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SOLUTION OF THE STOKES PROBLEM AS AN INVERSE PROBLEM [1]

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Abstract — In this study, we examine one of the approaches to the investigation and solution of the Stokes equations: an original problem is regarded as an inverse problem and then reduced to a problem of optimal control. Iteration algorithms are suggested and results of numerical experiments are presented.

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1. Introduction

In a number of physical processes of the fluid mechanics, the internal and pressure forces are the main forces that affect the particle velocity. In some mathematical models of these processes, like Stokes or Navier–Stokes equations, these forces are stated in equations as right-hand-side functions and pressure gradient [11]. Usually we consider the solution as both the unknown function of the velocity vector $\mathbf{u}(x)$ and the an unknown pressure function $p(x)$. However, in the theory of inverse problems statements very often arise, in which, besides a "regular" solution (for the Stokes equations we understand it as a function of the velocity vector), the functions of sources or some of them are sought [1]. Such problems belong to the class of inverse problems. Thus, if we assume this interpretation, it is possible to consider (to some extent!) the Stokes problem, the Navier–Stokes problem, etc., as inverse problems, since, along with the velocity vector of a fluid, we are also looking for the pressure function of pressure that determines one of the "source functions". But then an assumption arises

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that, in investigating and solving such problems, many special algorithms and approaches, developed in the classic theory of inverse problems [1], may prove to be very useful. More than that, it may happen that, ignoring these special approaches while examining and, especially, solving numerically these problems, many difficulties can be encountered (such a situation is observed in a number of problems of the fluid mechanics).

The approaches of the optimal control theory [1–3, 9, 10, 12], in turn, appeared to be fruitful in examining and solving the inverse problems. It should be noted here that many problems in the control theory are essentially inverse problems.

In view of the above assumptions, the authors examine in this study the following approach to the investigation and solution of problems of the given class exemplified by a boundary value problem for the Stokes systems: the pressure function is regarded as an "additional" unknown to the "main" component of a problem's solution, and the continuity condition from the Stokes system is considered as one of the "observation conditions" ("observation data", etc.), which is prescribed to close the system. Then, the problem, which is considered as an inverse one, is included in the family of optimal control problems depending on the regularization parameter. In the following, the problems of optimal control are examined and solved by classical methods [8, 12]. At the final stage, we must show that for a trivial regularization parameter we obtain the original problem and corresponding results (possibly, with additional restrictions on the initial data of the problem).

The above approach is suggested here to be applied to one of the boundary value problems for the Stokes equations. Having formulated the initial problem, the authors reduce it to a problem of optimal control, in which the pressure function $p(x)$ is a "control", while the conditions $\operatorname{div}\mathbf{u} = 0$ and $\int_\Omega p(x)dx = 0$ belong to the "observation conditions" and are included in the cost functional. Later, having investigated the solvability of the optimal control problem, the authors, as a consequence of the statements obtained, arrive at the well-known results on the solvability of the Stokes problem. This is regarded in this study as a proof that numerical algorithms applicable to the solution of the optimal control problem, can be chosen as algorithms for the initial problem as well.

The authors consider some examples of iteration processes for the solution of the problems in question, justify them, and estimate their convergence rates. The realization of these processes consists of a sequence of classical elliptic problems; it can be carried out by the well proved numerical methods (including the simplest ones), such as the method of finite elements and the method of finite differences. It should be noted that in solving these problems, the condition $\operatorname{div}\mathbf{u} = 0$ is absent; as is known [17], this condition leads to considerable complications of algorithms, especially in multi-dimensional problems and in the case of a domain of complex form (in addition, they impose certain restrictions on the order of employed finite elements in the method of finite elements).

In conclusion, we present some of the results of the numerical experiments; in performing them, the authors used classical difference schemes.

2. Statement of the problem

Let $\Omega \subset \mathbf{R}^d$ be a bounded domain with a regular (say, Lipschitz continuous) boundary Γ, $x = (x_1, \ldots, x_d)$. Denote by $\mathbf{u} = (u_1(x), \ldots, u_d(x)),\ \ldots,\ \mathbf{f} = (f_1(x), \ldots, f_d(x))$ the vector-functions defined on $\overline{\Omega} = \Omega \cup \Gamma$, and by $p(x),\ \ldots,\ g(x)$ the scalar functions. Later on we use the well-known Hilbert spaces $\mathrm{L}_2 \equiv \mathrm{L}_2(\Omega)$, $\mathrm{H}^1_0 \equiv \mathrm{H}^1_0(\Omega)$, $\mathrm{H}^{-1} \equiv (\mathrm{H}^1_0)^*$, $\ldots,$ $\mathrm{L}_2/\mathbf{R} = \{p(x) :$

$p \in \mathrm{L}_2, \int\limits_\Omega p dx = 0\}$ (see, e.g., [13, 17]).

Let us consider the following "generalized" Stokes problem: with $\mathbf{f}(x)$ prescribed, find $\mathbf{u}(x)$ and $p(x)$ such that

$$\begin{aligned} -a\Delta\mathbf{u} + b\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \int\limits_\Omega p(x)dx &= 0, \\ u|_\Gamma &= 0, \end{aligned} \tag{1}$$

where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$, $\operatorname{div}\mathbf{u} = \sum_{i=1}^d \partial u_i/\partial x_i$, $\Delta = \operatorname{div}\nabla = \sum_{i=1}^d \partial^2/\partial x_i^2$, $a = \text{const} > 0$, $b = \text{const} \geqslant 0$. If $b = 0$ then (1) is the "classical" Stokes problem.

Let $p(x)$ belong to the number of "additional unknowns", while the equation "$\operatorname{div}\mathbf{u} = 0$" is regarded as one of the "observation conditions". Then (1) can be reformulated as follows: find the function $\mathbf{u}(x)$ and the "additional unknown" $p(x)$ such that

$$A\mathbf{u} = \mathbf{f} - \nabla p \quad \text{in} \quad \Omega \tag{2}$$

and the following "observation conditions" hold:

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \quad \int\limits_\Omega p(x)dx = 0. \tag{3}$$

Here,

$$A\mathbf{u} = -a\Delta\mathbf{u} + b\mathbf{u} \quad \text{in} \quad \Omega, \quad \mathbf{u}|_\Gamma = 0.$$

Later on we consider the operator A only for the boundary condition "$\mathbf{u}|_\Gamma = 0$"; therefore, the latter, "as a rule", will not be written in equations and relations in which A is present. The form of equation (2) is understood as follows: with $\mathbf{f} \in (\mathrm{H}^{-1})^d$, $p \in \mathrm{L}_2$ find $\mathbf{u} \in (\mathrm{H}_0^1)^d$ such that

$$(A\mathbf{u}, \mathbf{v}) \equiv (a\nabla\mathbf{u}, \nabla\mathbf{v}) + (b\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + (p, \operatorname{div}\mathbf{v})_{\mathrm{L}_2(\Omega)} \quad \forall\mathbf{v} \in (\mathrm{H}_0^1)^d, \tag{4}$$

where (hereafter) $(\mathbf{u}, \mathbf{v}) \equiv (\mathbf{u}, \mathbf{v})_{(\mathrm{L}_2)^d}$, $\|\mathbf{u}\| \equiv (\mathbf{u}, \mathbf{u})^{1/2}$.

We investigate the solvability of problem (2), (3) using the "optimal control approach". To this end this aim let us introduce a multitude of problems of optimal control depending on the numerical parameter $\alpha \geqslant 0$, which and are stated as follows: with $\mathbf{f} \in (\mathrm{H}^{-1})^d$ prescribed find $\mathbf{u} \in (\mathrm{H}_0^1)^d$ and an unknown "control" $p \in \mathrm{L}_2$ such that

$$A\mathbf{u} = \mathbf{f} - \nabla p, \tag{5}$$

$$\inf_{p \in \mathrm{H}^k} J_{\alpha,k}(p, \mathbf{u}), \tag{6}$$

where

$$J_\alpha(p, \mathbf{u}) = \alpha\|p\|_{\mathrm{H}^k}^2 + \|\operatorname{div}\mathbf{u}\|_{\mathrm{L}_2}^2 + \left(\int\limits_\Omega p dx\right)^2, \quad k = 0, 1$$

and, for the sake of simplicity, the same notation is retained for the solution of problem (5), (6) as for the solution of (2), (3)).

As we see, there is no additional restriction to $\mathbf{u}$, p in (5), (6)) (except for $\mathbf{u} \in (\mathrm{H}_0^1)^d$, $p \in \mathrm{L}_2$ or $p \in H^1$). We prove in the following the solvability of problem (5), (6) for $\forall\alpha \geqslant 0$ and show that for $\alpha = 0$ its solution yields a solution to problem (1).

