

APPROXIMATE SOLUTION OF A SINGULAR INTEGRAL CAUCHY-KERNEL EQUATION OF THE FIRST KIND

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Abstract — In this paper we present a method for the numerical solution of Cauchy type singular integral equations of the first kind on a finite segment which is unbounded at the end points of the segment. Chebyshev polynomials of the first and second kinds are used to derive an approximate solution. Moreover, an estimation error is computed for the approximate solution.

Keywords: Singular integral equation, Cauchy kernel, approximate solution, Chebyshev polynomials of the first and second kinds.

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1. Introduction

Let us consider a singular integral equation of the form

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 k(x,t)\varphi(t)dt = f(x), \quad -1 < x < 1, \quad (1.1)$$

where $k(x,t)$ and $f(x)$ are given real-valued Hölder continuous functions and $\varphi(x)$ is the unknown function. Equations of form (1.1) arise from the formulation of the mixed boundary value problems in applied physics and engineering [1, 5, 3]. In a very large majority of physical problems, φ is generally either a "potential-type" quantity (such as displacement, temperature, electrostatic potential, velocity potential), or a "flux-type" quantity (such as stress, dislocation density, heat flux). We consider, however, that there may be situations in which at the end points the solution requires to have a stronger zero (e.g., stress function) or a stronger singularity (e.g., transverse shear in a fourth-order plate theory). In this paper, we consider the function $\varphi(x)$ which has integrable singularities at both ends $x = \pm 1$ (unbounded at both end-points $x = \pm 1$). In [10], an approximate solution of a singular integral equation of the second kind is presented and constructed using Chebyshev polynomials of the first and second kinds. It is well known (see [4, 7]) that the analytic solution of the characteristic singular integral equation

$$\int_{-1}^1 \frac{\varphi(t)dt}{t-x} = f(x), \quad -1 < x < 1, \quad (1.2)$$

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which is unbounded at both end-points $x = \pm 1$, is given by

$$\varphi(x) = \frac{-1}{\pi^2 \sqrt{1-x^2}} \int_{-1}^1 \frac{\sqrt{1-t^2} f(t)}{t-x} dt + \frac{C}{\pi \sqrt{1-x^2}}, \quad (1.3)$$

where C is an arbitrary constant and

$$C = \int_{-1}^1 \varphi(t) dt$$

is the uniqueness condition. Solving Eq. (1.1) for its characteristic part and defining the bounded function $g(x)$ by

$$g(x) = \sqrt{1-x^2} \varphi(x),$$

we find that it is equivalent to the Fredholm equation of the second kind

$$g(x) + \int_{-1}^1 g(t) K(x, t) dt = F(x), \quad (1.4)$$

with

$$F(x) = \frac{-1}{\pi^2} \int_{-1}^1 \frac{\sqrt{1-t^2} f(t)}{t-x} dt + \frac{C}{\pi}, \quad (1.5)$$

and

$$K(x, t) = \frac{-1}{\pi^2 \sqrt{1-t^2}} \int_{-1}^1 \frac{\sqrt{1-\tau^2} k(\tau, t)}{\tau-x} d\tau \quad (1.6)$$

(see [10]). Thus, Eq. (1.1) is solvable if the equivalent Fredholm equation is also solvable. Therefore, according to the Fredholm alternative theorem, if the homogeneous form of Eq.(1.4) ($F(x) \equiv 0$) is not solvable (i.e., if it has only a solution identically equal to zero), then the solution of the nonhomogeneous equation (1.4) exists and is equal to

$$g(x) = F(x) - \int_{-1}^1 \Gamma(x, t) F(t) dt, \quad (1.7)$$

where $\Gamma(x, t)$ is the resolvent of the kernel $K(x, t)$.

