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Realization of Primitive Branched Coverings over Closed Surfaces Following the Hurwitz Approach

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Abstract: Let V be a closed surface, $H \subseteq \pi_1(V)$ a subgroup of finite index ℓ and $\mathcal{D} = [A_1, \dots, A_m]$ a collection of partitions of a given number $d \geq 2$ with positive defect $v(\mathcal{D})$. *When does there exist a connected branched covering $f : W \to V$ of order d with branch data $\mathcal{D}$ and $f_\#(\pi_1(W)) = H$?*

It has been shown by geometric arguments [4] that, for $\ell = 1$ and a surface V different from the sphere and the projective plane, the corresponding branched covering exists (the data $\mathcal{D}$ is realizable) if and only if the data $\mathcal{D}$ fulfills the Hurwitz congruence $v(\mathcal{D}) \equiv 0 \mod 2$. In the case $\ell > 1$, the corresponding branched covering exists if and only if $v(\mathcal{D}) \equiv 0 \mod 2$, the number d/ℓ is an integer, and each partition $A_i \in \mathcal{D}$ splits into the union of ℓ partitions of the number d/ℓ. Here we give a purely algebraic proof of this result following the approach of Hurwitz [11].

The realization problem for the projective plane and $\ell = 1$ has been solved in [7,8]. The case of the sphere is treated in [1, 2, 12, 7].

Keywords: covering, branched covering of surfaces, branching order, Hurwitz problem, representations to the symmetry groups Σ_d

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1 Introduction

Branched coverings of surfaces have been studied by Hurwitz since 1891, see [11]. He gave a result for the existence of branched coverings and also for the classification of them. For the existence he postulated a condition on the branch data – that is the collection of the branching orders at the different branch points – and the connectedness of the covering surface. Very little has been done for the more refined, yet still natural question of existence of branched coverings between surfaces $p\colon W \to V$ under the additional condition that the image of the fundamental group $\pi_1(W)$ under $p_\#$ is a given subgroup $H \subset \pi_1(V)$ of finite index; in particular, for the main, or *primitive*, case where $H = \pi_1(V)$. In this paper we will extend results of the literature and give a full solution of this problem assuming that the target V is neither the sphere nor the projective plane. The question is transformed to a simple arithmetic one, see Theorem 2.4. Our arguments in the proofs are of purely algebraic nature following the approach of Hurwitz [11].

The next section contains a more detailed presentation of the problem, its history and the main results in an introductory form. In section 3, we consider the reduction to an algebraic group theoretical problem. In section 4, we introduce a "gluing" operation of homomorphisms to symmetric groups, which is used to answer the algebraic question in sections 5 (for primitive branched coverings over the torus), 6 (over the Klein bottle), and 7 (for the general case).

2 On the classification of branched coverings

Let us first recall some basic notions and facts. Let $f\colon W \to V$ be a branched covering of finite degree d between closed connected surfaces and let $x_1, \ldots, x_m \in V$ be the points over which the branching occurs. Assume that over x_i there are r_i points with branching orders $d_{i1}, \ldots, d_{ir_i}$ where these numbers form a *partition* of d, that is,

$$d = d_{i1} + \ldots + d_{ir_i}, \quad 1 \leq d_{ij} \in \mathbb{Z},\ i \in \{1, \ldots, m\}.$$

We denote this partition by $A_i = [d_{i1}, \ldots, d_{ir_i}]$ and call $\mathcal{D} = [A_1, \ldots, A_m]$ the *branch data* of the branched covering. The number

$$v(\mathcal{D}) = \sum_{i=1}^{m}\sum_{j=1}^{r_i}(d_{ij}-1) = \sum_{i=1}^{m}\left(-r_i + \sum_{j=1}^{r_i} d_{ij}\right) = \sum_{i=1}^{m}(d-r_i) = md - \sum_{i=1}^{m} r_i$$

is non-negative and is called the *defect* of the branched covering. It has the following important property:

$$v(\mathcal{D}) \equiv 0 \mod 2$$

which is called the *Hurwitz congruence* [11]. A proper branching happens if and only if $v(\mathcal{D}) > 0$. We choose to call a system $\mathcal{D} = [A_1, \ldots, A_m]$, with $A_i = [d_{i1}, \ldots, d_{ir_i}]$, of partitions of d with the aforementioned properties *virtual branch data* of *order* d. Finally, the branched covering $f\colon W \to V$ defines the subgroup $H = f_\#(\pi_1(W)) < \pi_1(V)$ of finite index; in fact, its system of conjugate subgroups is a suitable invariant of f. The covering is called *primitive* if $H = \pi_1(V)$.

Given a connected closed surface V, virtual branch data $\mathcal{D}$, and a subgroup H of $\pi_1(V)$ the following questions arise:

Problems 2.1.

(a) Does there exist a connected branched covering $f\colon W \to V$ with the branch data $\mathcal{D}$?
(b) Does there exist a primitive connected branched covering $f\colon W \to V$ with the branch data $\mathcal{D}$?
(c) Does there exist a connected branched covering $f\colon W \to V$ with the branch data $\mathcal{D}$ and $f_\#(\pi_1(W)) = H$?
(d) How many "different" connected branched coverings solve the considered problem?

Edmonds, Kulkarni and Stong [7] gave a positive answer to the first question for any surface $V \neq S^2$ and the full answer to questions (a) and (b) for the projective plane. Positive answers to questions (b) and (c) are given for all closed surfaces different from the sphere and the projective plane in [4]. The proof in [4] consists of constructing the corresponding branched coverings by figures for "small" cases and using a gluing procedure for the general case. Here we follow the Hurwitz approach [11] of constructing branched coverings using representations of the fundamental groups in symmetric groups. The suitable representations were found by looking at the figures from [4], but the formal algebraic proof of the claim given in this article appears simpler and more algorithmic than the geometric one.

