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Generalization of Weierstrass Canonical Integrals

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Abstract: In this paper we prove that a subharmonic function in $\mathbb{R}^m$ of finite λ-type can be represented (within some subharmonic function) as the sum of a generalized Weierstrass canonical integral and a function of finite λ-type which tends to zero uniformly on compacts of $\mathbb{R}^m$. The known Brelot-Hadamard representation of subharmonic functions in $\mathbb{R}^m$ of finite order can be obtained as a corollary from this result. Moreover, some properties of R-remainders of λ-admissible mass distributions are investigated.

Keywords: Subharmonic function, Weierstrass canonical integral, Brelot-Hadamard representation, subharmonic function of finite λ-type
MSC (2000): 31B05

1 Introduction

Throughout this paper $\mathbb{R}^m$ is the m-dimensional Euclidean space, S^m is the unit sphere in $\mathbb{R}^m$ centered at the origin and ω_m is its surface area.

Let Δ denote the Laplace operator. If u is a subharmonic function in $\mathbb{R}^m$, then Δu is non-negative in the sense of generalized functions, and $\mu_u = \frac{1}{d_m \omega_m}\Delta u$ is a positive measure, which is called the Riesz mass distribution associated with u (see, e.g. [1, pp. 55–58], [13, p.43]). Here $d_2 = 1$ and $d_m = m-2$ for $m > 2$.

For any integer $q \geq 0$, define

$$K_q(y;\zeta) = \begin{cases} ln\left|1-\frac{y}{\zeta}\right| + \sum\limits_{k=1}^{q} \frac{\left|\frac{y}{\zeta}\right|^k \cos k\varphi}{k}, & m=2, \\ -\frac{1}{|y-\zeta|^{m-2}} + \frac{1}{|\zeta|^{m-2}} \sum\limits_{k=0}^{q} \left|\frac{y}{\zeta}\right|^k C_k^{\nu}\left[\left(\frac{y}{|y|}, \frac{\zeta}{|\zeta|}\right)\right], & m>2, \end{cases}$$

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where φ is the angle between the radius-vectors of points $y, \zeta \in \mathbb{R}^2$; $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^m$, $m > 2$, and C_k^ν are the Gegenbauer polynomials [2, pp. 302, 329], [12, p.125] of degree k and order $\nu = (m-2)/2$.

Let μ be a mass distribution in $\mathbb{R}^m$ such that $0 \notin \operatorname{supp}\mu$ and $p = p_\mu$ denotes the least nonnegative integer number for which $\int\limits_0^\infty \frac{d\,\mu(t)}{t^{m+p-1}} < \infty$.

The function

$$J_p(y;\mu) = \int\limits_{|\zeta|<\infty} K_p(y;\zeta) d\mu(\zeta)$$

is called the Weierstrass canonical integral of genus p [1, p. 78], [13, pp. 67–68]. It is a subharmonic function and μ is its associated Riesz mass distribution (see, e.g. [3, p. 163]).

Let u be a subharmonic function in $\mathbb{R}^m$ which is harmonic in some neighborhood of the origin, with $u(O) = 0$, and let λ be a positive continuous increasing function on $(0, +\infty)$, which is called the function of growth.

Put $B(r, u) = \max\{u(y) : |y| \leqslant r\}$.

Definition 1.1. [4] A subharmonic function u is called a function of finite λ-type if there exist constants A and B such that

$$B(r, u) \leqslant A\,\lambda(Br)$$

for all $r > 0$. The class of such functions is denoted by Λ_s.

It is known (see [5], [13, pp. 68–69]) that in the case $\lambda(r) = r^\varrho$, $\varrho > 0$, the subharmonic function $u \in \Lambda_s$ is represented in the form of sum

$$u(y) = J_p(y;\mu_u) + \mathcal{P}_n(y), \tag{1}$$

where $p \leqslant \varrho$ and $\mathcal{P}_n$ is a harmonic polynomial of degree $n \leqslant \varrho$. If we denote

$$u_R(y) = \int\limits_{|\zeta|\leqslant R} K_p(y;\zeta)\, d\mu_u(\zeta) + \mathcal{P}_n(y) \quad (R > 0)$$

the sum (1) can be written as

$$u(y) = u_R(y) + \int\limits_{|\zeta|>R} K_p(y;\zeta)\, d\mu_u(\zeta).$$

In addition, the Riesz mass distributions associated with the functions u and u_R coincide in the ball $\{y \in \mathbb{R}^m : |y| \leqslant R\}$, the function $u - u_R$ converges to zero uniformly on compacts of $\mathbb{R}^m$ as $R \to \infty$. Moreover, each of functions $u, u_R, u - u_R$ belongs to the class Λ_s (see, e.g. [1, pp. 79–80]).