3. Variational equations

Let us assume that $\mathbf{u}$ and p are a solution to problem (5), (6) at $k = 0$. Then, setting zero the first variation of the functional, we obtain the following equations ("variational equations"):

$$\begin{aligned} &A\mathbf{u} = \mathbf{f} - \nabla p, \\ &\alpha(p, \tilde{p})_{\mathrm{L}_2} + (\operatorname{div} \mathbf{u}, \operatorname{div} \tilde{\mathbf{u}})_{\mathrm{L}_2} + \left(\int_\Omega p dx \right) \left(\int_\Omega \tilde{p} dx \right) = 0, \quad \forall \tilde{p} \in \mathrm{L}_2, \end{aligned} \tag{7}$$

where the function $\tilde{\mathbf{u}} \in (\mathrm{H}_0^1)^d$ is defined as a solution of the equation

$$A\tilde{\mathbf{u}} = -\nabla \tilde{p}. \tag{8}$$

If $\mathbf{f} \in (\mathrm{H}^{-1})^d$, $p \in \mathrm{L}_2$ and $\tilde{p} \in \mathrm{L}_2$, then the functions $\mathbf{u}$, $\tilde{\mathbf{u}}$ can be represented as

$$\mathbf{u} = G_0 \mathbf{f} - G_0 \nabla p, \quad \tilde{\mathbf{u}} = -G_0 \nabla \tilde{p}, \tag{9}$$

where $G_0 : (\mathrm{H}^{-1})^d \to (\mathrm{H}_0^1)^d$, $G_0 \nabla : \mathrm{L}_2 \to (\mathrm{H}_0^1)^d$ are linear bounded "resolution" operators of equations (7), (8). Substituting (9) to the second equation of system (7), we obtain the variational equation for p:

$$a_\alpha(p, \tilde{p}) = g(\tilde{p}) \quad \forall \tilde{p} \in \mathrm{L}_2, \tag{10}$$

where

$$\begin{aligned} a_\alpha(p, \tilde{p}) &\equiv \alpha(p, p)_{\mathrm{L}_2} + (\operatorname{div} G_0 \nabla p, \operatorname{div} G_0 \nabla \tilde{p}) + \left(\int_\Omega p dx \right) \left(\int_\Omega \tilde{p} dx \right), \\ g(\tilde{p}) &\equiv \int_\Omega (\operatorname{div} G_0 \mathbf{f})(\operatorname{div} G_0 \nabla \tilde{p}) dx. \end{aligned}$$

The following operator equation in L_2 can be obtained from (10):

$$\mathcal{A}_\alpha p = g, \tag{11}$$

where

$$\begin{aligned} (\mathcal{A}_\alpha p, \tilde{p})_{\mathrm{L}_2} &\equiv a_\alpha(p, \tilde{p}), & (g, \tilde{p})_{\mathrm{L}_2} &= g(\tilde{p}) \quad \forall p, \tilde{p} \in \mathrm{L}_2, \\ \mathcal{A}_\alpha &= \alpha I + \mathcal{A}_0, & \mathcal{A}_0 &= B^* \cdot B, \\ B^* = B &= -\operatorname{div} G_0 \nabla : \mathrm{L}_2 \to \mathrm{L}_2, & g &= -B \operatorname{div} G_0 \mathbf{f} \end{aligned}$$

and I is the identity operator.

We present another form of variational equations. To do this, we introduce the "adjoint" problem: with given $\mathbf{u}$ find $\mathbf{q}$ such that

$$A\mathbf{q} = -\nabla \operatorname{div} \mathbf{u}. \tag{12}$$

It is easy to check then that

$$(\operatorname{div} \mathbf{u}, \operatorname{div} \tilde{\mathbf{u}})_{\mathrm{L}_2} = (\operatorname{div} \mathbf{q}, \tilde{p})_{\mathrm{L}_2}. \tag{13}$$

From (7) and (12) we obtain the following form for the variational equations:

$$\begin{aligned} A\mathbf{u} &= \mathbf{f} - \nabla p, \\ A\mathbf{q} &= -\nabla \operatorname{div} \mathbf{u}, \\ \alpha p + \operatorname{div} \mathbf{q} + \int_\Omega p dx &= 0 \qquad \text{in} \quad \Omega. \end{aligned} \tag{14}$$

If $k = 1,\ \alpha > 0$, then the system of variational equations is given by

$$\begin{aligned} A\mathbf{u} &= \mathbf{f} - \nabla p, \\ A\mathbf{q} &= -\nabla \operatorname{div} \mathbf{u}, \\ \alpha A_1 p + \operatorname{div} \mathbf{q} + \int_\Omega p dx &= 0 \qquad \text{in} \quad \Omega, \end{aligned} \tag{15}$$

where $A_1 p \equiv -\Delta p + p$ in Ω, $\alpha \partial p / \partial n = 0$ on . The boundary value problems (14), (15) and their generalized statements coincide at $\alpha = 0$.

Our immediate goal is to examine the solvability of the above variational equations; then we consider the iteration algorithms for solving these equations, which are actually the algorithms for solving the initial problem (1).

4. Existence of solutions

Assume that $k = 0$. We note the evident features of the operator $\mathcal{A}_\alpha$ and of the form $a_\alpha(\cdot,\cdot)$, namely: their symmetricity, nonnegativeness for $\alpha \geqslant 0$, boundedness, and positive definiteness for $\alpha > 0$. They are also positive for $\alpha = 0$. In fact, if $\mathcal{A}_\alpha p = 0$ or $a_\alpha(p,p) = 0$ for some p, then $\mathbf{u} = -G_0 \nabla p$ is a solution to the problem

$$A\mathbf{u} = -\nabla p, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega, \qquad \int_\Omega p dx = 0.$$

But, this problem has the only trivial solution $\mathbf{u} = 0,\ p = 0$. Thus, $\operatorname{Ker}(\mathcal{A}_\alpha) = \{0\}\ \forall \alpha \geqslant 0$.

However, as a matter of fact, the operator $\mathcal{A}_\alpha$ and the form $a_\alpha(\cdot,\cdot)$ are positive definite $\forall \alpha \geqslant 0$. Let us prove this statement.

Let $\lambda_j^{(D)},\ j = 1, 2, \ldots$ be the eigenvalue of the following problem

$$-\Delta_0 \Phi_j = \lambda_j^{(D)} \Phi_j, \quad \|\Phi_j\| = 1,$$

where $\Delta_0 \Phi_j = \Delta \Phi_j$ in Ω and $\Phi_j \equiv 0$ on , while $\mu_j,\ j = 1, 2, \ldots$ are the eigenvalues of the operator "$\operatorname{div} \Delta_0^{-1} \nabla$". It is known that $\{\mu_j\}$ are positive and only depend on Ω only [7]; besides, $\max_j \mu_j \leqslant 1$ [5].

Lemma 1. *The operator $\mathcal{A}_0$ is positive definite and bounded on $L_2/\mathbf{R}$:*

$$\left(\frac{\lambda_1^{(D)} \mu_1}{a \lambda_1^{(D)} + b} \right)^2 \leqslant \frac{(\mathcal{A}_0 p, p)}{(p,p)} \leqslant \frac{1}{a^2}, \quad \forall (p \not\equiv 0) \in L_2/\mathbf{R}. \tag{16}$$

Proof. . Let $p \in \mathrm{L}_2/\mathbf{R}$. Then, according to the statements from [6,8], the operator B is positive definite on $\mathrm{L}_2/\mathbf{R} : (Bp,p)_{\mathrm{L}_2} \geqslant \gamma^2 \|p\|_{\mathrm{L}_2}$, $\gamma = \text{const} > 0$.

The system $\{\Phi_j\}$ is a basis in L_2, therefore

$$\begin{aligned}(A^{-1}\nabla p, \nabla p) &= \sum_{j=1}^{\infty} \frac{|(\nabla p, \Phi_j)|^2}{a\lambda_j^{(D)} + b} \geqslant \frac{\lambda_1^{(D)}}{a\lambda_1^{(D)} + b} \sum_{j=1}^{\infty} \frac{|(\nabla p, \Phi_j)|^2}{\lambda_j^{(D)}} = \\ &= (-\Delta_0^{-1}\nabla p, \nabla p) \frac{\lambda_1^{(D)}}{a\lambda_1^{(D)} + b} \geqslant \frac{\lambda_1^{(D)}\mu_1}{a\lambda_1^{(D)} + b}(p,p), \\ (A^{-1}\nabla p, \nabla p) &\leqslant \frac{1}{a}(-\Delta_0^{-1}\nabla p, \nabla p) \leqslant \frac{1}{a}(p,p)\end{aligned}$$

and we have (16). □

Note also that the boundedness of the operators B and A_0 can easily be established by virtue of the known results on the solvability of elliptic boundary value problems.

