Central European Science Journals
www.cesj.com

Central European Journal of Mathematics
Central European Science Journals
CEJM 3(2) 2005 331–341

On Bochner flat para-Kählerian manifolds

Dorota Łuczyszyn*

Institute of Mathematics,
Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

Received 30 October 2004; accepted 27 December 2004

Abstract: Let B be the Bochner curvature tensor of a para-Kählerian manifold. It is proved that if the manifold is Bochner parallel ($\nabla B = 0$), then it is Bochner flat ($B = 0$) or locally symmetric ($\nabla R = 0$). Moreover, we define the notion of the paraholomorphic pseudosymmetry of a para-Kählerian manifold. We find necessary and sufficient conditions for a Bochner flat para-Kählerian manifold to be paraholomorphically pseudosymmetric. Especially, in the case when the Ricci operator is diagonalizable, a Bochner flat para-Kählerian manifold is paraholomorphically pseudosymmetric if and only if the Ricci operator has at most two eigenvalues. A class of examples of manifolds of this kind is presented.

Keywords: Para-Kählerian manifold, Bochner conformal curvature, Bochner flat manifold
MSC (2000): 53C15, 53C50, 53C56

1 Preliminaries

A triple (M, J, g) is said to be a para-Kählerian manifold ([4, 5]) if M is a connected differentiable manifold of dimension $n = 2m$, J is a $(1, 1)$-tensor field and g is a pseudo-Riemannian metric of neutral signature on M such that

$$J^2 = I, \qquad g(X, JY) + g(Y, JX) = 0, \qquad \nabla J = 0 \tag{1}$$

for any $X, Y \in \mathfrak{X}(M)$, where I is the identity tensor field, $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on M and ∇ is the Levi-Civita connection with respect to g.

Let (M, J, g) be a para-Kählerian manifold. In the sequel, $W, X, Y, Z, \ldots$ will denote arbitrary elements of $\mathfrak{X}(M)$ if it is not otherwise stated. Let $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ be the curvature operator and R the Riemann curvature tensor given by $R(X, Y, Z, W) =$

* E-mail: Dorota.Luczyszyn@pwr.wroc.pl

$g(R(X,Y)Z,W)$. The symbols Q, S, r will denote the Ricci operator, the Ricci curvature tensor and the scalar curvature, respectively, and we assume the following convention

$$\mathrm{Tr}\,\{Z \longmapsto R(Z,X)Y\} = S(X,Y) = g(QX,Y), \qquad r = \mathrm{Tr}\,Q.$$

For these tensor fields, we have the following famous consequences of the second Bianchi identity

$$\sum\nolimits_i \varepsilon_i(\nabla_{e_i}R)(X,Y,Z,e_i) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z), \tag{2}$$

where $(e_i; i = 1,\ldots,n)$ stands for an orthonormal frame and $\varepsilon_i = g(e_i,e_i) = \pm 1$.

As it is well-known (cf. e.g. [1]), the Riemannian and Ricci curvature tensors of a para-Kählerian manifold have additionally the following algebraic properties

$$\begin{gathered} R(JX,JY) = -R(X,Y), \quad R(X,Y)J = JR(X,Y), \\ S(JX,Y) = -S(JY,X), \quad S(JX,JY) = -S(X,Y), \quad QJ = JQ. \end{gathered} \tag{3}$$

For $X,Y \in \mathfrak{X}(M)$, define the operator $X \wedge Y$ acting on vector fields by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \quad Z \in \mathfrak{X}(M).$$

The *Bochner curvature operator* B is defined by (see [1, 18])

$$\begin{aligned} B(X,Y) = R(X,Y) - \frac{1}{n+4}&(X \wedge (QY) + (QX) \wedge Y - (JX) \wedge (QJY) \\ &- (QJX) \wedge (JY) + 2g(JX,Y)QJ + 2g(QJX,Y)J) \\ &+ \frac{r}{(n+4)(n+2)}(X \wedge Y - (JX) \wedge (JY) + 2g(JX,Y)J), \end{aligned} \tag{4}$$

and the Bochner curvature $(0,4)$-tensor is given by $B(X,Y,Z,W) = g(B(X,Y)Z,W)$.