We shall generalize this result to the case of arbitrary functions $u \in \Lambda_s$, subharmonic in $\mathbb{R}^m$, $m \geqslant 3$, with a more general growth. The analogous generalization for entire functions f in the plane, such that $\ln|f| \in \Lambda_s$ was obtained by L.Rubel [6] and for subharmonic in $\mathbb{R}^2$ functions of finite λ-type by Ya.Vasylkiv [7].

Let $\alpha[\lambda]$ denote a lower order of λ defined by

$$\alpha[\lambda] = \varliminf_{r\to\infty} \frac{\ln \lambda(r)}{\ln r}.$$

Theorem 1.2. For every function $u \in \Lambda_s$ there exist a subharmonic function $h \not\equiv -\infty$, an unbounded set $\Re$ of positive numbers and a family $\{u_R : R \in \Re\}$ of subharmonic functions such that

1) the Riesz mass distributions associated with the functions u_R and $u+h$ coincide in the ball $\{y \in \mathbb{R}^m : |y| \leqslant R\}$ for all $R \in \Re$;
2) the difference $(u+h)-u_R$ tends to 0 uniformly on compacts of $\mathbb{R}^m$ as $R \to \infty$, $R \in \Re$;
3) $h, u_R, (u+h) - u_R \in \Lambda_s$ for all $R \in \Re$.

If $\alpha[\lambda] = \infty$, then we can take $h \equiv 0$ and if $\ln \lambda(r)$ is convex in $\ln r$, then we can take $h \equiv 0$ and $\Re = \{R : R \geqslant R_0\}$ for some $R_0 > 0$.

Definition 1.3. The family of functions $\{u_R : R \in \Re\}$ defined by the preceding theorem is called the generalized Weierstrass canonical integral of function u.

In the next section we obtain some auxiliary results, which will be used for the proof of Theorem.

2 The remainders

Definition 2.1. [4] A mass distribution μ in $\mathbb{R}^m$, $0 \notin \operatorname{supp}\mu$, is called λ-admissible, if there exist constants A, B and $l \in \mathbb{R}_+$ such that

$$\left| \int\limits_{r_1<|y|\leqslant r_2} C_k^{\nu}\left[\left(x, \frac{y}{|y|}\right)\right] \frac{d\mu(y)}{|y|^{k+2\nu}} \right| \leqslant A(k+1)^l \left[\frac{\lambda(Br_1)}{r_1^k} + \frac{\lambda(Br_2)}{r_2^k}\right] \tag{2}$$

for all $r_1, r_2 > 0$, $k \in \mathbb{Z}_+$, $x \in S^m$. Here and below, $\nu = (m-2)/2$.

Put $\bar{V}_R^m = \{y \in \mathbb{R}^m : |y| \leqslant R\}$.

Definition 2.2. A mass distribution μ_R $(R > 0)$ defined for any Borel set $G \subset \mathbb{R}^m$ by the equality $\mu_R(G) = \mu(G \setminus \bar{V}_R^m)$ is called the R-remainder of μ.

Let $\Re$ be a non-empty set of positive numbers.

Definition 2.3. If the set $\Re$ is unbounded, the family of remainders $\{\mu_R : R \in \Re\}$ is called complete.

Definition 2.4. A family of remainders $\{\mu_R : R \in \Re\}$, $0 \notin \operatorname{supp} \mu_R$, is called uniformly λ-admissible, if it satisfies the inequality (2) for all $r_1, r_2 > 0$, $k \in \mathbb{Z}_+$, $x \in S^m$ and $R \in \Re$.

A spherical harmonic or a spherical Laplace function of degree k ($k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$), denoted $Y^{(k)}$, is defined as the restriction of a homogeneous, harmonic polynomial of degree k on the unit sphere S^m (see, e.g. [8], [9]).

Let μ be a mass distribution in $\mathbb{R}^m$ such that $0 \notin \operatorname{supp} \mu$, and let $Y = \{Y^{(k)}(x)\}$ ($k \in \mathbb{Z}_+$, $Y^{(0)}(x) = 0$) be some sequence of spherical harmonics.

Definition 2.5. [4] The functions

$$c_k(x, r; Y, \mu) = Y^{(k)}(x)\, r^k + r^k \int\limits_{|\zeta| \leqslant r} C_k^\nu \left[\left(x, \frac{\zeta}{|\zeta|} \right) \right] \frac{d\mu(\zeta)}{|\zeta|^{k+2\nu}}$$

$$- \frac{1}{r^{k+2\nu}} \int\limits_{|\zeta| \leqslant r} |\zeta|^k C_k^\nu \left[\left(x, \frac{\zeta}{|\zeta|} \right) \right] d\mu(\zeta) \quad (k \in \mathbb{Z}_+)$$

are called the spherical harmonics of the pair (Y, μ).