Let us also remark that, according to the theorem of Gabai-Kazez [9], Problem 2.1 (c) is not only of interest in itself, but also plays an important role in the Nielsen theory to find the minimal number of roots in the homotopy class of the given mapping [3].

Two branched coverings $f_i\colon W_i \to V$ are considered as equivalent if there exists a homeomorphism $h\colon W_1 \to W_2$ such that $f_2 = f_1 \circ h$. A lower bound for the number asked for in Problem 2.1 (d) could be found from a solution of the following problem: What is the maximal number m such that the branched covering is the composition of m branched coverings of degree ≥ 2? It would also be of interest to find other invariants of branched coverings of geometric nature.

By the Hurwitz approach, with each branched covering f of degree d over a closed surface V with the set of branch points $B_f \subset V$ one associates a homomorphism $\varphi_f\colon \pi_1(V \setminus B_f) \to \Sigma_d$ called the *Hurwitz representation* realized by f, see 3.1. We first find out the necessary and sufficient algebraic conditions to guarantee that a representation is realized by a branched covering admitting the prescribed subgroup H, see Theorem 3.2. Then, for given virtual branch data and a subgroup of finite index in the fundamental group of the closed surface, we construct representations of the fundamental groups to the symmetric groups which provide the desired branched coverings.

Our main algebraic results are the following two implying the existence of primitive branched coverings over the torus and the Klein bottle. By the *commutator* and the *quasi-commutator* of two elements a, b we mean $[a, b] = aba^{-1}b^{-1}$ and $[a, b]_- = abab^{-1}$.

Theorem 2.2. For each partition $A = [d_1, \ldots, d_r]$ of the number d with a positive even defect $v(A) = (d_1 - 1) + \ldots + (d_r - 1) = d - r > 0$ there are permutations $\hat{a}, \hat{b} \in \Sigma_d$ with the following properties:

(a) The subgroup of Σ_d generated by $\hat{a}, \hat{b}$ acts transitively on $\{1, \ldots, d\}$.
(b) The commutator $[\hat{a}, \hat{b}]$ consists of cycles of the lengths $d_1, \ldots, d_r$.
(c) The symbol 1 is fixed under the action of $\hat{a}$.
(d) The symbol 1 is fixed under $\hat{b}[\hat{a}, \hat{b}]$ or under $\hat{b}[\hat{a}^2, \hat{b}]$.

Theorem 2.3. For each partition $A = [d_1, \ldots, d_r]$ of the number d with a positive even defect $v(A) = (d_1 - 1) + \ldots + (d_r - 1) = d - r > 0$ there are permutations $\hat{a}, \hat{b} \in \Sigma_d$ and a natural number q with the following properties:

(a) The subgroup of Σ_d generated by $\hat{a}, \hat{b}$ acts transitively on $\{1, \ldots, d\}$.
(b) The quasi-commutator $[\hat{a}, \hat{b}]_-$ consists of cycles of the lengths $d_1, \ldots, d_r$.
(c) The symbol 1 is fixed under the action of $\hat{a}$.
(d) The symbol 1 is fixed under $\hat{b}[\hat{a}, \hat{b}]_-^q$.

These two results of special nature can easily be joined to a geometric result on branched coverings over surfaces of arbitrary genus.

We say that the *subgroup H of $\pi_1(V)$ corresponds to the branched covering $f\colon W \to V$* if $H = f_\#(\pi_1(W))$.

Theorem 2.4. [4, Theorem 4.2] Let V be a closed surface different from the sphere and the projective plane, $H \subset \pi_1(V)$ a subgroup, and let $\mathcal{D} = [A_1, \ldots, A_m]$ be some virtual branch data of order d. Then the following two assertions are equivalent.

(1) The subgroup H corresponds to some connected branched covering between closed surfaces realizing the branch data $\mathcal{D}$.
(2) H is a subgroup of finite index ℓ such that $\ell | d$. For each $i \in \{1, \ldots, m\}$ there exist ℓ partitions

$$B_{i1} = [d_{i11}, \ldots, d_{i1r_{i1}}], \ldots, B_{i\ell} = [d_{i\ell 1}, \ldots, d_{i\ell r_{i\ell}}]$$

of the number d/ℓ such that

$$A_i = B_{i1} \sqcup \ldots \sqcup B_{i\ell} = [d_{i11}, \ldots, d_{i1r_{i1}}, \ldots, d_{i\ell 1}, \ldots, d_{i\ell r_{i\ell}}].$$

The algebraic version of this theorem follows. For a homomorphism $\varphi\colon \pi \to \Sigma_d$ of a group π, consider the corresponding action of π on the set $\{1, \ldots, d\}$. Denote by $\mathrm{Stab}_\varphi(k)$, $1 \le k \le d$ the stabilizer of the symbol k under this action. By $\langle\langle x_1, \ldots \rangle\rangle$ we denote the smallest normal subgroup containing the elements $x_1, \ldots$.

Theorem 2.5. Let $\pi = \langle a_1, \ldots, a_n, s_1, \ldots, s_m \mid \prod^* \cdot (s_1 \ldots s_m)\rangle$ where $n \ge 2$, $m \ge 1$, and $\prod^* = \prod_{i=1}^{n/2} [a_{2i-1}, a_{2i}]$ or $\prod^* = a_1^2 \cdot \ldots \cdot a_n^2$, and let $H < \pi/\langle\langle s_1, \ldots, s_m\rangle\rangle$ be a subgroup.