The tensor B has the same symmetry and antisymmetry properties as the usual curvature tensor and moreover the following additional ones

$$\begin{gathered} B(JX,JY) + B(X,Y) = 0, \quad B(JX,Y) + B(X,JY) = 0, \\ \mathrm{Tr}\{Z \mapsto B(Z,X)Y\} = \mathrm{Tr}\{Z \mapsto B(JZ,X)Y\} = 0, \\ \mathrm{Tr}_g\{(W,Z) \mapsto B(W,JZ)\} = 0. \end{gathered} \tag{5}$$

The Bochner curvature of para-Kählerian manifolds has been studied by many authors [1, 2, 6-10, 17-20], etc. In these papers, geometric interpretations and applications of this tensor can be found.

A big inspiration for the author to study the Bochner curvature tensor on para-Kählerian manifolds comes from the fact that there are many interesting results concerning the Bochner curvature of Kählerian manifolds (see e.g. [3]; for general theory of Kählerian manifolds, see e.g. [9, 21]). Note that the defining conditions of a Kählerian manifold remain the same as (1) except the first one which should be replaced by $J^2 = -I$, and the metric g is positive definite. Therefore, the researches in para-Kählerian manifolds are not simple translations of the investigations of Kählerian manifolds. However, certain results look similar.

Here, I would like to point out only the papers [12, 13, 14] concerning Kählerian manifolds and which are strictly related to our investigations. Precisely, our Theorem 2.1 is a para-Kählerian analogy of Theorem 2 from [12]; Proposition 3.1 corresponds to the Proposition from [13], p. 221; and Theorems 3.2 and 3.3 are related to, respectively, Theorems 2 and 3 from [14].

Finally, I would like to turn reader's attention to the paper [3], which is very important since it provides an explicit local classification of Bochner-Kähler (Bochner flat Kählerian) manifolds.

2 Bochner parallelity

A para-Kählerian manifold is called *Bochner flat* if its Bochner curvature tensor vanishes identically [1, 20]. In the papers [2, 6, 7] a geometric meaning of the Bochner flatness can be found; such manifolds are called there to be isotropic para-Kählerian.

A para-Kählerian manifold with $\nabla B = 0$ will be called *Bochner parallel*. It is obvious that Bochner flat ($B = 0$) as well as locally symmetric ($\nabla R = 0$) para-Kählerian manifolds are Bochner parallel.

Bochner parallel para-Kählerian manifolds were studied by N. Pušić in [18], where it is proved that such manifolds are Bochner flat or the gradient of the scalar curvature is isotropic (i.e., $g(\operatorname{grad} r, \operatorname{grad} r) = 0$). Generalizing this result, we show that a non Bochner flat, Bochner parallel para-Kählerian manifold is necessarily locally symmetric.

In our investigations, we need the important formula obtained in [18, eq. (1.8)] for the covariant derivative ∇S of a Bochner parallel para-Kählerian manifold. Namely,

$$\begin{aligned}(\nabla_X S)(Y,Z) = \frac{1}{2(n+2)}&(g(X,Z)dr(Y) + g(X,Y)dr(Z) + 2g(Y,Z)dr(X) \\ &- g(X,JY)dr(JZ) - g(X,JZ)dr(JY)).\end{aligned} \tag{6}$$

Theorem 2.1. If a para-Kählerian manifold is Bochner parallel, then it is Bochner flat or locally symmetric.

Proof. Let (M, J, g) be a para-Kählerian manifold. By the Ricci identity, we have the following general formula

$$\begin{aligned}&\left(\nabla^2_{UV} B - \nabla^2_{VU} B\right)(X,Y,Z,W) \\ &= -B(R(U,V)X,Y,Z,W) - B(X,R(U,V)Y,Z,W) \\ &\quad - B(X,Y,R(U,V)Z,W) - B(X,Y,Z,R(U,V)W),\end{aligned} \tag{7}$$

where $\nabla^2_{UV} = \nabla_U \nabla_V - \nabla_{\nabla_U V}$ is the second covariant derivative.