Corollary 1. *For $\alpha \geqslant 0$, the operator $\mathcal{A}_\alpha$ and the form $a_\alpha(\cdot,\cdot)$ are L_2-bounded and L_2-definite:*

$$(\alpha + C_1)\|p\|^2_{L_2} \leqslant (\mathcal{A}_\alpha p, p)_{L_2} = a_\alpha(p,p) \leqslant (\alpha + C_2)\|p\|^2_{L_2}, \tag{17}$$

where the positive constants C_1 and C_2 are independent of α and $p \in L_2$,

$$C_1 = \left(\frac{\lambda_1^{(D)}\mu_1}{a\lambda_1^{(D)} + b}\right)^2, \qquad C_2 = \frac{1}{a^2}.$$

Corollary 2. *If $\|\mathbf{f}\|_{(H^{-1})^d} < \infty$, then equation (10) has a unique solution $p \in L_2/\mathbf{R}$ for $\alpha \geqslant 0$; in addition,*

$$2\alpha\|p\|^2_{L_2} + C_1\|p\|^2_{L_2} \leqslant C_3\|\mathbf{f}\|^2_{(H^{-1})^d},$$

where the constant C_3 is independent of α.

(The proof of Corollary 2 follows from the results on the minimum problem of quadratic functionals, the estimate $\|\operatorname{div} G_0\mathbf{f}\|_{\mathrm{L}_2} \leqslant C\|\mathbf{f}\|_{(\mathrm{H}^{-1})^d}$, and the correlation $\|\operatorname{div} G_0\nabla p\|^2_{\mathrm{L}_2} \geqslant C_1\|p\|^2_{\mathrm{L}_2}$ $\forall p \in \mathrm{L}_2/\mathbf{R}$).

We now prove the following statement.

Lemma 2. *If $\tilde{\mathbf{u}} = -G_0\nabla\tilde{p}$ $\forall\tilde{p} \in L_2/\mathbf{R}$ (or $\forall\tilde{p} \in L_2$), then a space spanned by $div\,\tilde{\mathbf{u}}$ is dense in $L_2/\mathbf{R}$.*

Proof. Let us consider $F \in \mathrm{L}_2/\mathbf{R}$ such that $(F, \operatorname{div}\tilde{\mathbf{u}})_{\mathrm{L}_2} = 0$ $\forall\tilde{\mathbf{u}}$. We associate with F the solution to the following equation: $A\mathbf{q} = -\nabla F$. Then $0 = (F, \operatorname{div}\tilde{\mathbf{u}})_{\mathrm{L}_2} = (-\nabla F, \tilde{\mathbf{u}})_{\mathrm{L}_2} = (\operatorname{div}\mathbf{q}, \tilde{p})_{\mathrm{L}_2}$ $\forall\tilde{p} \in \mathrm{L}_2/\mathbf{R}$ and $\operatorname{div}\tilde{\mathbf{q}} = 0$. But the equations $A\mathbf{q} = -\nabla F$ and $\operatorname{div}\mathbf{q} = 0$ have in Ω only the trivial solution $\mathbf{q}$: $(A\mathbf{q}, \mathbf{q}) = (-\nabla F, \mathbf{q}) = (F, \operatorname{div}\mathbf{q})_{\mathrm{L}_2} = 0$, $\mathbf{q} = 0$. Therefore, $F = 0$, which completes the proof. □

Corollary 3. *Let the functions $\mathbf{u}_0 \equiv G_0(\mathbf{f} - \nabla p_0)$ for $\mathbf{f} \in (H^{-1})^d$, $p_0 \in L_2$ satisfy the relation*

$$(div\,\mathbf{u}_0, div\,\tilde{\mathbf{u}})_{L_2} + \left(\int_\Omega p_0 dx\right)\left(\int_\Omega \tilde{p} dx\right) = 0, \quad \forall\tilde{p} \in L_2,$$

where $\tilde{\mathbf{u}} = -G_0\nabla\tilde{p}$. *Then the equalities* $div\,\mathbf{u}_0 = 0$ *in* Ω *and* $\int\limits_\Omega p_0 dx = 0$ *take place where necessary.*

(Indeed, by virtue of Lemma 2 for $\tilde{p} \in \mathrm{L}_2/\mathbf{R}$ we obtain $\mathrm{div}\,\mathbf{u_0} = 0$. But then $\int\limits_\Omega p_0 dx = 0$).

We now formulate the main statements on the solvability of the problems under consideration, which for $\alpha = 0$ lead to the well-known results on the existence of a generalized solution to problem (1).

Theorem 1. *Let* $\mathbf{f} \in (H^{-1})^d$. *Then:*

(i) System (14) *has a unique solution* $(\mathbf{u}, p, \mathbf{q}) \in (\overset{\circ}{H}{}^1_0)^d \times L_2/\mathbf{R} \times (\overset{\circ}{H}{}^1_0)^d$ *for all* $\alpha > 0$ *and the following estimates are valid:*

$$\|\mathbf{u}\|_{(H_0^1)^d} + \|p\|_{L_2} \leqslant C\|\mathbf{f}\|_{(H^{-1})^d}, \tag{18}$$

$$\|\mathbf{q}\|_{(H_0^1)^d} + \|div\,\mathbf{u}\|_{L_2} \leqslant C\alpha\|\mathbf{f}\|_{(H^{-1})^d}, \tag{19}$$

where the constant C *is independent of* α.

(ii) For $\alpha = 0$, *system* (14) *has a unique solution* $\mathbf{u}_0 \in (H_0^1)^d$, $p_0 \in L_2/\mathbf{R}$, $\mathbf{q}_0 \equiv 0$; *what is more,* $div\,\mathbf{u}_0 = 0$ *in* Ω, *and* $\mathbf{u}_0$, p_0 *are generalized solutions of problem* (1) *and estimate* (18) *holds for them.*

(iii) For $\alpha \to 0$, *a strong convergence* $u \equiv u(\alpha) \to u_0$, $p = p(\alpha) \to p_0$ *takes place; in addition,*

$$\|u - u_0\|_{(H_0^1)^d} + \|p - p_0\|_{L_2} \leqslant C\alpha\|\mathbf{f}\|_{(H^{-1})^d}, \tag{20}$$

where $C = const$ *is independent of* α.

Proof. Let p be a solution to equation (10), which, as mentioned above, exists and is unique for all $\alpha \geqslant 0$; in addition, $\|p\|_{\mathrm{L}_2} \leqslant C\|\mathbf{f}\|_{(\mathrm{H}^{-1})^d}$. Let us determine the functions $\mathbf{u}$ and $\mathbf{q}$ as follows: $\mathbf{u} \equiv G_0\mathbf{f} - G_0\nabla p$, $\mathbf{q} \equiv -G_0\nabla\mathrm{div}\,\mathbf{u}$. We note that $\mathbf{u} \in (\mathrm{H}_0^1)^d$, $\mathbf{q} \in (\mathrm{H}_0^1)^d$ and they satisfy system (14). Also, we note that the following estimates are valid:

$$\|\mathbf{u}\|_{(\mathrm{H}_0^1)^d} \leqslant (C\|\mathbf{f}\|_{(\mathrm{H}^{-1})^d} + \|p\|_{\mathrm{L}_2}), \quad \|\mathbf{q}\|_{(\mathrm{H}_0^1)^d} \leqslant C\|\mathbf{u}\|_{(\mathrm{H}_0^1)^d}.$$

By integrating the third equation from (14) over Ω, we obtain

$$\alpha\int\limits_\Omega p dx + \int\limits_\Gamma \mathbf{nq} d\Gamma + \mathrm{mes}(\Omega)\int\limits_\Omega p dx = (\alpha + \mathrm{mes}(\Omega))\int\limits_\Omega p dx = 0$$

($\mathbf{n}$ is the unit vector of the external normal to Γ). Consequently, for $\alpha \geqslant 0$ we have $\int\limits_\Omega p dx = 0$ and $p \in \mathrm{L}_2/\mathbf{R}$.

Let us prove (19). From the third equation of (14) we obtain

$$\|\mathrm{div}\,\mathbf{q}\|_{\mathrm{L}_2} = \alpha\|p\|_{\mathrm{L}_2} \leqslant C\alpha\|\mathbf{f}\|_{(\mathrm{H}^{-1})^d}$$

since $\int\limits_\Omega p dx = 0$. Then we have from the second equation of (14):

$$\mathbf{q} = -A^{-1}\nabla\mathrm{div}\,\mathbf{u}, \quad \mathrm{div}\,\mathbf{q} = -\mathrm{div}\,A^{-1}\nabla\mathrm{div}\,\mathbf{u},$$
$$\|\mathrm{div}\,\mathbf{q}\|_{\mathrm{L}_2} = \|\mathrm{div}\,A^{-1}\nabla(\mathrm{div}\,\mathbf{u})\|_{\mathrm{L}_2} \geqslant C_1^{1/2}\|\mathrm{div}\,\mathbf{u})\|_{\mathrm{L}_2}.$$

Consequently, $\|\operatorname{div}\mathbf{u}\|_{\mathrm{L}_2} \leqslant C\alpha\|\mathbf{f}\|_{(\mathrm{H}^{-1})^d}$. But then we immediately obtain the inequalities $\|\mathbf{q}\|_{(\mathrm{H}_0^1)^d} \leqslant C\|\operatorname{div}\mathbf{u}\|_{\mathrm{L}_2} \leqslant C\alpha\|\mathbf{f}\|_{(\mathrm{H}^{-1})^d}$ as well. Estimate (19) is thus established.