Proposition 2.6. Let μ be a λ-admissible mass distribution in $\mathbb{R}^m$. Then

1) there exists λ-admissible mass distribution $\mu' \geqslant \mu$ whose family of remainders $\{\mu'_R : R \in \Re\}$ is complete and uniformly λ-admissible;

2) for every such remainder μ'_R there exists the sequence $Y_R = \{Y_R^{(k)}(x)\}$ ($k \in \mathbb{Z}_+$, $Y_R^{(0)}(x) = 0$, $R \in \Re$) of spherical harmonics such that

$$a) \quad |c_k(x, r; Y_R, \mu'_R)| \leqslant A(k+1)^l \lambda(Br) \tag{3}$$

for all $r > 0$, $k \in \mathbb{Z}_+$, $x \in S^m$, $R \in \Re$ and some positive constants A, B and $l \in \mathbb{R}_+$;

$$b) \quad \lim_{\Re \ni R \to \infty} c_k(x, r; Y_R, \mu'_R) = 0 \tag{4}$$

for all $r > 0, k \in \mathbb{Z}_+, x \in S^m$.

If $\alpha[\lambda] = \infty$, then we can take $\mu' = \mu$.

If $\ln \lambda(r)$ is convex in $\ln r$, then we can take $\mu' = \mu$ and $\{\mu'_R\} = \{\mu_R : R \geqslant R_0 > 0\}$.

The following lemma from [6] will be used in the proof of the last special case of Proposition.

Lemma 2.7. If $\ln \lambda(r)$ is convex in $\ln r$, then there is $R_0 > 0$ such that for every $R \geqslant R_0$ we can find $\sigma = \sigma(R) > 0$, for which

$$\frac{\lambda(BR)}{R^\sigma} = \inf_{r>0} \frac{\lambda(Br)}{r^\sigma}$$

holds. Here B is some positive constant.

Proof of statement 1) of Proposition. For $r > 0, k \in \mathbb{Z}_+, x \in S^m$ and any mass distribution μ in $\mathbb{R}^m$, we put

$$I_k(r; x, \mu) = \int\limits_{|y| \leqslant r} C_k^\nu \left[\left(x, \frac{y}{|y|} \right) \right] \frac{d\mu(y)}{|y|^{k+2\nu}},$$

$$I_k(r_1, r_2; x, \mu) = I_k(r_2; x, \mu) - I_k(r_1; x, \mu) \quad (r_1 \leqslant r_2).$$

It follows from the equation $C_0^\nu(t) = 1$ [2, p.176], [12, p.125] that the functions $I_0(r; x, \mu)$, $I_0(r_1, r_2; x, \mu)$ are independent of x.

We distinguish three cases.

I. Let $\ln \lambda(r)$ be a convex in $\ln r$ function, let μ be a λ-admissible mass distribution in $\mathbb{R}^m$ and the numbers R, σ are as in Lemma 1. We shall show that inequality (2) holds for R-remainder of μ at $R \geqslant R_0$, where R_0 is defined in Lemma 2.7.

If $r_1 \leqslant r_2 \leqslant R$, then $I_k(r_1, r_2; x, \mu_R) = 0$.

If $R \leqslant r_1 \leqslant r_2$, then $I_k(r_1, r_2; x, \mu_R) = I_k(r_1, r_2; x, \mu)$ and therefore the mass distribution μ_R satisfies inequality (2) for all $R \in \Re$.

If $r_1 \leqslant R \leqslant r_2$, then $I_k(r_1, r_2; x, \mu_R) = I_k(R, r_2; x, \mu)$. The last expression doesn't exceed

$$A(k+1)^l \left[\frac{\lambda(BR)}{R^k} + \frac{\lambda(Br_2)}{r_2^k} \right],$$

where A, B are some positive constants and $l \in \mathbb{R}_+$.

Let $k \geqslant \sigma$. Then, by Lemma 1,

$$\frac{\lambda(BR)}{R^k} = \frac{\lambda(BR)}{R^\sigma} \cdot \frac{1}{R^{k-\sigma}} \leqslant \frac{\lambda(Br_1)}{r_1^\sigma} \cdot \frac{1}{r_1^{k-\sigma}} = \frac{\lambda(Br_1)}{r_1^k}.$$

Suppose now that $k < \sigma$. In this case we have

$$\frac{\lambda(BR)}{R^k} = \frac{\lambda(BR)}{R^\sigma} \cdot \frac{1}{R^{k-\sigma}} \leqslant \frac{\lambda(Br_2)}{r_2^\sigma} \cdot \frac{1}{r_2^{k-\sigma}} = \frac{\lambda(Br_2)}{r_2^k}.$$

Thus

$$\frac{\lambda(BR)}{R^k} \leqslant \max \left\{ \frac{\lambda(Br_1)}{r_1^k}, \frac{\lambda(Br_2)}{r_2^k} \right\}$$

hence

$$|I_k(r_1, r_2; x, \mu_R)| \leqslant 2A(k+1)^l \left[\frac{\lambda(Br_1)}{r_1^k} + \frac{\lambda(Br_2)}{r_2^k} \right]$$

for all $r_1, r_2 > 0$, $k \in \mathbb{Z}_+$, $x \in S^m$, $R \in \Re$. If we choose $\mu' = \mu$ whose complete family of remainders is $\{\mu_R : R \geqslant R_0\}$, the statement 1) of Proposition is proved in the case I.