Assume that (M, J, g) is additionally Bochner parallel. Then $\nabla B = 0$, and also

$\nabla^2_{UV}B = 0$. Consequently, by (7), it follows

$$\begin{aligned} &B(R(U,V)X,Y,Z,W) + B(X,R(U,V)Y,Z,W) \\ &\quad + B(X,Y,R(U,V)Z,W) + B(X,Y,Z,R(U,V)W) = 0. \end{aligned}$$

Differentiating the above covariantly, we obtain

$$\begin{aligned} &B((\nabla_T R)(U,V)X,Y,Z,W) + B(X,(\nabla_T R)(U,V)Y,Z,W) \\ &\quad + B(X,Y,(\nabla_T R)(U,V)Z,W) + B(X,Y,Z,(\nabla_T R)(U,V)W) = 0, \end{aligned}$$

which can be rewritten as follows

$$\begin{aligned} &\sum\nolimits_j \varepsilon_j((\nabla_T R)(e_j,X,V,U)B(e_j,Y,Z,W) + (\nabla_T R)(e_j,Y,V,U)B(X,e_j,Z,W) \\ &+ (\nabla_T R)(e_j,Z,V,U)B(X,Y,e_j,W) + (\nabla_T R)(e_j,W,V,U)B(X,Y,Z,e_j)) = 0. \quad (8) \end{aligned}$$

Note that in virtue of (2) and (6), we have

$$\begin{aligned} \sum\nolimits_i \varepsilon_i(\nabla_{e_i}R)(X,Y,Z,e_i) &= \frac{1}{2(n+2)}(-g(X,Z)dr(Y) + g(Y,Z)dr(X) \\ &- 2g(X,JY)dr(JZ) - 2g(X,JZ)dr(JY) + 2g(Y,JZ)dr(JX)). \quad (9) \end{aligned}$$

Taking the trace of the equality (8) with respect to the pair T, U (that is, substituting $T = U = e_i$ into (8), multiplying it by ε_i and summing over $i = 1, \ldots, n$) and applying relations (5) and (9), we get

$$\begin{aligned} &g(V,X)B(\operatorname{grad} r,Y,Z,W) + g(V,Y)B(X,\operatorname{grad} r,Z,W) \\ &\quad + g(V,Z)B(X,Y,\operatorname{grad} r,W) + g(V,W)B(X,Y,Z,\operatorname{grad} r) \\ &\quad - B(V,Y,Z,W)dr(X) - B(X,V,Z,W)dr(Y) \\ &\quad - B(X,Y,V,W)dr(Z) - B(X,Y,Z,V)dr(W) \\ &\quad - B(JV,Y,Z,W)dr(JX) - B(X,JV,Z,W)dr(JY) \\ &\quad - B(X,Y,JV,W)dr(JZ) - B(X,Y,Z,JV)dr(JW) \\ &\quad + g(V,JX)B(J\operatorname{grad} r,Y,Z,W) + g(V,JY)B(X,J\operatorname{grad} r,Z,W) \\ &\quad + g(V,JZ)B(X,Y,J\operatorname{grad} r,W) + g(V,JW)B(X,Y,Z,J\operatorname{grad} r) = 0. \quad (10) \end{aligned}$$

Taking the trace of (10) with respect to the pair V, X, and using (5) and the first Bianchi identity, we find

$$B(\operatorname{grad} r,X,Y,Z) = 0. \quad (11)$$

The equalities (10), (11) are in fact proved in [18, eq. (1.9), (1.12)] with applying the local coordinates convention. We included the proofs of them for completness only.

In virtue of (11) and (5), we rewrite the relation (10) as follows

$$\begin{aligned} &B(V,Y,Z,W)dr(X) + B(X,V,Z,W)dr(Y) \\ &\quad + B(X,Y,V,W)dr(Z) + B(X,Y,Z,V)dr(W) \\ &\quad - B(V,JY,Z,W)dr(JX) - B(JX,V,Z,W)dr(JY) \\ &\quad - B(X,Y,V,JW)dr(JZ) - B(X,Y,JZ,V)dr(JW) = 0. \quad (12) \end{aligned}$$