2. Approximate solution

In this section, we will derive an approximate solution of (1.1). For this purpose, we will use the following approximation formula:

$$f(x) \simeq f_n(x) = \sum_{j=0}^n f_j U_j(x), \quad (2.1)$$

where U_j is the Chebyshev polynomial of second kind which is defined by the following recurrence relation:

$$\begin{aligned} U_0(x) &= 1, & U_1(x) &= 2x, \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x), & n &\geq 2. \end{aligned} \quad (2.2)$$

Substituting

$$\begin{aligned} T_0(x) &= U_0(x), & 2T_1(x) &= U_1(x), \\ 2T_j(x) &= U_j(x) - U_{j-2}(x), & j &\geq 2, \end{aligned}$$

and using Theorem 7.9 from [8], we obtain

$$f_j = \begin{cases} \frac{1}{n+1} \sum_{k=1}^{n+1} f(x_k)[T_j(x_k) - T_{j+2}(x_k)], & j = 0, 1, \dots, n - 2, \\ \frac{1}{n+1} \sum_{k=1}^{n+1} f(x_k)T_j(x_k), & j = n - 1, n, \end{cases} \tag{2.3}$$

where $T_j(t)$ denotes Chebyshev polynomials of the first kind and

$$x_k = \cos \frac{2k - 1}{2n + 2} \pi, \quad k = 1, 2, \dots, n + 1, \tag{2.4}$$

are the roots of $T_{n+1}(t)$. We will find an approximate solution of problem (1.1) as an exact solution of the following problem:

$$\int_{-1}^1 \frac{\varphi_{n+1}(t)}{t - x} dt + \int_{-1}^1 k_{n,n}(x, t)\varphi_{n+1}(t)dt = f_n(x), \tag{2.5}$$

$$\int_{-1}^1 \varphi_{n+1}(t)dt = C, \tag{2.6}$$

where $f_n(x)$ is given by (2.1). The approximate solution is given in the form

$$\varphi_{n+1}(x) = \frac{1}{\sqrt{1 - x^2}} g_{n+1}(x) = \frac{1}{\sqrt{1 - x^2}} \sum_{j=0}^{n+1} \alpha_j T_j(x), \tag{2.7}$$

with unknown coefficients α_j and $k_{n,n}(x, t)$ is expressed as

$$k_{n,n}(x, t) = \sum_{m=0}^n U_m(x) \sum_{j=0}^n \sigma_{m,j} T_j(t), \tag{2.8}$$

with

$$\sigma_{m,j} = \begin{cases} \frac{2}{n+1} \sum_{l=1}^{n+1} \frac{1}{n+1} \sum_{k=1}^{n+1} k(x_k, x_l)[T_m(x_k) - T_{m+2}(x_k)]T_j(x_l) & m = 0, 1, \dots, n - 2, \\ \frac{2}{n+1} \sum_{l=1}^{n+1} [\frac{1}{n+1} \sum_{k=1}^{n+1} k(x_k, x_l)T_m(x_k)]T_j(x_l) & m = n - 1, n, \end{cases}$$

where x_k is defined in the same way as in (2.4). If we substitute (2.1), (2.7) and (2.8) into (2.5) and use the relation (see [6], Lemma 8.3)

$$\int_{-1}^1 \frac{T_j(t)}{\sqrt{1 - t^2} t - x} dt = \begin{cases} \pi U_{n-1}(x), & n > 0 \\ 0, & n = 0 \end{cases} \tag{2.9}$$

then we will obtain

$$\pi \sum_{j=1}^{n+1} \alpha_j U_{j-1}(x) + \sum_{j=0}^{n+1} \alpha_j \sum_{m=0}^n \delta_{m,j} U_m(x) = \sum_{j=0}^n f_j U_j(x) \tag{2.10}$$

with

$$\delta_{m,j} = \sum_{k=0}^n \sigma_{m,k} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} T_k(t) T_j(t) dt.$$

By using the well-known relation

$$\frac{1}{\pi} \int_{-1}^1 \frac{T_j(t) T_k(t)}{\sqrt{1-t^2}} dt = \begin{cases} 0, & j \neq k, \\ \frac{1}{2}, & j = k \neq 0, \\ 1, & j = k = 0, \end{cases} \tag{2.11}$$

and (2.6), we obtain that $\alpha_0 = \frac{C}{\pi}$ and $\{\alpha_j\}_1^{n+1}$ are determined by solving the following system of linear equations:

$$\begin{aligned} \pi \alpha_1 + \sum_{j=0}^{n+1} \alpha_j \delta_{0,j} &= f_0 \\ \pi \alpha_2 + \sum_{j=0}^{n+1} \alpha_j \delta_{1,j} &= f_1 \\ &\vdots \\ \pi \alpha_{n+1} + \sum_{j=0}^{n+1} \alpha_j \delta_{n,j} &= f_n. \end{aligned} \tag{2.12}$$

3. Error bound and error estimation

We recall the following definition and theorem from ([9], pp 171).