II. Let $\alpha[\lambda] = \infty, \lambda(0) > 0$. For any positive σ, put

$$R_\sigma = \max \left\{ R : \lambda(BR)/R^\sigma = \inf_{r>0} \lambda(Br)/r^\sigma \right\},$$

where B is some positive constant. Since $\lim\limits_{r\to\infty} \lambda(Br)/r^\sigma = \infty$ for every $\sigma > 0$ and λ is a continuous function, the numbers R_σ are defined correctly and they are positive.

As it is shown in [6] R_σ is an increasing unbounded function of σ. Therefore the family of remainders $\{\mu_{R_\sigma} : \sigma > 0\}$ is complete. Analogously to case I it is easy to verify that this family is uniformly λ-admissible.

III. Let $\alpha[\lambda] < \infty$. Then there exists $d \in \mathbb{N}^* = \{1, 2, 3, \dots\}$ such that $\varliminf_{r\to\infty} \lambda(r)/r^b = 0$ for all $b \geqslant d$. We denote by $\Re$ the set of positive numbers R satisfying the relation

$$\frac{\lambda(BR)}{R^d} = \inf_{r\leqslant R} \left\{ \frac{\lambda(Br)}{r^d} \right\}.$$

It is obvious that $\lim\limits_{\Re\ni R\to\infty} \lambda(BR)/R^d = 0$.

Let construct the mass distribution μ' in the following way. Consider the function

$$Q(\eta;\xi) = D - \frac{\Gamma(\nu)}{2\pi^{\nu+1}} \sum_{j=1}^{d} (j+\nu) C_j^\nu[(\eta,\xi)], \tag{5}$$

where the constant D is chosen in such a way that $Q(\eta,\xi) \geqslant 0$ for all $\eta \in S^m$ and all $\xi \in S^m$.

Let μ be λ-admissible mass distribution in $\mathbb{R}^m$ and let φ be an arbitrary function in the class $C_0(\mathbb{R}^m)$ of continuous functions in $\mathbb{R}^m$ with compact support. Put

$$L(\varphi) = \int\limits_{\mathbb{R}^m} \Psi(\varphi; y) d\mu(y),$$

where

$$\Psi(\varphi; y) = \int\limits_{S^m} \varphi(t\eta)\, Q(\eta;\xi)\, dS(\eta) \quad (y = t\xi, \quad t = |y|, \quad \xi \in S^m).$$

It is easy to see that $L(\varphi)$ is a linear continuous positive functional defined on $C_0(\mathbb{R}^m)$. We continue the functional L on the class of semicontinuous functions as this is done in [3, pp. 105–114] and denote the obtained continuation by $\widetilde{L}$. If χ_G is the characteristic function of the set $G \subset \mathbb{R}^m$, we define the measure $\widetilde{\mu}$ associated with the functional $\widetilde{L}$ by $\widetilde{\mu}(G) = \widetilde{L}(\chi_G)$.

Let χ_1 and χ_2 be the characteristic functions of the balls $\bar{V}^m_{r_1}$ and $\bar{V}^m_{r_2}$ respectively. Since $0 \notin \operatorname{supp} \widetilde{\mu}$, then in some neighborhood of the point $y = 0$, which doesn't intersect with $\operatorname{supp} \widetilde{\mu}$, we change the function $C_k^\nu[(x, y/|y|)]/|y|^{k+2\nu}$ (x is fixed), so that it becomes continuous in $\bar{V}^m_{r_1}$ and hence in $\bar{V}^m_{r_2}$. Therefore we have

$$\begin{aligned} I_k(r_1, r_2; x, \widetilde{\mu}) &= I_k(r_2; x, \widetilde{\mu}) - I_k(r_1; x, \widetilde{\mu}) \\ &= \widetilde{L}\left(\chi_2 \frac{C_k^\nu\left[\left(x, \frac{y}{|y|}\right)\right]}{|y|^{k+2\nu}} \right) - \widetilde{L}\left(\chi_1 \frac{C_k^\nu\left[\left(x, \frac{y}{|y|}\right)\right]}{|y|^{k+2\nu}} \right). \end{aligned} \tag{6}$$

Denote

$$F_k(x; y) = \frac{C_k^\nu\left[\left(x, \frac{y}{|y|}\right)\right]}{|y|^{k+2\nu}},$$

$$F_k^+(x;y) = \max\{0;F_k\}, \quad F_k^-(x;y) = -\min\{0;F_k\}.$$

Then $F_k = F_k^+ - F_k^-$ and

$$\widetilde{L}(\chi_i F_k) = \widetilde{L}(\chi_i F_k^+) - \widetilde{L}(\chi_i F_k^-). \tag{7}$$

Here and below index i takes values 1,2.