Summing up (12) cyclically with respect to V, Y, X and using (5) and the first Bianchi identity, we find

$$B(V,Y,Z,W)dr(X) + B(X,V,Z,W)dr(Y) = B(X,Y,Z,W)dr(V). \tag{13}$$

Interchanging the pairs X, Y and Z, W in (13), we have

$$B(X,Y,V,W)dr(Z) + B(X,Y,Z,V)dr(W) = B(X,Y,Z,W)dr(V). \tag{14}$$

As consequences of (13), (14) and (5), we also obtain

$$\begin{aligned} &B(V,JY,Z,W)dr(JX) + B(JX,V,Z,W)dr(JY) \\ &\qquad = B(JX,JY,Z,W)dr(V) = -B(X,Y,Z,W)dr(V), \qquad (15) \\ &B(X,Y,V,JW)dr(JZ) + B(X,Y,JZ,V)dr(JW) \\ &\qquad = B(X,Y,JZ,JW)dr(V) = -B(X,Y,Z,W)dr(V). \qquad (16) \end{aligned}$$

In virtue of (13) - (16), from (12), we get

$$B(X,Y,Z,W)dr(V) = 0. \tag{17}$$

Let the manifold (M, J, g) be additionally non-Bochner flat. Since B is parallel, B is non-zero at every point of M. Now, from (17) it follows that $dr = 0$, i.e. the scalar curvature is constant. Then by (6) the Ricci tensor S is parallel. Consequently, using (4), we see that $\nabla R = 0$, that is, the manifold is locally symmetric. □

Results concerning locally symmetric para-Kählerian manifolds one can find in [15, 16]; and (globally) symmetric para-Kählerian spaces are classified in [8]; we refer [5] for more details about classifications of such spaces.

3 Paraholomorphic pseudosymmetry

The usual pseudosymmetry conditions for para-Kählerian manifolds were studied by the author in [11]. Below, we will define the new notion of paraholomorphic pseudosymmetry.

Let (M, J, g) be a para-Kählerian manifold. Besides the curvature operators $R(U, V)$, we will consider also the operators $R^1(U, V)$ acting on $\mathfrak{X}(M)$, which are defined for any $U, V \in \mathfrak{X}(M)$ by

$$R^1(U,V) = U \wedge V - JU \wedge JV - 2g(U,JV)J. \tag{18}$$

For a $(0, k)$-tensor field K on M, define $(0, k+2)$-tensor fields $R \cdot K$ and $R^1 \cdot K$ by

$$\begin{aligned} (R \cdot K)(U,V,X_1,\dots,X_k) &= \left(\nabla^2_{UV}K - \nabla^2_{VU}K\right)(X_1,\dots,X_k) \\ &= -\sum_{a=1}^{k} K(X_1,\dots,R(U,V)X_a,\dots,X_k), \qquad (19) \\ (R^1 \cdot K)(U,V,X_1,\dots,X_k) &= -\sum_{a=1}^{k} K(X_1,\dots,R^1(U,V)X_a,\dots,X_k). \qquad (20) \end{aligned}$$

We say that a $(0,k)$-tensor field K is of paraholomorphically pseudosymmetric type if the condition $R\cdot K = f\,R^1\cdot K$ is fulfilled on M with a certain function f.

A para-Kählerian manifold will be called to be
(i) *paraholomorphically pseudosymmetric* if its Riemann curvature tensor R is of paraholomorphically pseudosymmetric type, i.e.,

$$R\cdot R = f\,R^1\cdot R, \quad f : M \to \mathbb{R}; \tag{21}$$

(ii) *paraholomorphically Ricci-pseudosymmetric* if its Ricci curvature tensor S is of paraholomorphically pseudosymmetric type, i.e.,

$$R\cdot S = f\,R^1\cdot S, \quad f : M \to \mathbb{R}. \tag{22}$$

Clearly, the paraholomorphic pseudosymmetry implies the paraholomorphic Ricci-pseudosymmetry, the semisymmetry ($R\cdot R = 0$) implies the paraholomorphic pseudosymmetry and the Ricci semisymmetry ($R\cdot S = 0$) implies the paraholomorphic Ricci-pseudosymmetry.

Before we state necessary and sufficient conditions for a Bochner flat para-Kählerian manifold to be paraholomorphically pseudosymmetric, we prove the following auxiliary result:

Proposition 3.1. For a Bochner flat para-Kählerian manifold of dimension $n = 2m \geqslant 4$,

$$\begin{aligned}(n+4)\Big(n\nabla^2_{XY}r - (\Delta r)g(X,Y)\Big)\\ = 2n(n+2)S(QX,Y) - 2nrS(X,Y) - 2((n+2)\operatorname{Tr}Q^2 - r^2)g(X,Y),\end{aligned} \tag{23}$$

$\Delta = \sum_i \varepsilon_i \nabla^2_{e_ie_i}$ being the Laplace operator.