Definition 3.1. Let $r \geq 0, 0 < \mu \leq 1$. We say that a function $f(x), x \in [-1, 1]$ belongs to the class $W^r H^\mu$ if all the derivatives up to the order r inclusive exist and the r th derivative belongs to the Hölder class $H(\mu)$

$$|f^{(r)}(x') - f^{(r)}(x'')| \leq k |x' - x''|^\mu, \quad \forall x', x'' \in [-1, 1],$$

where the constants k and μ are independent of the choice of points x', x'' .

Theorem 3.1. Let the exact and approximate Fredholm equations

$$K(\varphi; x) \equiv \varphi(x) + \int_{-1}^1 k(x, t) \varphi(t) dt = f(x), \quad -1 \leq x \leq 1, \tag{3.1}$$

$$K_n(\varphi_n; x) \equiv \varphi_n(x) + \int_{-1}^1 k_n(x, t) \varphi_n(t) dt = f_n(x), \quad -1 \leq x \leq 1, \tag{3.2}$$

be given with specified continuous functions $k(x, t), k_n(x, t), f(x)$ and $f_n(x)$, where the first two functions can have integrable singularities in the neighborhood of the points ± 1 . Moreover, let the homogeneous case of Eq. (3.1) be not solvable and let the resolvent $\gamma(x, t)$ of the kernel $k(x, t)$ satisfy the estimate

$$\max_x \int_{-1}^1 |\gamma(x, t)| dt \leq \rho.$$

Further, suppose that $\varepsilon_1 B < 1$, with

$$\varepsilon_1 = \max_x \int_{-1}^1 \int_{-1}^1 |k(x, t) - k_n(x, t)| |k_n(t, \tau)| d\tau dt, \quad B = 1 + \rho.$$

Then Eq.(3.2) has a unique solution and

$$\|K_n^{-1}\| \leq \frac{1 + BK'_1}{1 - \varepsilon_1 B}, \quad K'_1 = \max_x \int_{-1}^1 |k_n(x, t)| dt.$$

Additionally, if we denote

$$\varepsilon_2 = \max_x \int_{-1}^1 |k(x, t) - k_n(x, t)| dt, \quad \varepsilon_3 = \|f - f_n\|,$$

then we have

$$\|\varphi - \varphi_n\| \leq \frac{1 + BK'_1}{1 - \varepsilon_1 B} (\varepsilon_2 B \|f\| + \varepsilon_3).$$

All the norms in the last theorem are uniform on the interval $[-1, 1]$. This theorem will be used to prove the following theorem.

Theorem 3.2. Suppose that the functions $f(x)$ and $k(x, t)$ belong to the class $W^r H^\mu$ (the second function of both variables) for some $r \geq 0$ and $0 < \mu \leq 1$, and are approximated by the polynomials $f_n(x)$ and $k_{n,n}(x, t)$ defined as in (2.1) and (2.8), respectively, and the homogeneous equation corresponding to (1.4) is not solvable. Then the system of linear equations (2.12) is nonsingular for sufficiently large n , and the estimate

$$\|g - g_{n+1}\| \leq M \frac{\ln^2(n)}{n^{r+\mu}} \tag{3.3}$$

holds, where M is a constant independent of n .

Proof. Problem (2.5)- (2.6) or system (2.12) is solvable if and only if the Fredholm equation

$$g_{n+1}(x) + \int_{-1}^1 g_{n+1}(t) K_n(x, t) dt = F_n(x), \tag{3.4}$$

with

$$F_n(x) = \frac{-1}{\pi^2} \int_{-1}^1 \frac{\sqrt{1-t^2} f_n(t) dt}{t-x} + \frac{C}{\pi}, \tag{3.5}$$

and

$$K_n(x, t) = \frac{-1}{\pi^2 \sqrt{1-t^2}} \int_{-1}^1 \frac{\sqrt{1-\tau^2} k_{n,n}(\tau, t) d\tau}{\tau-x} \tag{3.6}$$

is solvable. To prove the solvability of the last equation, it is sufficient, in view of theorem 3.1, to estimate the quantity

$$\varepsilon_1 = \max_x \int_{-1}^1 \int_{-1}^1 |K(x, t) - K_n(x, t)| |K_n(t, \tau)| d\tau dt.$$