Furthermore let $\sigma_1, \ldots, \sigma_m \in \Sigma_d$, where $\sigma_i \neq \mathrm{id}$ for at least one i, be some permutations such that $\prod_{i=1}^m \sigma_i$ is an even permutation. Denote by A_i the collection of the orders of the cycles of σ_i. Then the following two assertions are equivalent:

(1) There exists a homomorphism $\varphi\colon\ \pi \to \Sigma_d$ such that
  (a) the group $\varphi(\pi) < \Sigma_d$ acts transitively on $\{1, \ldots, d\}$,
  (b) $\varphi(s_i)$ is conjugate to σ_i in Σ_d,
  (c) the image of the composition $\mathrm{Stab}_\varphi(1) \hookrightarrow \pi \to \pi/\langle\langle s_1, \ldots, s_m\rangle\rangle$ is H.

(2) H is a subgroup of finite index ℓ such that $\ell | d$. For each $i \in \{1, \ldots, m\}$ there exist ℓ partitions

$$B_{i1} = [d_{i11}, \ldots, d_{i1r_{i1}}], \ldots, B_{i\ell} = [d_{i\ell 1}, \ldots, d_{i\ell r_{i\ell}}]$$

of the number d/ℓ such that

$$A_i = B_{i1} \sqcup \ldots \sqcup B_{i\ell} = [d_{i11}, \ldots, d_{i1r_{i1}}, \ldots, d_{i\ell 1}, \ldots, d_{i\ell r_{i\ell}}].$$

3 A reduction to algebra

In this section we transform the problem of constructing (primitive) branched coverings over surfaces into algebraic terms. First we describe the Hurwitz representation associated to a branched covering.

Hurwitz Representation 3.1. Let $f\colon\ W \to V$ be a d-fold branched covering of a connected closed surface W over V, let $B_f \subset V$ denote the set of branch points of f and $*_V \in V \setminus B_f$, $*_W \in f^{-1}(*_V)$ the basepoints. Take a small disk U arround $*_V$ such that $p^{-1}(U)$ consists of disjoint disks each of which is mapped homeomorphically to U. Enumerate the disks by $\{1, \ldots, d\}$ where the disk with label 1 contains $*_W$ in its interior. Moreover let $*_{Wi}$ be the point over $*_V$ in the i-th disk, in particular, $*_W = *_{W1}$. A closed path γ in $V \setminus B_f$ starting in $*_V$ admits, for each $i \in \{1, \ldots, d\}$, a uniquely determined lift $\tilde{\gamma}_i$ that starts in $*_{Wi}$. Adjoining to i the label of the endpoint of $\tilde{\gamma}_i$, we obtain a permutation $\varphi_f(\gamma)$ lying in the symmetric group Σ_d. This permutation remains the same when γ is continuously deformed in $V \setminus B_f$ such that the start and end of γ always stay at the basepoint $*_V$. Thus φ_f induces a homomorphism of the fundamental group of $V \setminus B_f$ to the symmetric group which we also denote by φ_f; now $\varphi_f\colon\ \pi_1(V \setminus B_f, *_V) \to \Sigma_d$ is called the *Hurwitz representation* associated to f. We can interpret this as an action of $\pi_1(V \setminus B_f, *_V)$ on $\{1, \ldots, d\}$. This action is transitive since W is connected. For details see [11], [5], [15, 6.7.2].

In the following we use the notation $\langle i\rangle\sigma = j$ if the permutation σ maps i to j. By $(i_1, i_2, \ldots, i_k)$ we denote the cyclic permutation sending i_j to i_{j+1}, $1 \leq j \leq k-1$ and i_k to i_1.

As above, let $\mathrm{Stab}_{\varphi_f}(k)$ denote the stabilizer of the symbol k under the action of $\pi_1(V \setminus B_f, *_V)$ on $\{1, \ldots, d\}$ corresponding to the Hurwitz representation φ_f, that is,

$$\mathrm{Stab}_{\varphi_f}(k) = \{a \in \pi_1(V \setminus B_f, *_V) \mid \langle k\rangle\varphi_f(a) = k\}.$$

Theorem 3.2. Let $i_f\colon (V \setminus B_f, *_V) \to (V, *_V)$ be the inclusion. Under the hypotheses from above,

$$f_\#(\pi_1(W, *_W)) = i_{f\#}(\mathrm{Stab}_{\varphi_f}(1)).$$

Proof. Consider the restriction $g = f|_{W \setminus f^{-1}(B_f)}$ and the inclusion $j_f\colon (W \setminus f^{-1}(B_f), *_W) \to (W, *_W)$. From 3.1, see also [13, §58], it follows that

$$g_\#(\pi_1(W \setminus f^{-1}(B_f), *_W)) = \mathrm{Stab}_{\varphi_f}(1) \subset \pi_1(V \setminus B_f, *_V).$$

Clearly, the homomorphisms

$$i_{f\#}\colon \pi_1(V \setminus B_f, *_V) \to \pi_1(V, *_V), \quad j_{f\#}\colon \pi_1(W \setminus f^{-1}(B_f), *_W) \to \pi_1(W, *_W)$$

are surjective; hence,

$$\begin{aligned} f_\#(\pi_1(W, *_W)) &= f_\# \circ j_{f\#}(\pi_1(W \setminus f^{-1}(B_f), *_W)) \\ &= i_{f\#} \circ g_\#(\pi_1(W \setminus f^{-1}(B_f), *_W)) = i_{f\#}(\mathrm{Stab}_{\varphi_f}(1)). \end{aligned}$$

□

From the definitions and Theorem 3.2, the following corollaries are direct consequences.