Since the balls $\bar{V}_{r_i}^m$ are compact subsets in $\mathbb{R}^m$, the functions χ_i are upper semicontinuous. Then, according to Theorem 1.4 from [3, p.22], there exist decreasing sequences $\{g_n^i\}$ of continuous functions such that $g_n^i \to \chi_i$ as $n \to \infty$. It is obvious that every sequence $\{g_n^i\}$ can be chosen so that the supports of functions g_n^i are contained in some compact. Further, since F_k^+ is a nonnegative continuous function, the sequences $\{g_n^i\, F_k^+\}$ are monotone (decreasing in n) sequences of continuous functions with compact supports, moreover, $g_n^i\, F_k^+ \to \chi_i\, F_k^+$ as $n \to \infty$. By Theorem 3.3 from [3, p.109] and definition of the functional L,

$$\widetilde{L}(\chi_i\, F_k^+) = \lim_{n\to\infty} L(g_n^i\, F_k^+) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^m} \Psi(g_n^i\, F_k^+; y)\, d\mu(y).$$

From the non-negativity of function Q we conclude that sequences $\{g_n^i\, F_k^+\, Q\}$ are monotone decreasing in n. Hence the sequences $\{\Psi(g_n^i\, F_k^+; y)\}$ are also monotone decreasing in n. Using Lebesgue's theorem about monotone convergence, we have

$$\widetilde{L}(\chi_i\, F_k^+) = \int\limits_{\mathbb{R}^m} \Psi(\chi_i\, F_k^+; y)\, d\mu(y).$$

Analogously $\widetilde{L}(\chi_i\, F_k^-) = \int\limits_{\mathbb{R}^m} \Psi(\chi_i\, F_k^-; y)\, d\mu(y)$.

Thus, taking into account equality (7), we obtain

$$\widetilde{L}(\chi_i\, F_k) = \int\limits_{\mathbb{R}^m} \Psi(\chi_i\, F_k; y)\, d\mu(y).$$

Hence, by means of (6), we find

$$\begin{aligned} I_k(r_1, r_2; x, \widetilde{\mu}) &= \int\limits_{\mathbb{R}^m} \left\{ \int\limits_{S^m} \chi_{1,2}(|y|\eta) \frac{C_k^\nu[(x,\eta)]}{|y|^{k+2\nu}} Q(\eta;\xi)\, dS(\eta) \right\} d\mu(y) \\ &= \int\limits_{r_1<|y|\leqslant r_2} \left\{ \frac{1}{|y|^{k+2\nu}} \int\limits_{S^m} C_k^\nu[(x,\eta)]\, Q(\eta;\xi)\, dS(\eta) \right\} d\mu(y), \end{aligned} \tag{8}$$

where $y = |y|\xi$, $\xi \in S^m$, and $\chi_{1,2}$ is a characteristic function of ring $\{y \in \mathbb{R}^m : r_1 < |y| \leqslant r_2\}$. Denote

$$D_k(x;\xi) = \int\limits_{S^m} C_k^\nu[(x,\eta)]\, Q(\eta;\xi)\, dS(\eta),$$

where the function Q is defined by relation (5).

At $k = 0$ in consequence of orthogonality of Gegenbauer polynomials [8, p. 179] we have $D_0(x;\xi) = D\omega_m$.

In the case $0 < k \leqslant d$, using the equalities [2, p. 238]

$$\int\limits_{S^m} C_k^\nu[(x,\eta)]\, C_j^\nu[(\eta,\xi)]\, dS(\eta) = \begin{cases} 0, & k \neq j, \\ \frac{2\pi^{\nu+1} C_k^\nu[(x,\xi)]}{(k+\nu)\Gamma(\nu)}, & k = j, \end{cases}$$

we get $D_k(x;\xi) = -C_k^\nu[(x,\xi)]$.

If $k > d$, then $D_k(x;\xi) = 0$.

Therefore from (8) we conclude that

$$I_k(r_1, r_2; x, \widetilde{\mu}) = \begin{cases} D\,\omega_m\, I_0(r_1, r_2; x, \mu), & k = 0, \\ -I_k(r_1, r_2; x, \mu), & 0 < k \leqslant d, \\ 0, & k > d. \end{cases}$$

Choose $\mu' = \mu + \widetilde{\mu}$. Then, by virtue of the previous relations, the equalities

$$I_k(r_1, r_2; x, \mu') = \begin{cases} (1 + D\,\omega_m) I_0(r_1, r_2; x, \mu), & k = 0, \\ 0, & 0 < k \leqslant d, \\ I_k(r_1, r_2; x, \mu), & k > d \end{cases}$$

hold. Therefore it remains to verify that the remainders $\mu'_R\,(R \in \Re)$ are uniformly λ-admissible when $k = 0$ and $k > d$. For this it is sufficient to consider the case $r_1 \leqslant R \leqslant r_2$.