By using (1.6) and (3.6), and the formula

$$\frac{1}{\pi} \int_{-1}^1 \sqrt{1-t^2} \frac{U_{n-1}(t)}{t-x} dt = -T_n(x), \quad -1 < x < 1, \quad n = 1, 2, \dots \tag{3.7}$$

(see [6], Lemma 8.3), we obtain

$$\begin{aligned} |K(x, t) - K_n(x, t)| &= \left| \frac{1}{\pi^2 \sqrt{1-t^2}} \int_{-1}^1 \sqrt{1-\tau^2} \frac{[k_{n,n}(\tau, t) - k(\tau, t)]}{\tau-x} d\tau \right| \\ &\leq \frac{1}{\pi^2 \sqrt{1-t^2}} \|k - k_{n,n}\| \max_x \left| \int_{-1}^1 \frac{\sqrt{1-\tau^2}}{\tau-x} d\tau \right| \\ &< \frac{1}{\pi \sqrt{1-t^2}} \|k - k_{n,n}\|. \end{aligned}$$

Since (see [8])

$$\|k - k_{n,n}\| = O\left(\frac{\ln^2(n)}{n^{r+\mu}}\right),$$

and

$$K'_1 = \max_x \int_{-1}^1 |K_n(x, t)| dt = \max_x \int_{-1}^1 |[K_n(x, t) - K(x, t)] + K(x, t)| dt = O(1),$$

(see [9]), it is easily seen that

$$\varepsilon_1 = O\left(\frac{\ln^2(n)}{n^{r+\mu}}\right).$$

Thus, system (3.4) is nonsingular for sufficiently large n. Moreover, we have the following estimates:

$$\varepsilon_3 = \|F - F_n\| = \max_x |F(x) - F_n(x)|,$$

by using (3.5), (1.5), and (3.7), we get

$$|F(x) - F_n(x)| = \left| \frac{1}{\pi^2} \int_{-1}^1 \frac{\sqrt{1-t^2}(f_n(t) - f(t))}{t-x} dt \right| \tag{3.8}$$

$$\leq \frac{1}{\pi^2} \|f - f_n\| \max_x \left| \int_{-1}^1 \frac{\sqrt{1-t^2}}{t-x} dt \right| \tag{3.9}$$

and since (see [8])

$$\|f - f_n\| \leq M_1 \frac{\ln(n)}{n^{r+\mu}},$$

we have

$$|F(x) - F_n(x)| < \frac{M_1}{\pi} \frac{\ln(n)}{n^{r+\mu}},$$

where M_1 denotes a constant independent of n , and

$$\varepsilon_3 = O\left(\frac{\ln(n)}{n^{r+\mu}}\right),$$

thus,

$$\varepsilon_2 = \max_x \int_{-1}^1 |K(x, t) - K_n(x, t)| dt = O\left(\frac{\ln^2(n)}{n^{r+\mu}}\right).$$

Using the above estimates, we conclude the proof of estimate (3.3). □

Now, we give an error estimation for the approximate solution of (1.1). Let $\varphi_{n+1}(t)$ be an approximate solution and $e_{n+1}(t) = \varphi_{n+1}(t) - \varphi(t)$ be the error function associated with $\varphi_{n+1}(t)$, where $\varphi(t)$ is the exact solution of (1.1). Since $\varphi_{n+1}(t)$ is an approximate solution, it satisfies

$$\int_{-1}^1 \frac{\varphi_{n+1}(t)}{t-x} dt + \int_{-1}^1 k(x, t)\varphi_{n+1}(t) dt = f(x) + H_n(x), \tag{3.10}$$

where $H_n(x)$ is a perturbation term and is obtained from

$$H_n(x) = \int_{-1}^1 \frac{\varphi_{n+1}(t)}{t-x} dt + \int_{-1}^1 k(x, t)\varphi_{n+1}(t) dt - f(x). \tag{3.11}$$

Subtracting (1.1) from (3.10) yields the equation

$$\int_{-1}^1 \frac{e_{n+1}(t)}{t-x} dt + \int_{-1}^1 k(x, t)e_{n+1}(t) dt = H_n(x) \tag{3.12}$$

for the error function $e_{n+1}(t)$. To find the approximation $\hat{e}_{n+1}(t)$ to $e_{n+1}(t)$, we can solve Eq. (3.12) in the same way as for (1.1). In this case, only the function $f(x)$ will be replaced by the perturbation term $H_n(x)$. Note that the integrals in the above equations are considered as the Cauchy principal value integrals.

