_i}|^p \ : \ 1 \leqslant k_1 < \cdots < k_n,\ n \in \mathbb{N}\right\}\right)^{\frac{1}{p}}$$

is the norm in J_p. For more information on James' spaces we refer the reader to [16, 35, 38, 42, 57].
It was an open question whether every finitely strictly singular operator has invariant subspaces. Some partial results in this direction were obtained in [2, 50]. We answer this question in the negative by showing that Read's operator in [56] is, in fact, finitely strictly singular. As an intermediate result, we prove that the formal inclusion operator from J_p to J_q with $1 \leqslant p < q < \infty$ is finitely strictly singular.
Suppose that $1 \leqslant p < q$. Since $||x||_{J_p}$ is defined as the supremum of ℓ_p-norms of certain sequences, the condition $||\cdot||_{\ell_q} \leqslant ||\cdot||_{\ell_p}$ implies that $||\cdot||_{J_q} \leqslant ||\cdot||_{J_p}$. It follows that $J_p \subseteq J_q$ and the formal inclusion operator $i_{p,q} : J_p \to J_q$ has norm 1.

Theorem 15.
If $1 \leq p < q < \infty$ then the formal inclusion operator $i_{p,q} : J_p \to J_q$ is finitely strictly singular.

Proof. Given any $x \in J_p$, then $|x_{i+1}-x_i|^q \leqslant \big(2||x||_\infty\big)^{q-p}|x_{i+1}-x_i|^p$ for every $i \in \mathbb{N}$, so that $||x||_{J_q} \leqslant \big(2||x||_\infty\big)^{1-\frac{p}{q}}||x||_{J_p}^{\frac{p}{q}}$. Fix an arbitrary $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $(k-1)^{\frac{1}{p}-\frac{1}{q}} > \frac{1}{\varepsilon}$. Suppose that E is a subspace of J_p with $\dim E = k$. By Theorem 14, there is a zigzag $z \in E$ of order k. By the definition of norm in J_p, we have $||z||_{J_p} \geqslant 2(k-1)^{\frac{1}{p}}$.
Put $y = \frac{z}{||z||_{J_p}}$. Then $y \in E$ with $||y||_{J_p} = 1$. Obviously, $||y||_\infty \leqslant \frac{1}{2}(k-1)^{-\frac{1}{p}}$, so that

$$||i_{p,q}(y)||_{J_q} = ||y||_{J_q} \leqslant (k-1)^{\frac{1}{q}-\frac{1}{p}}||y||_{J_p}^{\frac{p}{q}} < \varepsilon.$$

Hence, $i_{p,q}$ is finitely strictly singular. □

We will now use Theorem 15 to show that the strictly singular operator T constructed by Read in [56] is finitely strictly singular. Let us briefly outline those properties of T that will be relevant for our investigation. The underlying space $\mathcal{X}$ for this operator is defined as the ℓ_2-direct sum of ℓ_2 and $\mathcal{Y}$, $\mathcal{X} = (\ell_2 \oplus \mathcal{Y})_{\ell_2}$, where $\mathcal{Y}$ itself is the ℓ_2-direct sum of an infinite sequence of J_p-spaces $\mathcal{Y} = \big(\bigoplus_{i=1}^{\infty} J_{p_i}\big)_{\ell_2}$, with (p_i) a certain strictly increasing sequence in $(2, +\infty)$. The operator T is a compact perturbation of $0 \oplus W_1$, where $W_1 : \mathcal{Y} \to \mathcal{Y}$ acts as a weighted right shift, that is,

$$W_1(x_1, x_2, x_3, \dots) = (0, \beta_1 x_1, \beta_2 x_2, \beta_3 x_3, \dots), \quad x_i \in J_{p_i},$$

with $\beta_i \to 0$. Note that one should rather write $\beta_i i_{p_i,p_{i+1}} x_i$ instead of $\beta_i x_i$. Clearly, it suffices to show that W_1 is finitely strictly singular.
For $n \in \mathbb{N}$, define $V_n : \mathcal{Y} \to \mathcal{Y}$ via

$$V_n(x_1, x_2, x_3, \dots) = (0, \beta_1 x_1, \dots, \beta_n x_n, 0, 0 \dots), \quad x_i \in J_{p_i}.$$

It follows from $\beta_i \to 0$ that $||V_n - W_1|| \to 0$. Since finitely strictly singular operators from $\mathcal{Y}$ to $\mathcal{Y}$ form a closed subspace of $L(\mathcal{Y})$, it suffices to show that V_n is finitely strictly singular for every n. Given $n \in \mathbb{N}$, one can write

$$V_n = \sum_{i=1}^{n} \beta_i j_{i+1} i_{p_i,p_{i+1}} P_i,$$

where $P_i : \mathcal{Y} \to J_{p_i}$ is the canonical projection and $j_i : J_{p_i} \to \mathcal{Y}$ is the canonical inclusion. Thus, V_n is finitely strictly singular because finitely strictly singular operators form an operator ideal. This yields the following result.

Theorem 16.
Read's operator T is finitely strictly singular and has no nontrivial invariant subspace.

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