Proof. We can apply the formula (6). Covariant differentiation of (6) gives

$$\begin{aligned}2(n+2)\left(\nabla^2_{WX}S\right)(Y,Z) = g(X,Z)\nabla^2_{WY}r + g(X,Y)\nabla^2_{WZ}r\\ + g(Y,JX)\nabla^2_{WJZ}r + g(Z,JX)\nabla^2_{WJY}r + 2g(Y,Z)\nabla^2_{WX}r.\end{aligned}$$

Antisymmetrizing the last identity with respect to W and X, we obtain

$$\begin{aligned}2(n+2)\left(\left(\nabla^2_{WX}S\right)(Y,Z) - \left(\nabla^2_{XW}S\right)(Y,Z)\right)\\ = g(X,Z)\nabla^2_{WY}r + g(X,Y)\nabla^2_{WZ}r - g(W,Z)\nabla^2_{XY}r\\ - g(W,Y)\nabla^2_{XZ}r + g(Y,JX)\nabla^2_{WJZ}r + g(Z,JX)\nabla^2_{WJY}r\\ - g(Y,JW)\nabla^2_{XJZ}r - g(Z,JW)\nabla^2_{XJY}r.\end{aligned} \tag{24}$$

Taking the trace of the above equality with respect to the pair W, Z, we find

$$\begin{aligned}2(n+2)\sum\nolimits_i \varepsilon_i\left(\left(\nabla^2_{e_iX}S\right)(Y,e_i) - \left(\nabla^2_{Xe_i}S\right)(Y,e_i)\right)\\ = \nabla^2_{JXJY}r + (\Delta r)g(X,Y) - (n-1)\nabla^2_{XY}r,\end{aligned} \tag{25}$$

where we used the symmetry of $\nabla^2_{XY} r$. On the other hand, by the Ricci identity, we have

$$\sum_i \varepsilon_i \left(\left(\nabla^2_{e_i X} S\right)(Y, e_i) - \left(\nabla^2_{X e_i} S\right)(Y, e_i) \right) = S(QX, Y) - \sum_i \varepsilon_i R(e_i, X, Y, Qe_i). \quad (26)$$

Moreover, by $B = 0$, (4) and (3), we find

$$(n+4) \sum_i \varepsilon_i R(e_i, X, Y, Qe_i) = \frac{n}{n+2} r S(X, Y) + 4S(QX, Y) + \left(TrQ^2 - \frac{r^2}{n+2} \right) g(X, Y).$$

Applying the last relation into (26), and next using the obtained identity into (25), we get

$$\begin{aligned} &(n+4)\left(\nabla^2_{JXJY} r + (\Delta r) g(X, Y) - (n-1) \nabla^2_{XY} r\right) \\ &= -2nrS(X, Y) + 2n(n+2) S(QX, Y) - 2\left((n+2)\operatorname{Tr} Q^2 - r^2\right) g(X, Y). \quad (27) \end{aligned}$$

Substituting, respectively, JX and JY instead of X and Y into (27), and adding the obtained relation to (27), we get with the help of (3)

$$\nabla^2_{JXJY} r = -\nabla^2_{XY} r. \quad (28)$$

Consequently, (27) turns into (23). □

Theorem 3.2. For a Bochner flat para-Kählerian manifold (M, J, g), the following conditions are equivalent:
(i) the manifold is paraholomorphically pseudosymmetric;
(ii) the manifold is paraholomorphically Ricci-pseudosymmetric;
(iii) for the Ricci curvature tensor S and a function $f : M \to \mathbb{R}$, it holds

$$\nabla^2_{XY} r - \frac{1}{n}(\Delta r) g(X, Y) = -2(n+2) f \left(S(X, Y) - \frac{r}{n} g(X, Y) \right); \quad (29)$$

(iv) for the Ricci operator Q and a function $f : M \to \mathbb{R}$, it holds

$$Q^2 + \left((n+4) f - \frac{r}{n+2} \right) Q - \frac{1}{n} \left(TrQ^2 - \frac{r^2}{n+2} + (n+4) r f \right) I = 0. \quad (30)$$

Proof. Suppose that (M, J, g) is a Bochner flat para-Kählerian manifold.