Corollary 3.3. Let $f\colon W \to V$ be a branched covering of order d between two connected closed surfaces, $B_f \subset V$ the set of branch points, and $\varphi_f\colon \pi_1(V \setminus B_f, *_V) \to \Sigma_d$ the Hurwitz representation for f. Then the following conditions are equivalent:

(a) f is primitive;
(b) the composition $\mathrm{Stab}_{\varphi_f}(k) \hookrightarrow \pi_1(V \setminus B_f, *_V) \to \pi_1(V, *_V)$ is surjective for each symbol $k \in \{1, \ldots, d\}$;
(c) the composition $\mathrm{Stab}_{\varphi_f}(k) \hookrightarrow \pi_1(V \setminus B_f, *_V) \to \pi_1(V, *_V)$ is surjective for some symbol $k \in \{1, \ldots, d\}$. □

Corollary 3.4. Let $f\colon W \to V$ be a branched covering of order d between two connected closed surfaces, $B_f \subset V$ the set of branch points, $\varphi_f\colon \pi_1(V \setminus B_f, *_V) \to \Sigma_d$ the Hurwitz representation for f, and $H < \pi_1(V, *_V)$ a subgroup. Then the following conditions are equivalent:

(a) the subgroup H corresponds to the branched covering f;
(b) the image of the composition $\mathrm{Stab}_{\varphi_f}(1) \hookrightarrow \pi_1(V \setminus B_f, *_V) \to \pi_1(V, *_V)$ is H. □

Remark that the fundamental groups $\pi_1(V \setminus B_f, *_V)$ and $\pi_1(V, *_V)$ are isomorphic to the groups π and $\pi/\langle\langle s_1, \ldots, s_m \rangle\rangle$ considered in Theorem 2.5 respecting the projections $\pi_1(V \setminus B_f, *_V) \to \pi_1(V, *_V)$ and $\pi \to \pi/\langle\langle s_1, \ldots, s_m \rangle\rangle$.

It follows from Corollary 3.4 that the Theorems 2.4 and 2.5 are equivalent.

4 A Gluing Operation on Homomorphisms to Symmetric Groups

From two representations $\varphi_1\colon G \to \Sigma_n$ and $\varphi_2\colon G \to \Sigma_m$ we easily construct the direct sum $\varphi_1 \times \varphi_2\colon G \to \Sigma_{n+m}$, but it is not of geometric interest since in the corresponding covering the source consists of *two* connected components corresponding to the two given representations. To get a connected source we use a gluing procedure [4, Section 2] for the two covering surfaces and, thus, have to find in both surfaces a non-separating simple loop such that both curves are mapped to the same power of a loop of the target. We describe an algebraic version of the gluing operation, but only for the groups $G = G_\pm$ where $G_+ = \langle a, b, c \mid [a, b]c^{-1}\rangle$ and $G_- = \langle a, b, c \mid abab^{-1}c^{-1}\rangle$, the fundamental groups of the torus and the Klein bottle minus a "small" disk. We also assume that the two loops are mapped homeomorphically to the same loop of the standard homotopy class a.

Notation 4.1. The image of an element $g \in G$ under a representation φ to Σ_d is denoted by $\hat{g}$; similarly, $\hat{g}_i$ denotes $\varphi_i(g)$ for $i = 1, 2$. To denote a permutation we write it either explicitly or by adding a "ˆ".

Construction of a Gluing Operation 4.2. Let the permutations $\hat{a}_1$ and $\hat{b}_1$ generate a subgroup transitively acting on $\{1, \ldots, n\}$ and assume that the symbol i_1 stays invariant under $\hat{a}_1$. Similarly, let $\hat{a}_2, \hat{b}_2$ generate a subgroup acting transitively on $\{n+1, \ldots, n+m\}$ and let i_2 be fixed under $\hat{a}_2$. Clearly, for $n \geq 2$ it follows from the transitivity that $\hat{b}_1$ does not fix i_1. The element $\hat{a} \in \Sigma_{n+m}$ is defined as the direct sum of $\hat{a}_1$ and $\hat{a}_2$, that is, it operates on the first n symbols like $\hat{a}_1$ and on the last m like $\hat{a}_2$. In the following the direct sum of $\hat{a}_1$ and $\hat{a}_2$ is denoted by $\hat{a}_1 \times \hat{a}_2$. The permutation $\hat{b} \in \Sigma_{n+m}$ is defined as the direct sum $\hat{b}_1 \times \hat{b}_2$, followed by the transposition (i_1, i_2) of the symbols i_1 and i_2, that is, $\hat{b} = (\hat{b}_1 \times \hat{b}_2) \circ (i_1, i_2)$. For elements $\hat{c}, \hat{c}_1, \hat{c}_2$ related to the $\hat{a}, \hat{b}, \ldots$ as in the presentations of G_+ or G_-, it will be shown below (Proposition 4.4) that $\hat{c} = \hat{c}_1 \times \hat{c}_2$ is the permutation corresponding to $c = [a, b]$ or $c = abab^{-1}$. The result of the gluing operation on the representations φ_1 and φ_2 is the representation $\varphi\colon G \to \Sigma_{n+m}$ with $\varphi(a) = \hat{a} = \hat{a}_1 \times \hat{a}_2$, $\varphi(b) = \hat{b} = (\hat{b}_1 \times \hat{b}_2) \circ (i_1, i_2)$.

Proposition 4.3. If the subgroups generated by $\hat{a}_1, \hat{b}_1$ and $\hat{a}_2, \hat{b}_2$ transitively act on $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$, respectively, then the subgroup generated by $\hat{a}, \hat{b}$ transitively acts on $\{1, \ldots, n+m\}$.