Let $\alpha = 0$. Then the statements on the solvability of (10) formulated above remains valid, and the functions p_0, $\mathbf{u}_0 \equiv G_0\mathbf{f} - G_0\nabla\mathbf{p}_0$, and $\mathbf{q}_0 \equiv -G_0\nabla\operatorname{div}\mathbf{u}_0$ satisfy system (14), which takes in this case the form

$$\begin{aligned} A\mathbf{u}_0 &= \mathbf{f} - \nabla p_0, \\ A\mathbf{q}_0 &= -\nabla\operatorname{div}\mathbf{u}_0, \\ \operatorname{div}\mathbf{q}_0 &+ \int_\Omega p_0 dx = 0. \end{aligned} \tag{21}$$

It follows from the last equation of (21) that $\int_\Omega p_0 dx = 0$ and $\operatorname{div}\mathbf{q}_0 = 0$ in Ω. Make the inner product by multiplying the second equation of (21) by $\mathbf{q}_0$ in $(\mathrm{L}_2)^d$; we obtain $(A\mathbf{q}_0, \mathbf{q}_0) = (\operatorname{div}\mathbf{u}_0, \operatorname{div}\mathbf{q}_0) = 0$. Consequently, $\mathbf{q}_0 \equiv 0$; but then $\operatorname{div}\mathbf{u}_0 = 0$ in Ω. Thus, for $\alpha = 0$ the functions $\mathbf{u}_0$ and p_0 are a generalized solution to problem (1), and the well-known estimate (18) is valid for these functions.

Let us estimate the convergence rate of p to p_0 as $\alpha \to 0$. From (10) we obtain the following correlations:

$$\begin{aligned} &\alpha(p, \tilde{p})_{\mathrm{L}_2} + (\operatorname{div} G_0\nabla(p - p_0), \operatorname{div} G_0\nabla\tilde{p})_{\mathrm{L}_2} = 0, \quad \forall\tilde{p} \in \mathrm{L}_2, \\ &C_1\|p - p_0\|^2_{\mathrm{L}_2} \leqslant \|\operatorname{div} G_0\nabla(p - p_0)\|^2_{\mathrm{L}_2} \leqslant \alpha\|p\|_{\mathrm{L}_2}\|p - p_0\|_{\mathrm{L}_2} \leqslant C\alpha\|\mathbf{f}\|_{(\mathrm{H}^{-1})^d}\|p - p_0\|_{\mathrm{L}_2}, \\ &\|p - p_0\|_{\mathrm{L}_2} \leqslant C\alpha\|\mathbf{f}\|_{(\mathrm{H}^{-1})^d}/C_1. \end{aligned}$$

Since $(\mathbf{u} - \mathbf{u}_0)$ is the solution of the equation $A(\mathbf{u} - \mathbf{u}_0) = -\nabla(p - p_0)$, then we conclude that estimate (20) is correct. (We note that $\operatorname{div}\mathbf{u}_0 = 0$ in Ω, therefore estimate (19) is easily established from estimate (20), which can also be regarded as one of the consequences of (20).)

Thus, all the statements have been established. □

Remark 1. Proving Theorem 1, we did not use the classical results on the solvability of the Stokes problem. We tried to obtain exactly these results on the basis of the variational equation (10), to which we had reduced the investigation of the optimal control problem (5), (6). So, we can consider a certain algorithm applied to (5), (6) as the algorithm for solving (1). Actually, Theorem 1 presents a justification of such a procedure.

Remark 2. If we assume $\int_\Omega p dx = 0$ in (14) and eliminate p and q, then we obtain the following equation for $\mathbf{u}$:

$$A\mathbf{u} + \frac{1}{\alpha}\nabla\operatorname{div} G_0\nabla\operatorname{div}\mathbf{u} = \mathbf{f}. \tag{22}$$

Let us present, for comparison, the "penalty approximation" of the Stokes equations:

$$A\mathbf{u} - \frac{1}{\alpha}\nabla\operatorname{div}\mathbf{u} = \mathbf{f}, \tag{23}$$

which is used in the approximate solution of the Stokes problems [17].

Let us now consider system (15), where the problem for p is also understood in generalized form. Let, for example, the boundary Γ be sufficiently smooth or convex. Then, if $\mathbf{f} \in (\mathrm{L}_2)^d$ it is possible to establish, repeating the above statements, the existence of solutions $\mathbf{u} \subset (\mathrm{H}^2 \cap \mathrm{H}_0^1)^d$, $\mathbf{q} \subset (\mathrm{H}^2 \cap \mathrm{H}_0^1)^d$, and $p \in \mathrm{H}^1$ (and find corresponding estimates similar to those presented above) for $\alpha \geqslant 0$. However, if $\mathbf{f} \in (\mathrm{H}^{-1})^d$, one would expect that $\mathbf{u}$ and $\mathbf{q}$ are classed with $(\mathrm{H}_0^1)^d$ for - $\alpha \geqslant 0$, while $p \in \mathrm{H}^1$ for $\alpha > 0$ and $p \in \mathrm{L}_2$ for $\alpha = 0$.

The generalized statement of the problem for p can be written as follows: find $p \in H^1$ such that

$$a_\alpha^{(1)}(p, \tilde{p}) = g(\tilde{p}), \quad \forall\, \tilde{p} \in H^1, \tag{24}$$

where

$$a_\alpha^{(1)}(p, \tilde{p}) = \alpha(p, \tilde{p})_{\mathrm{H}^1} + \operatorname{div} G_0 \nabla p, \operatorname{div} G_0 \nabla \tilde{p}) + \left(\int\limits_\Omega p dx\right)\left(\int\limits_\Omega \tilde{p} dx\right).$$

Problem (24) has a unique solution $p \in H^1 \ \ \forall \alpha > 0$ and $\int\limits_\Omega p dx = 0$, i.e. $p \in H^1/\mathbf{R}$. If $\alpha = 0$, then $p = p_0$, where p_0 is the solution of (1) and $p_0 \in H^1$ as $\mathbf{f} \in (L_2)^d$.

Let us evaluate some estimates. Let $0 = \lambda_0^{(N)} < \lambda_1^{(N)} \leqslant \ldots$ be an eigenvalue of the following eigenvalue problem

$$-\Delta \psi_j = \lambda_j^{(N)} \psi_j \ \text{ in } \ \Omega, \quad \frac{\partial \psi_j}{\partial n} = 0 \text{ on } \Gamma, \quad \|\psi_j\| = 1. \tag{25}$$

The system of eigenfunctions $\{\psi_j\}$ is a basis in $L_2/\mathbf{R}$ and $H^1/\mathbf{R}$. We consider the following norm $\|p\|_1 \equiv (\nabla p, \nabla p)^{1/2} = \left(\sum\limits_{j=1}^{\infty} \lambda_j^{(N)} (p, \psi_j)^2\right)^{1/2}$ which is equivalent to $\|\cdot\|_{\mathrm{H}^1}$ in $H^1/\mathbf{R}$.

Lemma 3. . *The operator $\mathcal{A}_\alpha : (H^1/\mathbf{R}) \to (H^1/\mathbf{R})$ and the form $a_\alpha(\cdot,\cdot)$ are H^1-bounded and H^1- elliptic on $H^1/\mathbf{R}$, $\quad \forall \alpha > 0$:*

$$\alpha \|p\|_1^2 \leqslant (\mathcal{A}_\alpha p, p) = a_\alpha(p, p) \leqslant \left(\alpha\left(1 + \frac{1}{\lambda_1^{(N)}}\right) + \frac{1}{a(a\lambda_1^{(D)} + b)}\right)\|p\|_1^2, \tag{26}$$

i.e.

$$Sp(\mathcal{A}_\alpha) \in \left[\alpha, \alpha\left(1 + \frac{1}{\lambda_1^{(N)}}\right) + \frac{1}{a(a\lambda_1^{(D)} + b)}\right]. \tag{27}$$

Proof. Since $\|\nabla p\|^2 \geqslant \lambda_1^{(N)} \|p\|^2, \quad \forall p \in H^1/\mathbf{R}$, then

$$(\mathcal{A}_0 p, p) = (B \cdot B^{1/2} p, B^{1/2} p) \leqslant \frac{1}{a}\|B^{1/2} p\|^2 \leqslant \frac{1}{a(a\lambda_1^{(D)} + b)}\|\nabla p\|^2,$$

$$\alpha \|p\|_1^2 \leqslant \alpha \|p\|^2_{\mathrm{H}^1} + (\mathcal{A}_0 p, p) \leqslant \left(\alpha\left(1 + \frac{1}{\lambda_1^{(N)}}\right) + \frac{1}{a(a\lambda_1^{(D)} + b)}\right)\|p\|_1^2$$

and (27) is valid. □

Theorem 2. *Suppose that* $\mathbf{f} \in (L_2)^d$ *and* Γ *is smooth or convex; then:*

(i) System (15) *has a unique solution* $(\mathbf{u}, p, \mathbf{q}) \in (H^2 \cap H_0^1)^d \times (H^1/\mathbf{R}) \times (H^2 \cap H_0^1)^d$ $\forall \alpha > 0$ *and* $\mathbf{u} = \mathbf{u}_0$, $p = p_0$, $\mathbf{q} = \mathbf{q}_0 = 0$ *as* $\alpha = 0$, *where* $(\mathbf{u_0}, p_0, \mathbf{q_0})$ *is a solution of* (1).