At first, we remark that in virtue of (4) and $B = 0$, the equivalence (i) ⇔ (ii) is rather obvious. Precisely, the pseudosymmetry conditions (21) and (22) hold simultanously with the same function f.

To prove the remaining equivalences, first recall that

$$(R \cdot S)(W, X, Y, Z) = (\nabla^2_{WX} S)(Y, Z) - (\nabla^2_{XW} S)(Y, Z).$$

Next, by applying the above relation, (24), (18) - (20) with $K = S$, we find the following auxiliary formula

$$\begin{aligned} &2(n+2)\left(R \cdot S - f R^1 \cdot S\right)(Y, X, Z, W) \\ &= -g(Y, W)\nabla^2_{ZX} r + g(X, W)\nabla^2_{ZY} r - g(Z, Y)\nabla^2_{WX} r + g(Z, X)\nabla^2_{WY} r \\ &\quad - g(Z, JY)\nabla^2_{JWX} r + g(Z, JX)\nabla^2_{JWY} r - g(W, JY)\nabla^2_{JZX} r \\ &\quad + g(W, JX)\nabla^2_{JZY} r + 2(n+2) f (g(Z, X) S(Y, W) - g(Y, Z) S(X, W) \\ &\quad + g(X, W) S(Y, Z) - g(Y, W) S(X, Z) + g(X, JZ) S(JY, W) \\ &\quad - g(Y, JZ) S(JX, W) + g(X, JW) S(JY, Z) - g(Y, JW) S(JX, Z)), \quad (31) \end{aligned}$$

f being an arbitrary function on M.

(ii) $\Leftrightarrow$ (iii) Assume that the manifold is paraholomorphically (Ricci-)pseudosymmetric. Contracting (31) with respect to X, Z and using (22), we get

$$(n-1)\nabla^2_{WY}r - \nabla^2_{JWJY}r - (\Delta r)g(W,Y) = -2(n+2)f(nS(W,Y) - rg(W,Y)),$$

which with help of (28) turns into (29). Conversely, having (29), equality (31) yields (22).

(iii) $\Leftrightarrow$ (iv) Having in mind (23), we check easily that (29) and (30) are equivalent. $\square$

Now, we are going to present the case of the Ricci operator is diagonalizable.

Theorem 3.3. Let (M, J, g) be an $n(=2m)$-dimensional Bochner flat para-Kählerian manifold. Assume that at every point of the manifold, the Ricci operator Q is diagonalizable and not proportional to the identity operator. Then (M, J, g) is paraholomorphically pseudosymmetric if and only if the Ricci operator has two different eigenvalues; and in this case if $\lambda \neq \mu$ are the eigenvalues of Q, then the characteristic function f is related to them by

$$2(m+1)(m+2)f + (m-p+1)\lambda + (p+1)\mu = 0. \tag{32}$$

Proof. Let us assume that the Ricci operator Q is diagonalizable at a point p of the para-Kählerian manifold (M, J, g). By the commuting rule $QJ = JQ$, there exits an adapted basis $(e_i, 1 \leqslant i \leqslant n = 2m)$ of the tangent space T_pM consisting eigenvectors of Q so that it holds

$$g(e_\alpha, e_\alpha) = -g(e_{\alpha'}, e_{\alpha'}) = 1, \quad , Je_\alpha = e_{\alpha'}, \quad Je_{\alpha'} = e_\alpha, \quad Qe_\alpha = \lambda_\alpha e_\alpha, \quad Qe_{\alpha'} = \lambda_\alpha e_{\alpha'}$$

for $1 \leqslant \alpha \leqslant m$, $\alpha' = \alpha + m$, and $g(e_i, e_j) = 0$ otherwise.

Assume that the manifold (M, J, g) is paraholomorphically pseudosymmetric. By the relation (30), there are at most two different eigenvalues of Q. Since Q is not proportional to the identity operator, there are exactly two eigenvalues of Q, say $\lambda \neq \mu$, λ being of multiplicity $2p$ $(1 \leqslant p \leqslant m-1)$ and μ of multiplicity $2m-2p$. By (30), the eigenvalues λ and μ must fulfil the equality

$$\lambda + \mu + (n+4)f - \frac{r}{n+2} = 0.$$

Hence, since $r = 2p\lambda + 2(m-p)\mu$, the relation (32) follows.