Proof. Observe that the orbit of any symbol $i \leq n$ contains i_2. In fact, there is a word in $\hat{a}_1, \hat{b}_1$ which maps i to the symbol $\langle i_1\rangle\hat{b}_1^{-1}$. Take the shortest word with this property. If we replace in this word $\hat{a}_1$ by $\hat{a}$ and $\hat{b}_1$ by $\hat{b}$ then the obtained word in $\hat{a}$ and $\hat{b}$ also maps the symbol i to $\langle i_1\rangle\hat{b}_1^{-1}$. If we next apply once more $\hat{b}$ then we obtain the symbol i_2. A consequence is that there is a word in $\hat{a}, \hat{b}$ that maps i into i_2. Analogously, there exists a word in $\hat{a}, \hat{b}$ that transforms a symbol $j \geq n+1$, in particular i_2, into the symbol i_1. Hence, every symbol can be sent to i_1 and this shows the transitivity. □

In the study of branched coverings over nonorientable surfaces an important role is played by the *quasi-commutator*

$$[a,b]_- = abab^{-1};$$

now a corresponds to a two-sided, but b to a one-sided curve. To unify the notions of the commutator and the quasi-commutator, let us consider a more general analog of the commutator. For integers r and s, let $[a,b]_{rs} = a^r b a^s b^{-1}$. Then $[a,b]_{1,-1} = [a,b]$ and $[a,b]_{1,1} = [a,b]_-$.

Proposition 4.4. For any integers r and s,

$$[\hat{a},\hat{b}]_{rs} = [\hat{a}_1,\hat{b}_1]_{rs} \times [\hat{a}_2,\hat{b}_2]_{rs}.$$

Proof. Consider a symbol $i \le n$. We identify $\hat{a}_1, \hat{b}_1, \hat{a}_2, \hat{b}_2$ with the elements

$$\hat{a}_1 = \hat{a}_1 \times \mathrm{id},\ \hat{b}_1 = \hat{b}_1 \times \mathrm{id},$$

$$\hat{a}_2 = \mathrm{id} \times \hat{a}_2,\ \hat{b}_2 = \mathrm{id} \times \hat{b}_2$$

of Σ_{m+n}. Then

$$\langle i\rangle \hat{a}^r \hat{b} \hat{a}^s \hat{b}^{-1} = \langle i\rangle \hat{a}_1^r \hat{b}_1 \circ (i_1, i_2) \circ \hat{a}^s \hat{b}^{-1}.$$

Suppose that $\langle i\rangle \hat{a}_1^r \hat{b}_1 \neq i_1$, thus $\langle i\rangle \hat{a}^r \hat{b} \hat{a}^s \hat{b}^{-1} = \langle i\rangle \hat{a}_1^r \hat{b}_1 \hat{a}_1^s \hat{b}^{-1}$. Since the symbol i_1 is fixed under $\hat{a}_1$ it follows that $\langle i\rangle \hat{a}_1^r \hat{b}_1 \hat{a}_1^s \neq i_1$. Hence, under the action of $\hat{b}^{-1}$ the transposition (i_1, i_2) is not applied to the obtained element and, thus, $\langle i\rangle \hat{a}_1^r \hat{b}_1 \hat{a}_1^s \hat{b}^{-1} = \langle i\rangle \hat{a}_1^r \hat{b}_1 \hat{a}_1^s \hat{b}_1^{-1}$. For $\langle i\rangle \hat{a}_1^r \hat{b}_1 = i_1$ we obtain

$$\begin{aligned}\langle i\rangle \hat{a}^r \hat{b} \hat{a}^s \hat{b}^{-1} &= \langle i\rangle \hat{a}_1^r \hat{b}_1\,(i_1,i_2)\,\hat{a}^s \hat{b}^{-1} = \langle i_1\rangle (i_1,i_2)\,\hat{a}^s \hat{b}^{-1}\\ &= \langle i_2\rangle \hat{a}^s \hat{b}^{-1} = \langle i_2\rangle \hat{b}^{-1} = \langle i_1\rangle \hat{b}_1^{-1} = \langle i_1\rangle \hat{a}_1^s \hat{b}_1^{-1}\\ &= \langle i\rangle \hat{a}_1^r \hat{b}_1 \hat{a}_1^s \hat{b}_1^{-1}.\end{aligned}$$

For $i \ge n+1$ a similar consideration takes place. □

5 Realization of Primitive Branched Coverings over the Torus

Proof of Theorem 2.2 *by induction on* r. We assume that $d_1 \ge d_2 \ge \ldots \ge d_r$.

Case $r = 1$: From the condition that the defect $v(A)$ is even and > 0 it follows that the number d is odd and > 2; hence, $d = 2k+1$ with $k > 0$. Put

$$\hat{a} = \begin{pmatrix} 1 \ldots k & k+1 \ldots & 2k & 2k+1 \\ 1 \ldots k & k+2 \ldots & 2k+1 & k+1 \end{pmatrix},$$

$$\hat{b} = \begin{pmatrix} 1 & \ldots & k & k+1 & k+2 \ldots 2k+1 \\ 2k+1 & \ldots & k+2 & k+1 & 1 \ldots k \end{pmatrix}.$$

Next we check the claims (a) – (d).

(b): By a direct calculation it follows that

$$(*) \qquad [\hat{a}, \hat{b}] = (1,\ 2, \ldots,\ 2k,\ 2k+1).$$

(a) is a direct consequence of $(*)$, claim (c) is obvious.

(d): $\langle 1\rangle \hat{b}[\hat{a}, \hat{b}] = \langle 2k+1\rangle [\hat{a}, \hat{b}] = 1$ by $(*)$.