Let $k = 0$. Then

$$|I_0(r_1, r_2; \mu'_R)| = (1 + D\,\omega_m)|I_0(R, r_2; \mu)| \leqslant (1 + D\,\omega_m)|I_0(r_1, r_2; \mu)| \leqslant A[\lambda(Br_1) + \lambda(Br_2)],$$

where A and B are some positive constants.

If $k > d$, we have

$$|I_k(r_1, r_2; x, \mu'_R)| = |I_k(R, r_2; x, \mu)| \leqslant A(k+1)^l \left[\frac{\lambda(BR)}{R^k} + \frac{\lambda(Br_2)}{r_2^k}\right]$$

for $A, B > 0$ and $l \in \mathbb{R}_+$. But

$$\frac{\lambda(BR)}{R^k} = \frac{\lambda(BR)}{R^d} \cdot \frac{1}{R^{k-d}} \leqslant \frac{\lambda(Br_1)}{r_1^d} \cdot \frac{1}{r_1^{k-d}} = \frac{\lambda(Br_1)}{r_1^k}$$

and the proof of statement 1) of Proposition is completed.

Proof of statement 2) of Proposition. At first let prove relation (3). Let $k = 0$. For arbitrary mass distribution μ in $\mathbb{R}^m$, define

$$N(r, \mu) = (m-2) \int\limits_0^r \frac{n(t, \mu)}{t^{m-1}}\, dt,$$

where $n(t,\mu) = \int\limits_{|\tau|\le t} d\mu(\tau)$. Then the necessary relation can be obtained from the equation $c_0(x,r;Y_R,\mu'_R) = N(r,\mu'_R)$ (see [4]) and the uniform λ-admissibility of family of remainders $\{\mu'_R : R \in \Re\}$.

Suppose now that $k \in \mathbb{N}^*$. In this case we shall choose the sequence $\left\{Y_R^{(k)}(x)\right\}$ of spherical harmonics in the same way as in [10] and [4].

If $\varliminf\limits_{r\to\infty} \lambda(Br)r^{-k} > 0$ holds for all k, we put $p[\lambda] = \infty$, otherwise we put

$$p[\lambda] = \min\left\{k : \varliminf_{r\to\infty} \lambda(Br)r^{-k} = 0\right\}.$$

Let $1 \leqslant k < p[\lambda]$. Then $\inf\left\{\lambda(Br)r^{-k} : r > 0\right\} > 0$. Therefore for such k there exists r_k such that $\lambda(Br_k)r_k^{-k} \leqslant 2\lambda(Br)r^{-k}$ for all $r > 0$. In this case we choose $Y_R^{(k)}(x) = -I_k(r_k;x,\mu'_R)$.

Suppose that $k \geqslant p[\lambda]$. Then there is the sequence $\{\varrho_j\}$, $\varrho_j \uparrow \infty$ as $j \to \infty$ such that

$$\lim_{j\to\infty} \lambda(B\varrho_j)\,\varrho_j^{-p[\lambda]} = 0. \tag{9}$$

Since

$$|I_k(\varrho_i;x,\mu'_R) - I_k(\varrho_j;x,\mu'_R)| \leqslant A(k+1)^l\left[\frac{\lambda(B\varrho_i)}{\varrho_i^k} + \frac{\lambda(B\varrho_j)}{\varrho_j^k}\right]$$

for $A, B > 0$ and $l \in \mathbb{R}_+$, then from (9) we find that the sequence $\{I_k(\varrho_j;x,\mu'_R)\}_{j\in\mathbb{N}^*}$ is a Cauchy sequence for fixed x, k and R. Therefore for $k \geqslant p[\lambda]$ we put $Y_R^{(k)}(x) = -\lim\limits_{j\to\infty} I_k(\varrho_j;x,\mu'_R)$. By virtue of such choice of sequence $Y_R = \left\{Y_R^{(k)}(x)\right\}$, we have

$$\left|Y_R^{(k)}(x) + \int\limits_{|y|\leqslant r} C_k^\nu\left[\left(x,\frac{y}{|y|}\right)\right]\frac{d\mu'_R(y)}{|y|^{k+2\nu}}\right| = |I_k(r;x,\mu'_R) - I_k(r_k;x,\mu'_R)|$$

$$\leqslant A(k+1)^l\left[\frac{\lambda(Br_k)}{r_k^k} + \frac{\lambda(Br)}{r^k}\right] \leqslant 3A(k+1)^l\,\frac{\lambda(Br)}{r^k}$$