Conversely, assume that λ and μ are two different eigenavlues of Q, with λ being of multiplicity $2p$ $(1 \leqslant p \leqslant m-1)$ and μ of multiplicity $2m-2p$, and L_S is given by (32). Then also $r = 2p\lambda + 2(m-p)\mu$ and consequently

$$(n+4)f - \frac{r}{n+2} = -(\lambda + \mu).$$

Moreover, since $\mathrm{Tr}(Q^2) = 2p\lambda^2 + 2(m-p)\mu^2$, we have

$$\frac{1}{n}\Big(TrQ^2 - \frac{r^2}{n+2} + (n+4)rf\Big) = -\lambda\mu.$$

In view of the last two equalities, the equation (30) takes the following form

$$Q^2 - (\lambda + \mu)Q + \lambda\mu I = 0,$$

which shows that this equation is just fulfilled under our assumptions. Therefore, by Theorem 3.2, the manifold is paraholomorphically pseudosymmetric. □

Corollary 3.4. Any 4-dimensional Bochner flat para-Kählerian manifold with diagonalizable Ricci operator is paraholomorphically pseudosymmetric.

Proof. In this dimension, at every point of the manifold, the Ricci operator has one eigenvalue of multiplicity 4, or two different eigenvalues each of multiplicity 2. In the first case, the Ricci operator has the shape $Q = (r/4)I$, consequently $R \cdot S = 0$, which is just the semisymmetry condition. In the second case, by Theorem 3.3, the manifold is paraholomorphically Ricci-pseudosymmetric. □

4 Example

Let $(x^\alpha, x^{\alpha'}, z, t)$, $1 \leqslant \alpha \leqslant m$, $\alpha' = \alpha + m$, be the Cartesian coordinates in the Cartesian space $\mathbb{R}^{2m+2}$. Let (a, b) be an open interval and $h : (a, b) \to \mathbb{R}$ be a function such that $h'(t) \neq 0$ at every $t \in (a, b)$. Define a pseudo-Riemannian metric of signature $(m+1, m+1)$ on $M = \mathbb{R}^{2m+1} \times (a, b) \subset \mathbb{R}^{2m+2}$ by assuming

$$\begin{aligned} g_{\alpha\beta} &= e^{2h(t)}\delta_{\alpha\beta}, & g_{\alpha'\beta'} &= e^{2h(t)}\left(4h'^2(t)x_\alpha x_\beta - \delta_{\alpha\beta}\right), \\ g_{\alpha'(2m+1)} &= -2h'^2(t)e^{2h(t)}x_\alpha, & g_{(2m+1)(2m+1)} &= h'^2(t)e^{2h(t)}, \\ g_{(2m+2)(2m+2)} &= -e^{2h(t)}, & g_{ij} &= 0 \quad \text{otherwise.} \end{aligned}$$

Define also a $(1,1)$-tensor field J on M by assuming

$$\begin{aligned} J^\alpha_{\beta'} &= J^{\alpha'}_\beta = \delta^\alpha_\beta, & J^{2m+1}_\alpha &= 2x_\alpha, & J^{2m+2}_{\alpha'} &= -2h'(t)x_\alpha, \\ J^{2m+1}_{2m+2} &= 1/h'(t), & J^{2m+2}_{2m+1} &= h'(t), & J^i_j &= 0 \quad \text{otherwise.} \end{aligned}$$

By straighforward computations, one verifies that (J, g) is a para-Kählerian structure, which is paraholomorphically pseudosymmetric with $f = h''(t)e^{-2h(t)}$ as the structure function realizing (20). Moreover, it can also be checked that the structure (J, g) is Bochner flat if and only if function h fulfils the ordinary differential equation

$$h''' - 6h'h'' + 4h'^3 = 0.$$

The function

$$h(t) = -(1/2)\ln(t^2 + a), \quad a = \text{constant},$$

is a concrete function realizing this equation. In this case, the structure (J, g) is Bochner flat and paraholomorphically pseudosymmetric.

Acknowledgment

The author is greatly indebted to Prof. Zbigniew Olszak for his help during the preparation of this paper.