Case $r \geq 2$: If there is an odd number d_j then we consider the partitions $A' = [d_1, \ldots, d_{j-1}, d_{j+1}, \ldots, d_r]$ and $A'' = [d_j]$ of the numbers

$$d' = d - d_j = d_1 + \cdots + d_{j-1} + d_{j+1} + \cdots + d_r \quad \text{and} \quad d'' = d_j.$$

If $d_r = 1$ we put $j = r$. Since the defect $v([d_j]) = d_j - 1$ of the partition $[d_j]$ is even and

$$v(A) = v(A') + v(A''),$$

both considered partitions have an even defect. Furthermore, $v(A') > 0$ and $v(A'') = d_j - 1 \geq 0$, thus the induction hypothesis can be applied to both partitions except for the case $d'' = d_j = 1$ where $v(A'') = 0$. But for the trivial permutation the properties (a) – (d) are easily checked. By induction hypothesis, there are pairs of permutations $\hat{a}_1$, $\hat{b}_1$ and $\hat{a}_2$, $\hat{b}_2$ realizing the corresponding partitions A' and A'' of d' and d''. Since these pairs of permutations have the property (c), we can apply the gluing operation 4.2 to them. As the result of this operation, we obtain a pair of permutations $\hat{a}$, $\hat{b}$. Let us check the properties (a) – (d) for them.

The property (a) follows from Proposition 4.3; (b) follows from Proposition 4.4; and (c) is a consequence of $\langle 1\rangle \hat{a} = \langle 1\rangle \hat{a}_1 = 1$. To check the truth of (d) it suffices to use $i_1 = 1$ for the gluing operation. In fact, using Proposition 4.4, we have

$$\begin{aligned} \langle 1\rangle \hat{b} &= \langle \langle 1\rangle \hat{b}_1\rangle (1, i_2) = \langle 1\rangle \hat{b}_1 \quad \Longrightarrow \\ \langle 1\rangle \hat{b}[\hat{a}, \hat{b}] &= \langle \langle 1\rangle \hat{b}_1\rangle \left([\hat{a}_1, \hat{b}_1] \times [\hat{a}_2, \hat{b}_2]\right) = \langle 1\rangle \hat{b}_1[\hat{a}_1, \hat{b}_1] = 1 \qquad \text{or} \\ \langle 1\rangle \hat{b}[\hat{a}^2, \hat{b}] &= \langle \langle 1\rangle \hat{b}_1\rangle \left([\hat{a}_1^2, \hat{b}_1] \times [\hat{a}_2^2, \hat{b}_2]\right) = \langle 1\rangle \hat{b}_1[\hat{a}_1^2, \hat{b}_1] = 1, \end{aligned}$$

in dependence on the equality from (d) fulfilled by $\hat{a}_1$, $\hat{b}_1$.

Now let all d_i be even. Then the number d is also even and, thus, r too.

Consider the case $r = 2$. It follows from the hypothesis that $d = 2k$ for some $k \geq 2$ and that d_1 and d_2 are even. Thus $\ell = (d_1 - d_2)/2$ is an integer with $0 \leq \ell \leq k-2$; now $d_1 = k + \ell$ and $d_2 = k - \ell$.

Next we consider a more general situation assuming only that $d_2 > 1$, that is, $\ell \leq k-2$. In other words, for the next steps we do not need that d_1 and d_2 are even, but only that $d_2 > 1$.

For $0 \leq \ell \leq k-2$ we define

$$\hat{a} = \begin{pmatrix} 1 \dots k-1 & k & k+1 \dots 2k-1 & 2k \\ 1 \dots k-1 & k+1 & k+2 \dots & 2k & k \end{pmatrix},$$

$$\hat{b} = \begin{pmatrix} 1 & \dots k-1 & k & k+1 \dots k+\ell & k+\ell+1 & k+\ell+2 \dots & 2k \\ 2k-1 & \dots k+1 & k & 1 \quad \dots \quad \ell & 2k & \ell+1 & \dots k-1 \end{pmatrix}.$$

The columns with the upper symbols $k+1, \dots, k+\ell$ in the last formula are ignored if $\ell = 0$.

Next we check the claims (a) – (d).

(b): By a direct calculation we obtain

$$[\hat{a}, \hat{b}] = \begin{pmatrix} 1 \dots k+\ell-1 & k+\ell & k+\ell+1 \dots 2k-1 & 2k \\ 2 \dots \quad k+\ell & 1 & k+\ell+2 \dots \quad 2k & k+\ell+1 \end{pmatrix}$$
$$= (1,\ 2, \dots, k+\ell)(k+\ell+1, \dots, 2k).$$

(a): The transitivity follows from the fact that the commutator consists of two cycles and that $\hat{a}$ maps the symbol $2k$ from the second cycle to the symbol k of the first cycle.
The property (c) is obvious.
(d): This follows from $\langle 1 \rangle \hat{b} = 2k-1$ and $\langle 2k-1 \rangle [\hat{a}^2, \hat{b}] = 1$.

If $r \geq 3$ then $r \geq 4$ and $d_r \geq 2$. Since any two permutations from Σ_d are conjugate in Σ_d if they admit (up to a permutation) the same systems of lengths of their cycles, we can apply the gluing operation to pairs of permutations realizing the partitions $A' = [d_1, \dots, d_{r-2}]$ and $A'' = [d_{r-1}, d_r]$ of the numbers $d' = d - d_{r-1} - d_r = d_1 + \dots + d_{r-2}$ and $d'' = d_{r-1} + d_r$. □

Corollary 5.1. Let $\hat{c} \in \Sigma_d$ be a non-trivial even permutation. Then there are permutations $\hat{a}, \hat{b}$ with the following properties:

(a) The permutations $\hat{a}, \hat{b}$ generate a subgroup of Σ_d which transitively acts on $\{1, \dots, d\}$.
(b) $\hat{c} = [\hat{a}, \hat{b}]$.
(c) The symbol 1 is fixed under the action of $\hat{a}$.
(d) The symbol 1 is fixed under $\hat{b}[\hat{a}, \hat{b}]$ or under $\hat{b}[\hat{a}^2, \hat{b}]$. □