(ii) For $\alpha \to 0$ *the convergence* $\mathbf{u} \to \mathbf{u_0}$, $p \to p_0$ *takes place and the following estimates hold:*

$$\|p - p_0\|_{L_2} \leqslant (\alpha/(2C_1))^{1/2} K_1, \qquad \|\mathbf{u} - \mathbf{u}_0\|_{(H^1)^d} \leqslant (\alpha/(2aC_1))^{1/2} K_1, \tag{28}$$

where C_1 *is the constant from* (17) *and* K_1 *is a constant such that* $\|p_0\|_{H^1} \leqslant K_1 < \infty$.

(iii) In addition, if div$\mathbf{f} \in L_2$, *then*

$$\begin{aligned} \|p - p_0\|_{H^1} &\leqslant (\alpha/C_1)(\|\mathit{div}\,\mathbf{f}\|_{L_2} + K_0), \\ \|\mathbf{u} - \mathbf{u_0}\|_{(H^2)^d} &\leqslant (\alpha/(aC_1))(\|\mathit{div}\,\mathbf{f}\|_{L_2} + K_0), \end{aligned} \tag{29}$$

where K_0 *is a constant such that* $\|p_0\|_{L_2} \leqslant K_0 < \infty$.

Proof. Consider the following equalities:

$$\begin{aligned} \alpha(p, \tilde{p})_{\mathrm{H}^1} + (\mathcal{A}_0 p, \tilde{p})_{L_2} &= (g, \tilde{p})_{L_2}, \quad (\mathcal{A}_0 p_0, \tilde{p})_{L_2} = (g, \tilde{p})_{\mathrm{L}_2}, \\ \alpha(p, \tilde{p})_{\mathrm{H}^1} + (\mathcal{A}_0(p - p_0), \tilde{p})_{L_2} &= 0, \quad \forall \tilde{p} \in H^1, \\ \alpha(p, p)_{\mathrm{H}^1} + (\mathcal{A}_0(p - p_0), p - p_0)_{L_2} &= \alpha(p, p_0)_{\mathrm{H}^1}, \\ \alpha(p - p_0, p - p_0)_{\mathrm{H}^1} + (\mathcal{A}_0(p - p_0), p - p_0)_{L_2} &= -\alpha(p_0, p - p_0)_{\mathrm{H}^1}. \end{aligned}$$

From the first of them we have: $\|p\|_{H^1} \leqslant \|p_0\|_{H^1} \leqslant K_1, \quad \forall \alpha \geqslant 0$. Then

$$\begin{aligned} \alpha\|p\|^2_{\mathrm{H}^1} + C_1\|p - p_0\|^2_{L_2} &\leqslant \alpha\|p\|_{\mathrm{H}^1}\|p_0\|_{\mathrm{H}^1}, \\ \|p - p_0\|_{L_2} &\leqslant (\alpha/(2C_1))^{1/2}\|p_0\|_{\mathrm{H}^1} \end{aligned}$$

and (with the use of equations (1))

$$\|\mathbf{u} - \mathbf{u}_0\|_{(H^1)^d} \leqslant (\alpha/(2aC_1))^{1/2}\|p_0\|_{\mathrm{H}^1}.$$

We now consider the following equations:

$$\alpha\tilde{p} + \tilde{\mathcal{A}}_0\tilde{p} = \tilde{g}, \quad \tilde{\mathcal{A}}_0\tilde{p}_0 = \tilde{g}, \quad \tilde{p} - \tilde{p}_0 = -\alpha(\alpha\tilde{\mathcal{A}}_0^{-1} + I)^{-1}\tilde{\mathcal{A}}_0^{-1}\tilde{p}_0,$$

where $\tilde{p} = A_1^{1/2}p$, $\quad \tilde{p}_0 = A_1^{1/2}p_0$, $\quad \tilde{g} = A_1^{-1/2}g$, $\quad \tilde{\mathcal{A}}_0 = A_1^{-1/2}\mathcal{A}_0 A_1^{-1/2}$. From the last equality we obtain:

$$\begin{aligned} \|p - p_0\|_{\mathrm{H}^1} = \|\tilde{p} - \tilde{p}_0\|_{\mathrm{L}_2} &\leqslant \alpha\|\tilde{\mathcal{A}}_0^{-1}\tilde{p}_0\|_{L_2} \\ &\leqslant (\alpha/C_1)\|\tilde{p}_0\|_{\mathrm{H}^1} = (\alpha/C_1)\|A_1 p_0\|_{\mathrm{L}_2} = (\alpha/C_1)\|p_0 - \Delta p_0\|_{\mathrm{L}_2}. \end{aligned}$$

Since $\quad \Delta p_0 = \operatorname{div}\mathbf{f}$, then

$$\|p - p_0\|_{\mathrm{H}^1} \leqslant (\alpha/C_1)\|p_0 - \operatorname{div}\mathbf{f}\|_{\mathrm{L}_2} \leqslant (\alpha/C_1)(\|p_0\|_{\mathrm{L}_2} + \|\operatorname{div}\mathbf{f}\|_{\mathrm{L}_2}) \leqslant (\alpha/C_1)(K_0 + \|\operatorname{div}\mathbf{f}\|_{\mathrm{L}_2}).$$

From these correlations and equations (1) we also obtain the second estimate from (29). □

Corollary 4. *If* $b = 0, \mathbf{f} \in (L_2)^d, \operatorname{div}\mathbf{f} \in L_2$ *then*

$$\begin{aligned} \|p - p_0\|_{H^1} &\leqslant \frac{\alpha a^2}{\mu_1^2}(\|\mathit{div}\,\mathbf{f}\|_{L_2} + \frac{Ca}{\mu_1^2}\|\mathbf{f}\|_{(H^{-1})^d}), \\ \|\mathbf{u} - \mathbf{u}_0\|_{(H^2)^d} &\leqslant \frac{\alpha a^2}{\mu_1^2}(\|\mathit{div}\,\mathbf{f}\|_{L_2} + \frac{Ca}{\mu_1^2}\|\mathbf{f}\|_{(H^{-1})^d}), \end{aligned}$$

where the constant C *does not depend on* α *and* a.

(The proof of this corollary follows from (29) and from the estimate: $K_0 \leqslant C\|\mathbf{f}\|_{(H^{-1})^d}/(aC_1)$).

5. On the numerical solution of the inverse problems

The transition from problem (1) to problem (5), (6) allows the formulation of the iteration algorithms for its solutions. These algorithms imply the solution of "conventional elliptic equations" at each step, using the well-known numerical methods for which the "$\operatorname{div}\mathbf{u} = 0$" condition does not have to be satisfied. For example, one can use the simplest finite elements, which are especially important in dealing with multi-dimensional problems and domains with a curvilinear boundary (where, as is known, there are certain difficulties with the construction of finite elements satisfying the condition "$\operatorname{div}\mathbf{u} = 0$"). What is more, the finite difference schemes can also be successfully applied to the realization of these algorithms.

The idea of the construction of algorithms for an approximate solution of equation (1) implies the application of the well-known iteration methods to equation (11) and subsequent formulation of the steps in terms of the operators of system (14).

Method 1. Consider the following iteration process applied to (14):

$$p^{k+1} = p^k - \tau_k(\mathcal{A}_\alpha p^k - g), \quad k = 0, 1, 2. \dots \quad . \tag{30}$$

This process can be realized as follows:

Step 0 (computing g): Solve problems to $\mathbf{u}_0, \mathbf{q}_0$ and find g :

$$A\mathbf{u}_0 = \mathbf{f}, \quad A\mathbf{q}_0 = -\nabla\operatorname{div}\mathbf{u}_0, \quad g = -\operatorname{div}\mathbf{q}_0. \tag{31}$$

If we regard p^k as already found, the further computations will consist in the following:

Step 1: Consider the problems

$$A\mathbf{u}^k = -\nabla p^k, \quad A\mathbf{q}^k = -\nabla\operatorname{div}\mathbf{u}^k \tag{32}$$

and find

$$w^k = \alpha p^k + \operatorname{div}\mathbf{q}^k + \int_\Omega p^k dx. \tag{33}$$

(Note that if $p^k \in \mathrm{L}_2/\mathbf{R}$, then $w^k \in \mathrm{L}_2/\mathbf{R}$).