References

[1] C.L. Bejan: "The Bochner curvature tensor on a hyperbolic Kähler manifold", In: *Colloquia Mathematica Societatis Jànos Bolyai*, Vol. 56, Differential Geometry, Eger (Hungary), 1989, pp. 93–99,

[2] A. Bonome, R. Castro, E. García-Río, L. Hervella and R. Vázquez-Lorenzo: "On the paraholomorphic sectional curvature of almost para-Hermitian manifolds", *Houston J. Math.*, Vol. 24, (1998), pp. 277–300.

[3] R.L. Bryant: "Bochner-Kähler metrics", *J. Amer. Math. Soc.*, Vol. 14(3), (2001), pp. 623–715.

[4] V. Cruceanu, P. Fortuny and P.M. Gadea: "A survey on paracomplex geometry", *Rocky Mountain J. Math.*, Vol. 26, (1996), pp. 83–115.

[5] P.M. Gadea, V. Cruceanu and J. Muñoz Masqué: "Para-Hermitian and para-Kähler manifolds", *Quaderni Inst. Mat., Fac. Economia, Univ. Messina*, Vol. 1, (1995), pp. 72.

[6] G. Ganchev and A. Borisov: "Isotropic sections and curvature properties of hyperbolic Kaehlerian manifolds", *Publ. Inst. Math.*, Vol. 38, (1985), pp. 183–192.

[7] E. García-Río, L. Hervella and R. Vázquez-Lorenzo: "Curvature properties of para-Kähler manifolds", In: *New developments in differential geometry (Debrecen, 1994)*, Math. Appl., Vol. 350, Kluwer Acad. Publ., Dordrecht, 1996, pp. 193–200.

[8] S. Kaneyuki and M. Kozai: "Paracomplex structures and affine symmetric spaces", *Tokyo Math. J.*, Vol. 8, (1985), pp. 81–98.

[9] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry*, Vol. I, II, John Wiley & Sons, New York-London, 1963, 1969.

[10] D. Łuczyszyn: "On Bochner semisymmetric para-Kählerian manifolds", *Demonstr. Math.*, Vol. 34, (2001), pp. 933–942.

[11] D. Łuczyszyn: "On pseudosymmetric para-Kählerian manifolds", *Beiträge Alg. Geom.*, Vol. 44, (2003), pp. 551–558.

[12] M. Matsumoto and S. Tanno: "Kählerian spaces with parallel or vanishing Bochner curvature tensor", *Tensor N.S.*, Vol. 27, (1973), pp. 291–294.

[13] Z. Olszak: "Bochner flat Kählerian manifolds", In: *Differential Geometry, Banach Center Publication*, Vol. 12, PWN-Polish Scientific Publishers, Warsaw, 1984, pp. 219–223.

[14] Z. Olszak: "Bochner flat Kählerian manifolds with a certain condition on the Ricci tensor", *Simon Stevin*, Vol. 63, (1989), pp. 295–303.

[15] E.M. Patterson: "Riemann extensions which have Kähler metrics", *Proc. Roy. Soc. Edinburgh (A)*, Vol. 64, (1954), pp. 113–126.

[16] E.M. Patterson: "Symmetric Kähler spaces", *J. London Math. Soc.*, Vol. 30, (1955), pp. 286–291.

[17] N. Pušić: "On an invariant tensor of a conformal transformation of a hyperbolic Kaehlerian manifold", *Zb. Rad. Fil. Fak. Niš, Ser. Mat.*, Vol. 4, (1990), pp. 55–64.

[18] N. Pušić: "On *HB*-parallel hyperbolic Kaehlerian spaces", *Math. Balkanica N.S.*, Vol. 8, (1994), pp. 131–150.

[19] N. Pušić: "On *HB*-recurrent hyperbolic Kaehlerian spaces", *Publ. Inst. Math. (Beograd) N.S.*, Vol. 55, (1994), pp. 64–74.

[20] N. Pušić: "On *HB*-flat hyperbolic Kaehlerian spaces", *Mat. Vesnik*, Vol. 49, (1997), pp. 35–44.

[21] R.O. Wells: *Differential analysis on complex manifolds*, Graduate Texts in Mathematics, Vol. 65, Springer-Verlag, New York-Berlin, 1980.