Notice, the usual proof [7,8,12] of the existence theorem of a branched covering with given branch data of even defect over the torus and surfaces of higher genus is obtained as the geometric equivalent (according to the Hurwitz criterion) of the algebraic fact that *each even permutation $\hat{c}$ is the commutator of two permutations where one of these permutations is a large cycle and the other one admits a fixed symbol.* Corollary 5.1 of Theorem 2.2 gives a new proof that *each even permutation is the commutator of* *two permutations generating a transitive subgroup.* In our representation of the even permutation $\hat{c}$ as a commutator of $\hat{a}$ and $\hat{b}$, the permutation $\hat{a}$ has exactly $v(\hat{c})/2+e$ fixed symbols, where e is the number of cycles of length 1 of the permutation $\hat{c}=[\hat{a},\hat{b}]$.

6 Realization of Primitive Branched Coverings over the Klein Bottle

Proof of Theorem 2.3 *by induction on* r. We assume that $d_1 \geq \ldots \geq d_r$.

Case $r=1$. From the condition that the defect $v(A)$ is even it follows that $d=2k+1$ for some integer $k \geq 1$. Consider

$$\hat{a}=\begin{pmatrix} 1 \ldots k & k+1 \ldots & 2k & 2k+1 \\ 1 \ldots k & k+2 \ldots & 2k+1 & k+1 \end{pmatrix},$$
$$\hat{b}=\begin{pmatrix} 1 & 2 & \ldots k+1 & k+2 \ldots 2k+1 \\ 2k+1 & k+1 \ldots & 2k & 1 \quad \ldots \quad k \end{pmatrix}.$$

(b): By a direct calculation we obtain

$$[\hat{a},\hat{b}]_- = (1,\ 2,\ldots,\ 2k,\ 2k+1).$$

(a) is a direct consequence of (b), claim (c) is obvious.

(d): $\hat{b}$ maps the symbol 1 to the symbol $2k+1$ and this is mapped by $[\hat{a},\hat{b}]_-$ back into 1.

Case $r \geq 2$. If there is an odd d_j, then the assertion is obtained by the same arguments as in the proof of Theorem 2.2. Therefore, we may assume that all numbers d_j are even. Then d and thus, r are even.

Consider the case $r=2$. From the conditions it follows that $d=2k$ for some $k \geq 2$ and that d_1, d_2 are even. Put $\ell=(d_1-d_2)/2$, thus $d_1=k+\ell$ and $d_2=k-\ell$.

Now we obtain more general conclusions which are possible for $d_2>1$, that is, $\ell \leq k-2$. In other words, for the following discussion we may only assume that $d_2>1$.

For $0 \leq \ell \leq k-2$ we define

$$\hat{a}=\begin{pmatrix} 1 \ldots k-1 & k & k+1 \ldots 2k-1 & 2k \\ 1 \ldots k-1 & k+1 & k+2 \ldots \quad 2k & k \end{pmatrix},$$
$$\hat{b}=\begin{pmatrix} 1 \ldots \quad k & k+1 \ldots k+\ell & k+\ell+1 & k+\ell+2 \ldots & 2k \\ k \ldots 2k-1 & 1 \quad \ldots \quad \ell & 2k & \ell+1 & \ldots k-1 \end{pmatrix}.$$

The columns with the upper symbols $k+1,\ldots,k+\ell$ in the last formula are ignored if $\ell=0$.

(b): A direct calculation gives

$$[\hat{a},\hat{b}]_- = \begin{pmatrix} 1 \dots k+\ell-1 & k+\ell & k+\ell+1 \dots 2k-1 & 2k \\ 2 \dots \quad k+\ell & 1 & k+\ell+2 \dots \quad 2k & k+\ell+1 \end{pmatrix}$$
$$= (1,\ 2,\dots,k+\ell)(k+\ell+1,\dots,2k).$$

Now (a) follows from the facts that the quasi-commutator consists of two cycles and that under the action of $\hat{a}$ the symbol $2k$ from the second cycle goes into the symbol k from the first cycle. (c) is obvious.

(d): Under the action of $\hat{b}$ the symbol 1 goes into k and this one is mapped by $[\hat{a},\hat{b}]_-^{\ell+1}$ to 1.

The case $r \geq 3$, i.e. $r \geq 4$ can be handled in the same way as at the end of the proof of the Theorem 2.2. □

Corollary 6.1. Let $\hat{c} \in \Sigma_d$ be a non-trivial even permutation. Then there are permutations $\hat{a},\hat{b}$ and a natural number q with the following properties:

(a) The permutations $\hat{a},\hat{b}$ generate a subgroup of Σ_d which transitively acts on $\{1,\dots,d\}$.
(b) $\hat{c} = [\hat{a},\hat{b}]_-$.
(c) The symbol 1 is fixed under the action of $\hat{a}$.
(d) The symbol 1 is fixed under $\hat{b}[\hat{a},\hat{b}]_-^q$. □

Thus, each non-trivial even permutation $\hat{c} \in \Sigma_d$ is the quasi-commutator of two permutations which generate a subgroup of Σ_d acting transitively on $\{1,\dots,d\}$. Since

$$[\hat{a},\hat{b}]_- = \hat{a}_1^2\hat{b}^{-2} \quad \text{with} \quad \hat{a}_1 = \hat{a}\hat{b}$$

we also obtain that $\hat{c}$ is the product of the squares of two permutations which generate a subgroup of Σ_d acting transitively on $\{1,\dots,d\}$.