for $1 \leqslant k < p[\lambda]$ and

$$\left|Y_R^{(k)}(x) + \int\limits_{|y|\leqslant r} C_k^\nu\left[\left(x,\frac{y}{|y|}\right)\right]\frac{d\mu'_R(y)}{|y|^{k+2\nu}}\right| = \lim_{j\to\infty}|I_k(r;x,\mu'_R) - I_k(\varrho_j;x,\mu'_R)|$$

$$\leqslant A(k+1)^l\left[\frac{\lambda(Br)}{r^k} + \lim_{j\to\infty}\frac{\lambda(B\varrho_j)}{\varrho_j^k}\right] \leqslant A(k+1)^l\,\frac{\lambda(Br)}{r^k}$$

for $k \geqslant p[\lambda]$. Therefore

$$|c_k(x,r;Y_R,\mu'_R)| \leqslant 3A(k+1)^l\lambda(Br) + \frac{1}{r^{2\nu}}\left|\int\limits_{|y|\leqslant r}\left(\frac{|y|}{r}\right)^k C_k^\nu\left[\left(x,\frac{y}{|y|}\right)\right]d\mu'_R(y)\right|.$$

Applying inequality $n(r,\mu')/(2r^{2\nu}) \leqslant N(2r,\mu')$ (see [4]) to the last addend, we find

$$\frac{1}{r^{2\nu}}\left|\int\limits_{|y|\leqslant r}\left(\frac{|y|}{r}\right)^k C_k^\nu\left[\left(x,\frac{y}{|y|}\right)\right]d\mu'_R(y)\right| \leqslant \frac{1}{r^{2\nu}}C_k^\nu(1)\int\limits_{|y|\leqslant r}d\mu'_R(y)$$

$$= C_k^\nu(1)\,\frac{n(r,\mu'_R)}{r^{2\nu}} \leqslant 2\,C_k^\nu(1)\,N(2r,\mu'_R).$$

Hence, taking into account relation $C_k^\nu(1) = O\left(k^{2\nu-1}\right)$, $k\to\infty$ (see [9]) and uniform λ-admissibility of family of remainders $\{\mu'_R : R\in\Re\}$, we obtain estimate (3) for some possibly other constants A, B and $l = m-3$.

It remains to proof the relation (4). Since the integrals

$$\int\limits_{|y|\leqslant r} C_k^\nu\left[\left(x,\frac{y}{|y|}\right)\right]\frac{d\mu'_R(y)}{|y|^{k+2\nu}}, \qquad \int\limits_{|y|\leqslant r}|y|^k\, C_k^\nu\left[\left(x,\frac{y}{|y|}\right)\right]d\mu'_R(y)$$

are equal to zero for all $R>r$, then it is sufficient to show that $Y_R^{(k)}(x)\to 0$ as $\Re\ni R\to\infty$. The last is obvious from the definition of spherical harmonics $Y_R^{(k)}$ possibly with the exception of the case $k\geqslant p[\lambda]$, $p[\lambda]<\infty$. In this case by the inequation (3), we have $\left|Y_R^{(k)}(x)\,r^k\right| \leqslant A(k+1)^l\lambda(Br)$ for $r<R$ and therefore (lettinq $r\to R$)

$$\left|Y_R^{(k)}(x)\right| \leqslant A(k+1)^l\,\frac{\lambda(BR)}{R^k} \leqslant A(k+1)^l\frac{\lambda(BR)}{R^{p[\lambda]}}$$

since $k\geqslant p[\lambda]$.

From the construction of family $\Re$ (with $d=p[\lambda]$ in section III of the proof of statement 1) of Proposition) it follows that $\lim\limits_{\Re\ni R\to\infty}\lambda(BR)/R^{p[\lambda]}=0$, from which $\lim\limits_{\Re\ni R\to\infty}Y_R^{(k)}(x) = 0$. This completes the proof of Proposition.

Let $f\in L^1(S^m)$, then the series $\sum\limits_{k=0}^{\infty}Y^{(k)}(x;f)$ is called its Fourier-Laplace series. Here

$$Y^{(k)}(x;f) = a_1^{(k)}Y_1^{(k)}(x) + a_2^{(k)}Y_2^{(k)}(x) + \ldots + a_{\gamma_k}^{(k)}Y_{\gamma_k}^{(k)}(x),$$

$\{Y_1^{(k)}, Y_2^{(k)},\ldots,Y_{\gamma_k}^{(k)}\}$ is the orthonormal base, $a_i^{(k)} = (f, Y_i^{(k)}) \qquad (i=1,2,\ldots,\gamma_k)$.
In the case $m=2$ we have the trigonometric Fourier series.

Denote $u_r(x) = u(rx)$, $r>0$, $x\in S^m$.

Definition 2.8. [10] The functions

$$c_k(x,r;u) = Y^{(k)}(x;u_r) \quad (k\in\mathbb{Z}_+,\ x\in S^m)$$

are called the spherical harmonics associated with the function u.