Step 2: Find τ_k; if the optimization of τ_k is carried out, then, possibly, an auxiliary problem should be solved; if we take

$$\tau_k = \tau = \frac{2}{2\alpha + C_2 + C_1} = \frac{2a^2}{2\alpha a^2 + 1 + \mu_1^2/(1 + b/(a\lambda_1^{(D)}))^2}, \tag{34}$$

where $C_1,\ C_2$ are the constants from (17), then we obtain "the simplest optimal algorithm" and τ is not recalculated. Choosing τ, we find p^{k+1}:

$$p^{k+1} = p^k - \tau(w^k - g). \tag{35}$$

(Note again that if $p^k \in \mathrm{L}_2/\mathbf{R}$, then $p^{k+1} \in \mathrm{L}_2/\mathbf{R}$, since g and w_k are from $\mathrm{L}_2/\mathbf{R}$).

After that, we return to **Step 1** with the new approximation p^{k+1}. The process continues until a proper termination criterion is attained.

It is known from the theory of iteration processes [14] that, if τ_k is chosen in the form of (34), then $p^k \to p$ as $k \to \infty$ with the rate

$$\|p^k - p\|_{\mathrm{L}_2} \leqslant \mathbf{C}((1-\xi)/(1+\xi))^k \to 0, \quad k \to \infty, \tag{36}$$

where the constant $\mathbf{C}$ is independent of k and

$$\xi = (\alpha + C_1)/(\alpha + C_2), \qquad (1-\xi)/(1+\xi) = (C_2 - C_1)/(2\alpha + C_2 + C_1).$$

Remark 3. If $b = 0$, then

$$C_1 = \left(\frac{\mu_1}{a}\right)^2, \quad C_2 = \frac{1}{a^2}, \quad \tau = \frac{2a^2}{2\alpha a^2 + 1 + \mu_1^2}, \quad \xi = \frac{\alpha a^2 + \mu_1^2}{\alpha a^2 + 1}.$$

For $\alpha = 0$ we have: $\tau = 2a^2/(1+\mu_1^2)$, $\xi = \mu_1^2$, i.e. the convergence of (30) does not depend on a. Note that in the case of $d = 2$, $\Omega = \{(x_1, x_2) : 0 < x_i < L_i,\ i = 1, 2\}$, $l = \max(L_1/L_2, L_2/L_1)$, the estimate of μ_1 is known [5]: $\mu_1 \geqslant (304 l^2)^{-1}$. In this case we have: $\tau \cong 2a^2$, $\xi \cong (304 l^2)^{-2}$.

Remark 4. If $b = 1/\Delta t$ $(\Delta t \to +0)$, then

$$\tau \cong (2a^2)(2\alpha a^2 + 1 + (\mu_1 a \lambda_1^{(D)} \Delta t)^2)^{-1}, \quad \xi \cong a^2(\alpha + (\lambda_1^{(D)} \mu_1 \Delta t)^2)(1 + \alpha a^2)^{-1}.$$

Thus, if $\alpha \to +0$ and $\Delta t \to +0$, then the convergence of (30) is slow.

Note also that in the equations presented (here and before) the terms $\int_\Omega p^k dx$ can be omitted. But then, during the realization of iteration processes, "compulsory" orthogonalization of computed $\{p^k\}$ to unity should be carried out (the property that is lost in real numerical computations) in order to keep these functions in $\mathrm{L}_2/\mathbf{R}$.

Method 2. We present one of the simple iteration algorithms based on the method of minimal residuals; when applied to (11), it has the form:

$$\begin{aligned} p^0 &= 0, & \xi^0 &= -g, \\ p^{k+1} &= p^k - \tau_k \xi^k, & \xi^{k+1} &= \xi^k - \tau_k w^k, \end{aligned} \tag{37}$$

$$w^k = A_\alpha \xi^k, \quad \tau_k = (w^k, \xi^k)_{\mathrm{L}_2} / \|w^k\|^2_{\mathrm{L}_2}, \quad k = 0, 1, 2, \dots \ .$$

If g has already been computed, the realization of (37) is carried out as follows.

Step 1:

$$\begin{aligned} A\mathbf{u}^k &= -\nabla \xi^k, \quad A\mathbf{q}^k = -\nabla \mathrm{div}\, \mathbf{u}^k, \\ w^k &= \alpha \xi^k + \mathrm{div}\, \mathbf{q}^k + \int_\Omega \xi^k dx. \end{aligned} \tag{38}$$

Step 2:

$$\begin{aligned} \tau_k &= \frac{(w^k, \xi^k)_{\mathrm{L}_2}}{\|w^k\|^2_{\mathrm{L}_2}}, \\ \xi^{k+1} &= \xi^k - \tau_k w^k, \quad p^{k+1} = p^k - \tau_k \xi^k. \end{aligned} \tag{39}$$

Then we go back to **Step 1** with the new approximation p^{k+1}.

The operator $\mathcal{A}_\alpha$ is symmetric and positive definite; therefore, the estimate of the asymptotic convergence rate is also determined by expression (36).

Method 3. Let us consider (24) as the equation in H^1. Then it can be rewritten as follows:

$$\mathcal{A}_\alpha^{(1)} p \equiv \alpha p + A_1^{-1} \mathcal{A}_0 p = A_1^{-1} g. \tag{40}$$

Consider the following iterative process for this equation

$$p^{k+1} = p^k - \tau(\mathcal{A}_\alpha^{(1)} p - A_1^{-1} g), \quad k = 0, 1, 2, \dots \tag{41}$$

The steps of (41) are as follows:

Step 0: We solve (31), find $g = -\text{div}\,\mathbf{q_0}$ and solve

$$-\Delta g_1 + g_1 = g \text{ in } \Omega, \quad \frac{\partial g_1}{\partial n} = 0 \text{ on } \Gamma. \tag{42}$$

As a result, we have $g_1 \equiv A^{-1}g$.

For given g_1 and p_k we calculate p^{k+1}.

Step 1. Solve the problems

$$\begin{gathered} A\mathbf{u}^k = -\nabla p^k, \quad A\mathbf{q}^k = -\nabla \text{div}\,\mathbf{u}^k, \\ -\Delta w_1^k + w_1^k = \text{div}\,\mathbf{q}^k + \int\limits_\Omega p^k dx \text{ in } \Omega, \quad \frac{\partial w_1^k}{\partial n} = 0 \text{ on } \Gamma \end{gathered} \tag{43}$$

and find

$$w^k = \alpha p^k + w_1^k. \tag{44}$$

Step 2. We take

$$\tau = \frac{2}{\alpha(2 + 1/\lambda_1^{(N)}) + a^{-1}(a\lambda_1^{(D)} + b)^{-1}} \tag{45}$$

and calculate

$$p^{k+1} = p^k - \tau(w^k - g_1). \tag{46}$$

Then we return to **Step 1** with the new approximation p^{k+1}.

If τ is calculated by (45), then according to the theory of iterative processes, the following estimate holds:

$$\|p^k - p\|_{\mathrm{H}^1} \leqslant C((1-\xi)/(1+\xi))^k, \tag{47}$$

where

$$\xi = \frac{\alpha}{\alpha(1 + 1/\lambda_1^{(N)}) + a^{-1}(a\lambda_1^{(D)} + b)^{-1}} \tag{48}$$

and the constant C is independent of k.

Remark 5. If $b \to +0$ and $\alpha \to +0$, then the convergence rate of (41) can be slow. If $b = 1/\Delta t$ ($\Delta t \to +0$), then $\xi \cong \alpha/(\alpha(1 + 1/\lambda_1^{(N)}) + \Delta t/a)$ and $\xi \cong 1/(2 + 1/\lambda_1^{(N)})$ as $\alpha = \Delta t/a$. If $\alpha = \Delta t/a$ is sufficiently small, then, according to (28), (29) we also have an approximation to the solution of (1), but the convergence rate is independent of α and (practically) of a.

6. Results of numerical experiments

Let the following values be taken in problem (1): $d = 2$, $\Omega = \{-\pi/2 < x \equiv x_1 < 3/2\pi,\ -\pi < x_2 \equiv y < \pi\}$, $\mathbf{u} \equiv (u_1, u_2) \equiv (u, v)$, and the function $\mathbf{f} = (f_1, f_2)$ have the components:

$$\begin{aligned} f_1(x, y) &= (2a + b)\sin x \sin y + (a + b)\sin y + \cos x \cos(2y), \\ f_2(x, y) &= (2a + b)\cos x \cos y + (a + b)\cos x - 2\sin x \sin(2y). \end{aligned} \tag{49}$$

The exact solution of the problem is of the form: $u(x, y) = (1 + \sin x)\sin y$, $v(x, y) = \cos x(1 + \cos y)$, $p(x, y) = \sin x \cdot \cos(2y)$. To solve the problem, algorithm (31)–(35) is used both without optimizing the parameter τ and with optimizing it.