7 The General Case

Proof of Theorem 2.5 *for the primitive case* $H = \pi/\langle\langle s_1,\dots,s_m\rangle\rangle$. Remark that, in this case, the condition (2) of Theorem 2.5 is always true. So, we must prove that (1) is always true too.

First consider the case $\pi = \langle a,b,s_1,\dots,s_m \mid [a,b]\cdot(s_1\dots s_m)\rangle$ where $m \geq 1$.

Assume $m = 1$ and denote $A_1 = [d_1,\dots,d_r]$. According to Corollary 5.1 there are two permutations $\hat{a},\hat{b} \in \Sigma_d$ such that the commutator $[\hat{a},\hat{b}]$ consists of cycles of lengths $d_1,\dots,d_r$. The Hurwitz representation φ of the group $\pi = \langle a,b,c \mid [a,b]c^{-1}\rangle$ is given by $a \mapsto \hat{a}$, $b \mapsto \hat{b}$, $c \mapsto [\hat{a},\hat{b}]$. Now the properties (a) and (b) follow from the assertions (a) and (b) of Corollary 5.1. By the assertions (c) and (d) of Corollary 5.1, the symbol 1 is fixed under the actions of $\hat{a}$ and $\hat{b}[\hat{a}^q,\hat{b}]$ for an appropriate integer q, thus a, $b[a^q,b] \in \mathrm{Stab}_\varphi(1)$.

Therefore, the composition $\mathrm{Stab}_\varphi(1) \hookrightarrow \pi \to \pi/\langle\langle c\rangle\rangle$ is surjective. So, the property (c) is fulfilled.

Let $m \geq 2$ and denote $A_i = [d_1^i, \ldots, d_r^i]$. We take some permutations $\hat{s}_i \in \Sigma_d$ consisting of cycles of lengths $d_1^i, \ldots, d_{r_i}^i$, respectively. If the permutation $\hat{c} = \hat{s}_1 \cdot \ldots \cdot \hat{s}_m \neq \hat{e}$, $\hat{e}$ the identity, we apply Corollary 5.1 to it. If $\hat{s}_1 \cdot \ldots \cdot \hat{s}_m = \hat{e}$ then one of the following three cases is possible:

(1) in some $\hat{s}_i$ there exists a cycle of length ≥ 3;
(2) all cycles have length ≤ 2, but $d \geq 3$;
(3) $d = 2$.

In the first case, replace $\hat{s}_i$ by $\hat{s}_i^{-1}$. In the second case, permute in $\hat{s}_i \neq \hat{e}$ one symbol appearing in a cycle of length 2 with a symbol appearing in another cycle. In both cases, the new $\hat{s}_i$ have cycles of the same length, but the product of $\hat{s}_i$ is not the identity, so Corollary 5.1 can be applied. Now the Hurwitz representation $\varphi\colon \pi \to \Sigma_d$ is given by $a \mapsto \hat{b}$, $b \mapsto \hat{a}$, $s_i \mapsto \hat{s}_i$, thus $[a,b]\cdot(s_1 \ldots s_m) \mapsto [\hat{a},\hat{b}]^{-1}\hat{s}_1 \ldots \hat{s}_m = \hat{e}$. As for the case $m = 1$, the assertions (a) – (d) of Corollary 5.1 imply the required properties (a) – (c).

In the third case, there is a $\hat{s}_i \neq \hat{e}$. Take arbitrary permutations $\hat{a}, \hat{b} \in \Sigma_2$. The required properties (a) – (c) are easily checked.

Assume that $\pi = \langle a_1, \ldots, a_{2g}, s_1, \ldots, s_m \mid \prod^* \cdot (s_1 \ldots s_m)\rangle$ where $m \geq 1$, and $\prod^* = \prod_{i=1}^g [a_{2i-1}, a_{2i}]$. For definition of the Hurwitz representation $\pi \to \Sigma_d$, we map a_1, a_2, $s_1, \ldots, s_m$ as above and map $a_3, \ldots, a_{2g}$ to the identity permutation.

For the case $\pi = \langle a_1, \ldots, a_n, s_1, \ldots, s_m \mid \prod^* \cdot (s_1 \ldots s_m)\rangle$, where $m \geq 1$, $n \geq 2$, and $\prod^* = a_1^2 \cdot \ldots \cdot a_n^2$, we proceed as before, but using Corollary 6.1. □

Proof of Theorems 2.5 and 2.4 *in the general case.* Since the Theorems 2.5 and 2.4 are equivalent, we have obtained Theorem 2.4 for the primitive case $H = \pi_1(V)$ and it remains to prove it for the general case.

(1) $\Longrightarrow$ (2): Let $f\colon W \to V$ be a branched covering with $f_\#(\pi_1(W)) = H$. Consider the unbranched covering $p\colon \bar{V} \to V$ corresponding to the subgroup H. Then f lifts to $\bar{f}\colon W \to \bar{V}$. Now, for any branch point $x \in B_f$, the union of the branch data (with respect to $\bar{f}$) of ℓ points $\{y_1, \ldots, y_\ell\} = p^{-1}(x)$ gives the branch data for f at x.

(2) $\Longrightarrow$ (1): Let $p\colon \bar{V} \to V$ be the unbranched covering which corresponds to the subgroup H. Consider the virtual branch data $\bar{\mathcal{D}} = [B_{11}, \ldots, B_{1\ell}, \ldots, B_{m1}, \ldots, B_{m\ell}]$. Since $\chi(\bar{V}) = \ell \cdot \chi(V) \leq 0$, the surface $\bar{V}$ is different from the sphere and the projective plane. It follows from Theorem 2.4 for the primitive case that there is a connected primitive branched covering $h\colon W \to \bar{V}$ which realizes $\bar{\mathcal{D}}$. Therefore $p \circ h$ is the required covering. □

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