Lemma 2.9. Let $\Re$ be an unbounded set of positive numbers, and let $\{g_R : R\in\Re\}$ $(g_R(0)=0)$ be a family of subharmonic functions such that

a) the Riesz mass distribution associated with the function g_R is equal to 0 in the ball $\bar{V}_R^m$;
b) $|\, c_k(x, r; g_R)| \leqslant A(k+1)^l \lambda(Br)$
for some positive constants $A, B,\ l \in \mathbb{R}_+$ and for all $r > 0,\ x \in S^m,\ k \in \mathbb{Z}_+,\ R \in \Re$;
c) $\lim\limits_{\Re \ni R \to \infty} c_k(x, r; g_R) = 0$
for all $k \in \mathbb{Z}_+,\ r > 0$ and $x \in S^m$.

Then $\lim\limits_{\Re \ni R \to \infty} g_R(y) = 0$ uniformly on compacts of $\mathbb{R}^m$.

Proof. Since by virtue of condition a) the function g_R is harmonic in the ball $\bar{V}_R^m$, we can apply Poisson-Jensen's formula [3, pp. 139–140] to it. For $r < r^* < R$ we obtain

$$g_R(rx) = \frac{(r^*)^{2\nu}}{\omega_m} \int\limits_{S^m} \frac{[(r^*)^2 - r^2]\, g_R(r^*\xi)\, dS(\xi)}{[(r^*)^2 - 2r^*r(x, \xi) + r^2]^{\nu+1}} \qquad (x \in S^m),$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the scalar product in $\mathbb{R}^m$. Expanding the Poisson integral in series in spherical harmonics (see [10]), we have

$$g_R(rx) = \sum_{k=0}^{\infty} \left(\frac{r}{r^*}\right)^k Y^{(k)}(x; (g_R)_{r^*}).$$

Choose $r^* = 2r$. Then $|g_R(rx)| \leqslant \sum\limits_{k=0}^{\infty} 2^{-k}\, |c_k(x, 2r; g_R)|$. For fixed $r > 0$ and $x \in S^m$ the series $\sum\limits_{k=0}^{\infty} 2^{-k}\, |c_k(x, 2r; g_R)|$ is functional one, defined on $\Re$. From condition b) by Weierstrass indication this series converges uniformly on $\Re$. Let $S(R)$ be its sum. Then

$$\lim_{R\to\infty} S(R) = \lim_{R\to\infty} \sum_{k=0}^{\infty} 2^{-k}\, |c_k(x, 2r; g_R)| = \sum_{k=0}^{\infty} 2^{-k} \lim_{R\to\infty} |c_k(x, 2r; g_R)| = 0.$$

Therefore we get $\lim\limits_{\Re \ni R \to \infty} g_R(rx) = 0$ uniformly in $r \leqslant r_0 < r^*$.

3 Proof of Theorem

Let $\mu = \mu_u$ and let $\mu', \Re, Y_R$ and $c_k(x, r; Y_R, \mu'_R)$ be such as in Proposition. By Theorem 1 from [4], in consequence of λ-admissibility of mass distribution μ', there exists function $u^* \in \Lambda_s$ whose Riesz mass distribution is μ'. We shall assume that $u^*(0) = 0$. According to Lemma 4 from [4], there are subharmonic functions v_R, with $v_R(0) = 0$, such that $c_k(x, r; v_R) = c_k(x, r; Y_R, \mu'_R)$ for all $r > 0,\ x \in S^m,\ k \in \mathbb{Z}_+$ and also $\mu_{v_R} = \mu'_R$ for every $R \in \Re$. Since the mass distributions μ'_R are λ-admissible, then by Theorem 1 from [4] functions v_R belong to the class Λ_s. Moreover, by Lemma 2 when $\Re \ni R \to \infty$ these functions tend to 0 uniformly on compacts of $\mathbb{R}^m$.

Put $u_R = u^* - v_R$ and $h = u^* - u$. It is obvious that the functions u_R and h are subharmonic and their Riesz mass distributions satisfy condition 1) of Theorem. Since

$(u+h)-u_R = u^* - u_R = v_R$, condition 2) is true. Condition 3) we obtain from [11, Theorem 1].

The last conditions of Theorem immediately follow from Proposition.

The well-known representation of subharmonic functions in $\mathbb{R}^m$ of finite order can be obtained as a corollary from Theorem, if we take $\lambda(r) = r^\beta$, $\beta > \varrho$, $[\beta] = q$ (with ϱ and q as in the introduction), $h \equiv 0$, $\Re = \{R : R \geqslant R_0\}$ at some $R_0 > 0$, and $u_R(y) = \int\limits_{|\zeta|\leqslant R} K(y;\zeta)\, d\mu(\zeta) + \Phi_R(y)$, where $K(y;\zeta) = -|y-\zeta|^{-2\nu}$, and $\Phi_R(y)$ is harmonic in y for every $R \in \Re$.

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