In the experiments presented below, the elliptic problems from (31)–(35) were approximated with finite differences on uniform grids; the derivatives were approximated by the sequential application of the following approximations of the derivatives of the function of one variable $f(x)$:

$$\begin{aligned} f'(x_i) &= \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + \mathrm{O}(h^2), \\ f''(x_i) &= \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{h^2} + \mathrm{O}(h^2), \\ f'(x_0) &= \frac{-f(x_0) + 4f(x_1) - 3f(x_2)}{2h} + \mathrm{O}(h^2), \\ f'(x_N) &= \frac{3f(x_N) - 4f(x_{N-1}) + f(x_{N-2})}{2h} + \mathrm{O}(h^2). \end{aligned}$$

The computations were carried out for a different number of iterations and the accuracy obtained was investigated. The condition $\|p^{k+1} - p^k\| > \varepsilon$ while $\|p^{N_{it}+1} - p^{N_{it}}\| \leqslant \varepsilon$ for a prescribed ε (as well as its combinations with other criteria) can be used as one of the termination criteria for the iteration processes. The difference analogs of the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathrm{L}_2}$: $\|\cdot\|_{\mathrm{inf}}$ and $\|\cdot\|_{\mathrm{H}_2}$, were taken for the norm $\|\cdot\|$.

Experiment 1. Let us calculate the numerical solution of (1) by Method 1. Note that for the chosen cost functional at $k = 0$ the "regularizing influence" of the parameter α would not be expected (at least for coefficients $a \sim b \sim 1$), which was confirmed by the calculations. Besides, it may be expected that the accuracy of the values of $p(x, y)$, at the boundary grid points will be lower than the accuracy in the internal grid points (see below the tables with a mark "Variant 1").

Therefore, in order to refine the boundary values of $p(x, y)$ computed at each iteration, we used the following approach at each iteration ("the interpolation from internal values"): in the internal grid points, $p(x_i, y_i)$ were computed by formula (35), and then the boundary values were calculated by the simplest interpolations of the type of $f(x_0) \cong f(x_1)$, $f(x_0) \cong 2f(x_1) - f(x_2)$, where x_0 is a boundary point (see below the tables with a mark "Modified Variant 1", where the interpolation $f(x_0) \cong 2f(x_1) - f(x_2)$ was used). For the numerical solution of the presented test problem we took $\alpha = 0$ and $\tau = 0.05$.

The computational results for the case of (49) and for a number of other cases are presented below in tables and figures.

The numerical results presented in Tables 1-4 point to the accuracy of the finite difference approximations. These results have been derived for nonoptimal τ (i.e. we did not have to take into account the number of iterations in these calculations). The results presented in Figures (for Methods 1–3) demonstrate the dependence of number of iteration for some parameters of the problem.

Note that the accuracy of the numerical velocity vector in all Experiments 1–3 is similar to the accuracy presented in Tables, while this characteristic for $p(x, y)$ is strongly dependent on α and b: it is not sufficient for α tending to 1 or large b.

Table 1. Method : "Method 1 (Variant 1)"; the grid : $N \times N = 50 \times 50$; $a = 1, b = 1, \alpha = 0., \tau = 0.05$; the first number - $\|\cdot\|_{inf}$ on all nodes of the grid, the second one - $\|\cdot\|_{inf}$ on internal nodes, the third number - $\|\cdot\|_{H_2}$

Iterations	$\|div\ \mathbf{u}\|$	$\|u^k - u\|$	$\|v^k - v\|$	$\|p^k - p\|$
100	0.1294	0.0244	0.0326	0.5145
	0.10497	0.0244	0.0326	0.4067
	0.1768	0.0525	0.0724	0.2632
200	0.08355	0.0146	0.0175	0.4999
	0.07675	0.0146	0.0175	0.3168
	0.0810	0.0319	0.0339	0.2732
500	0.04999	0.00592	0.00627	0.4999
	0.03516	0.00592	0.00627	0.13025
	0.0256	0.01528	0.01266	0.12775
1000	0.0176	0.02698	0.00208	0.49989
	0.00873	0.02698	0.00208	0.0345
	0.00783	0.00687	0.0061	0.0765
2000	0.00316	0.00205	0.00159	0.49987
	0.00152	0.00205	0.00159	0.0197
	0.00118	0.0051	0.00452	0.0710

Table 2. Method : "Method 1 (Modified Variant 1)"; grid : $N \times N = 50 \times 50$; $a = 1, b = 1, \alpha = 0., \tau = 0.05$; first number - $\|\cdot\|_{inf}$ on all nodes of the grid, second number - $\|\cdot\|_{inf}$ on internal nodes, third number - $\|\cdot\|_{H_2}$

Iterations	$\|div\ \mathbf{u}\|$	$\|u^k - u\|$	$\|v^k - v\|$	$\|p^k - p\|$
100	0.1196	0.0240	0.0320	0.5408
	0.1045	0.0240	0.0320	0.414
	0.172	0.0513	0.0711	0.452
200	0.0854	0.0141	0.0169	0.49993
	0.0498	0.0141	0.0169	0.3292
	0.1041	0.0309	0.0330	0.2631
500	0.0363	0.00566	0.00606	0.2580
	0.0158	0.00566	0.00606	0.156
	0.0299	0.0147	0.0122	0.107
1000	0.0125	0.00262	0.00193	0.060
	0.0050	0.00262	0.00193	0.051
	0.0097	0.0066	0.00587	0.0428
2000	0.0040	0.0020	0.0016	0.0307
	0.0026	0.0020	0.0016	0.0237
	0.0039	0.0051	0.00452	0.0380

Table 3. Method : "Method 1 (Variant 1)"; grid: $N \times N = 100 \times 100$; $a = 1, b = 1, \alpha = 0., \tau = 0.05$; first number - $\| \cdot \|_{inf}$ on all nodes of the grid, second number - $\| \cdot \|_{inf}$ on internal nodes, third number - $\| \cdot \|_{H_2}$

Iterations	$\|div\ \mathbf{u}\|$	$\|u^k - u\|$	$\|v^k - v\|$	$\|p^k - p\|$
	0.1272	0.02449	0.0319	0.542
100	0.1142	0.02449	0.0319	0.498
	0.1721	0.0522	0.0702	0.4628
	0.0889	0.0147	0.0172	0.5305
200	0.0490	0.0147	0.0172	0.445
	0.0784	0.0314	0.0326	0.275
	0.0658	0.00624	0.00632	0.49998
500	0.0545	0.00624	0.00632	0.2457
	0.0259	0.0141	0.0119	0.123
	0.02838	0.00189	0.00198	0.49998
1000	0.0176	0.00189	0.00198	0.06866
	0.00838	0.00468	0.00444	0.0506
	0.0050	0.00056	0.00041	0.4999
2000	0.00227	0.00056	0.00041	0.0184
	0.0011	0.00137	0.00121	0.0331

Table 4. Method : "Method 1 (Modified Variant 1)"; grid : $N \times N = 100 \times 100$; $a = 1, b = 1, \alpha = 0., \tau = 0.05$; first number - $\| \cdot \|_{inf}$ on all nodes of the grid, second number - $\| \cdot \|_{inf}$ on internal nodes, third number - $\| \cdot \|_{H_2}$

Iterations	$\|div\ \mathbf{u}\|$	$\|u^k - u\|$	$\|v^k - v\|$	$\|p^k - p\|$
	0.125	0.0244	0.0318	0.6015
100	0.114	0.0244	0.0318	0.5
	0.1706	0.0520	0.0701	0.462
	0.0855	0.0146	0.0171	0.627
200	0.0842	0.0146	0.0171	0.448
	0.0780	0.0313	0.0325	0.275
	0.0554	0.00622	0.00631	0.4418
500	0.0536	0.00622	0.00631	0.2565
	0.0256	0.00141	0.0119	0.120
	0.0207	0.0018	0.00196	0.161
1000	0.0181	0.00186	0.00l96	0.0874
	0.0081	0.00463	0.00440	0.0383
	0.00261	0.00055	0.00041	0.01399
2000	0.00211	0.00055	0.00041	0.01205
	0.00089	0.00135	0.00119	0.00834

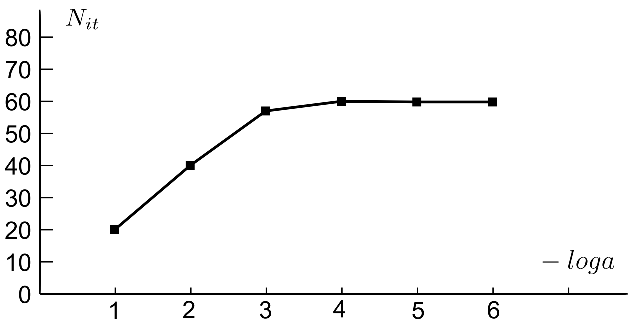

Figure 1. Number of iterations N_{it} in Method 1 (Modified Variant 1) as a function of α; $\tau = \tau_{opt}$, $a = 1$, $b = 1$, $\|P^{k+1} - P^k\|_{inf} \geqslant \varepsilon = 0.0005$, $N \times N = 50 \times 50$