4. Numerical Examples

In this section, we give some examples to clarify the accuracy of the presented method. In the following examples, the condition $\int_{-1}^1 \varphi(x) dx = 0$ is imposed to obtain a unique solution. The results of the examples are reported in Tables 1 and 2.

Example 4.1.

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 \sin(x)\cos(t)\varphi(t) dt = x^4 + 5x^3 + 2x^2 + x - (11/8), \quad -1 < x < 1. \tag{4.1}$$

Table 4.1. Numerical results of Example 4.1

n	$\varphi_{n+1}(x)$	$ \hat{e}_{n+1}(x) $	x
5	2.0241660837963	0.33680683635524e-6	-0.95
8	2.0241664221628	0.20510529632967e-8	
12	2.0241664219585	0.645947555494553e-10	
13	2.0241664217930	0.17224622506631e-8	
5	0.18282114676789	0.28078395790797e-7	-0.7
8	0.18282086442408	0.44373506961785e-9	
12	0.18282086386804	0.23868466338372e-10	
13	0.18282086387058	0.24785764263011e-9	
5	-0.31664632455373	0.31932739109603e-8	-0.1
8	-0.31664629193818	0.55584965857738e-10	
12	-0.31664629235929	0.29751953075183e-11	
13	-0.31664629239493	0.24785764263011e-10	
5	-0.62026672817518	0.27926244431865e-6	0.25
8	-0.62026661878874	0.13851676292616e-9	
12	-0.62026661836141	0.37296578231014e-11	
13	-0.620266618233020	0.29488788610435e-10	
5	-0.85099822660039	0.14481448588835e-6	0.6
8	-0.85099838345414	0.24797135908398e-9	
12	-0.85099838404798	0.12437958797499e-10	
13	-0.85099838382479	0.27382607962309e-10	
5	0.31798103759054	0.64908809776833e-6	0.9
8	0.3179811769847	0.88377404143995e-9	
12	0.31798117756039	0.61385063935078e-10	
13	0.31798117800892	0.41208174607233e-10	

Example 4.2.

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 \arctan(x^2 + 1)(t^3 + t^5)\varphi(t) dt = x \cos(x), \quad -1 < x < 1. \quad (4.2)$$

Table 4.2. Numerical results of Example 4.2

n	$\varphi_{n+1}(x)$	$ \hat{e}_{n+1}(x) $	x
6	0.29520673390836	0.73552646154989e-5	-0.95
9	0.29521196905766	0.29173738112653e-9	
12	0.29521196905247	0.28405896377144e-11	
6	0.022215588033061	0.32492755158884e-5	-0.7
9	0.022211235322308	0.76810974376475e-11	
12	0.022211235437605	0.14453687159640e-11	
6	-0.13680710869492	0.33006513401788e-6	-0.1
9	-0.13681222324737	0.69459372213244e-12	
12	-0.13681222323174	0.21312023413973e-12	
6	-0.11085126002452	0.13180485410804e-5	0.3
9	-0.11084769796870	0.11703001441979e-10	
12	-0.11084769796692	0.11573926265697e-12	
6	0.29520673390836	0.73552646154989e-5	0.95
9	0.29521196905766	0.29174323800867e-9	
12	0.29521196905244	0.28464465225946e-11	

Example 4.3. ([2], P. 189)

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt = \pi e^{-x}, \quad -1 < x < 1. \quad (4.3)$$

Table 4.3. Numerical results of Example 4.3

n	$\ e\ _{\tilde{H}_p^0}$ T. Hartmann, E.P. Stephan	$\ \hat{e}_{n+1}\ _\infty$
3	0.170	0.0059
4	0.277e-1	0.00017
6	0.339e-3	0.6638e-5
8	0.199e-5	0.3539e-7
10	0.688e-8	0.1377e-9
11	0.343e-9	0.5365e-10

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