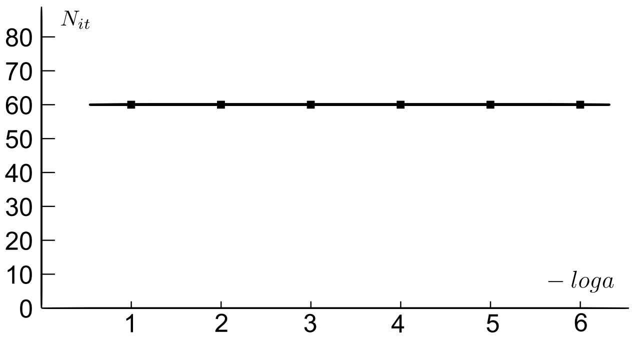

Figure 2. Number of iterations N_{it} in Method 1 (Modified Variant 1) as a function of a; $\tau = \tau_{opt}$, $\alpha = 0$, $b = 0$, $\|P^{k+1} - P^k\|_{inf} \geqslant \varepsilon = 0.0005$

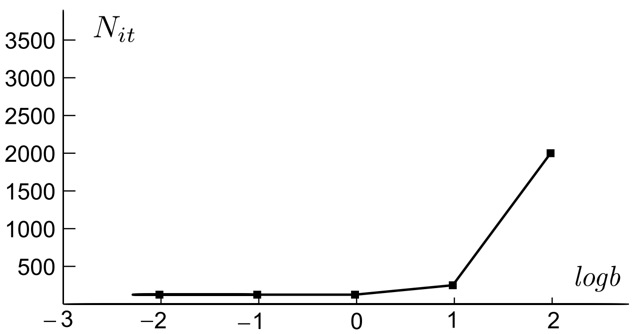

Figure 3. Number of iterations N_{it} in Method 1 (Modified Variant 1) as a function of b; $\tau = \tau_{opt}, a = 1$, $\alpha = 0.0001$, $\|P^{k+1} - P^k\|_{inf} \geqslant \varepsilon = 0.0005$, $N \times N = 50 \times 50$

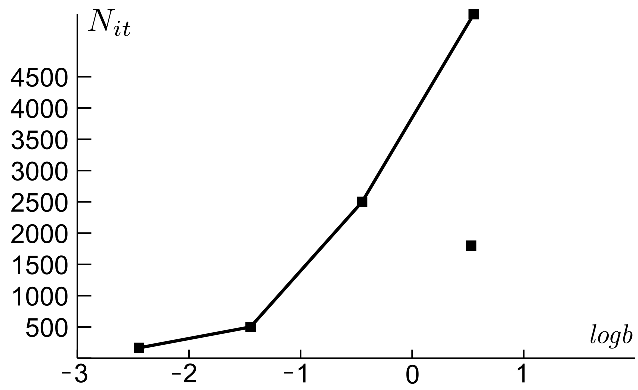

Figure 4. Number of iterations N_{it} in Method 1 (Modified Variant 1) as a function of b; $\tau = \tau_{opt}$, $a = 0.01$, $\alpha = 0.0001$, $\|P^{k+1} - P^k\|_{inf} \geqslant \varepsilon = 0.0005$

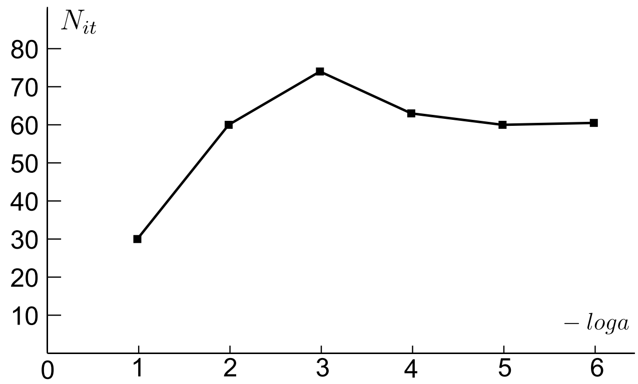

Figure 5. Number of iterations N_{it} in Method 2 as a function of α; $a = 1$, $b = 1$, $\|P^{k+1} - P^k\|_{inf} \geqslant \varepsilon = 0.0005$, $N \times N = 50 \times 50$

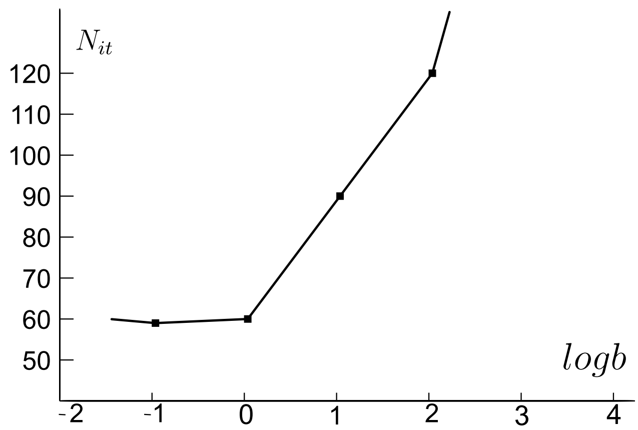

Figure 6. Number of iterations N_{it} in Method 2 as a function of b; $\alpha = 0.0001$, $a = 1$, $\|P^{k+1} - P^k\|_{inf} \geqslant \varepsilon = 0.0005$

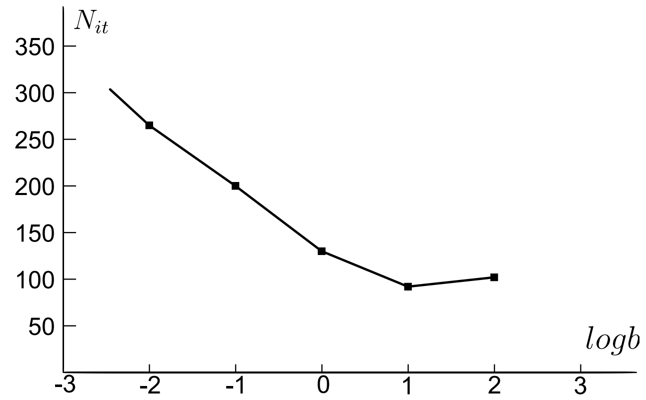

Figure 7. Number of iterations N_{it} in Method 3 as a function of b; $a = 1, \alpha = 0$, $\|P^{k+1} - P^k\|_{inf}/\|P^1 - P^0\|_{inf} \geqslant \varepsilon = 0.0005$, $N \times N = 50 \times 50$

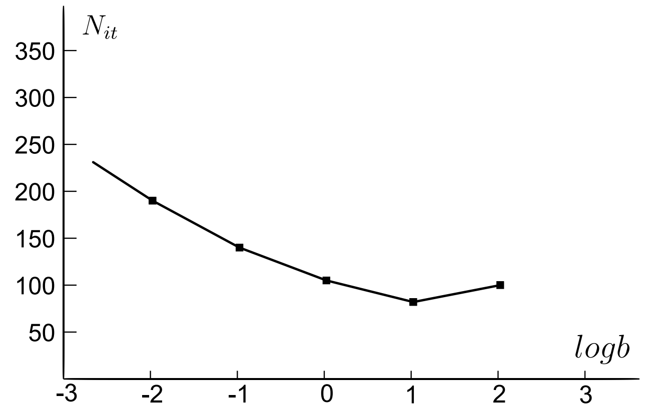

Figure 8. Number of iterations N_{it} in Method 3 as a function of b; $a = 1$, $\alpha = 0.001/(1 + b)$, $\|P^{k+1} - P^k\|_{inf}/\|P^1 - P^0\|_{inf} \geqslant \varepsilon = 0.0005$, $N \times N = 50 \times 50$

Conclusion

From the above considerations we can assume that for classical Stokes problem the iterative processes of types (31))-(35), (37) can be used. If we solve the generalized Stokes problem with large value of b, then the application of (41) can be reasonable.

The statements suggested in this study and results of numerical experiments allow the assumption that the approach considered here may be extended to other problems: the unsteady Stokes problem, linearized Navier–Stokes problem, and some others [4, 14–16].

It is easy to see that more effective iteration algorithms can be formulated and justified. It can be based, for example, on the iteration processes of conjugate gradients or on the generalized minimal residuals method.

A rich arsenal of effective numerical methods can be applied for the numerical solution of problems that arise at each step of a chosen iteration algorithm: the method of finite elements, the method of finite differences, spectral method, domain decomposition technique, and many others.

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