Central European Science Journals
w w w . c e s j . c o m

Central European Journal of Mathematics
Central European Science Journals
CEJM 2(4) 2004 573–583

On the existence of solutions for nonlinear impulsive periodic viable problems

Tiziana Cardinali[1]*, Raffaella Servadei[2]†

[1] *Department of Mathematics and Computer Science, University of Perugia,*
via Vanvitelli 1, Perugia 06123 Italy

[2] *Department of Mathematics, University of Rome 'Tor Vergata',*
via della Ricerca Scientifica, Roma 00133 Italy

Received 18 May 2004; accepted 8 August 2004

Abstract: In this paper we prove the existence of periodic solutions for nonlinear impulsive viable problems monitored by differential inclusions of the type $x^{'}(t) \in F(t, x(t)) + G(t, x(t))$. Our existence theorems extend, in a broad sense, some propositions proved in [10] and improve a result due to Hristova-Bainov in [13].

Keywords: impulsive periodic viable differential inclusions and equations, canonical domain, Bouligand contingent cone, lower semicontinuity and upper semicontinuity of set valued maps
MSC (2000): 34A37, 34A60, 34B15

---

1 Introduction

Impulsive differential equations, a new branch of the theory of ordinary differential equations, describe evolution processes which at a certain moment change their state rapidly. In the mathematical simulation of such processes it is convenient to assume that this change takes place momentarily and that the process changes its state by jump. Processes of such character are observed in numerous fields of science and technology: mechanics, population dynamics, theoretical physics, industrial robotics, pharmacokinetics, chemical technology, biotechnology, multiple-phase economic dynamics, stock management in production theory and so on. The qualitative investigation of impulsive differential equations began in 1960 with the work of Mil'man - Myshkis (see [16]). Some monographs related

---

* E-mail: tiziana@dipmat.unipg.it
† E-mail: servadei@mat.uniroma2.it

to this new subject have appeared (see Samoilenko - Perestyuk [17], Lakshmikantham - Bainov - Simeonov [15], Bainov - Simeonov [2, 3], Bainov - Covachev [1]). Recently, the attention has been given to impulsive differential inclusions and interesting results concerning the existence of solutions for Cauchy problems and for periodic problems were obtained (see, for instance, Benchohra - Boucherif [4, 5], Benchohra - Henderson - Ntouyas [7], Hristova - Bainov [13], Watson [18], Benchohra - Henderson - Ntouyas - Ouahabi [8, 9], Frigon - O'Regan [12]).

In this paper we study the existence of periodic solutions for some impulsive viable problems. In particular, in section 3, we consider the following problem:

$$(\mathcal{P})\begin{cases} x^{'}(t) \in F(t,x(t)) + G(t,x(t)) \text{ a.e. } & t \in [0,T] \setminus \{t_1,\dots,t_p\} \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } \ k \in \{1,\dots,p\} \\ x(0) = x(T), & \end{cases}$$

where $\overline{\Omega} \subset \mathbb{R}^N, N \geq 1$, is a canonical domain, $F, G : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N), \mathcal{P}(\mathbb{R}^N) := \{S : S \subseteq \mathbb{R}^N\}$, are set valued maps, $0 = t_0 < t_1 < \cdots < t_{p+1} = T$, $I_k : \overline{\Omega} \to \mathbb{R}^N$ is an impulse function for $k \in \{1,\dots,p\}$, and $x(t^+) = \lim\limits_{s \to t^+} x(s)$, and we want the solutions of the problem to remain in the fixed canonical domain $\overline{\Omega}$.

We get existence results for the impulsive viable problem $(\mathcal{P})$ (see Theorem 3.1 and Corollary 3.2) by using a result obtained in [10] (see Theorem 3.2).

We note that in [11] the authors have studied the impulsive problem $(\mathcal{P})$ considering the particular situation $N = 1$ and the not viable case. In this paper we study the viable problem in the N-dimensional case. We point out that in order to obtain that the solution of the impulsive problem $(\mathcal{P})$ satisfies the viability property (i.e. $x(t) \in \overline{\Omega}$ for any $t \in \overline{\Omega}$) it is necessary to require the Nagumo-type tangential condition.

In section 4 we obtain, as a consequence of Theorem 3.1 and Corollary 3.2, two existence theorems (see Corollaries 4.2 and 4.3) for the impulsive periodic viable problem without the perturbation G:

$$(\mathcal{F})\begin{cases} x^{'}(t) \in F(t,x(t)) & \text{a.e. } \ t \in [0,T] \setminus \{t_1,\dots,t_p\} \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } \ k \in \{1,\dots,p\} \\ x(0) = x(T). & \end{cases}$$

These results extend, in a broad sense, Theorems 3.1 and 3.2 of [10] (see Remark 4.6). We observe that, with the hypothesis of lower semicontinuity on F, in literature there are only results about the existence of solutions for impulsive problems without viability (see [8, 9]). Moreover, we remark that, in the field of single valued maps, from Theorem 3.1 we obtain Theorem 4.7. This proposition improves a result due to Hristova - Bainov (see [13], Theorem 2), because for us the single valued map f is not necessarily continuous on $[0,T] \times \overline{\Omega}$, but only continuous with respect to the second variable (see Remark 4.8). Moreover, we do not require a Lipschitz condition on f (see Remark 4.9).

2 Preliminaries

Let $\Omega \subset \mathbb{R}^N, N \geq 1$, be a domain with boundary $\partial\Omega$ and let us denote by $\overline{\Omega} = \Omega \cup \partial\Omega$ its closure and by *int* $\overline{\Omega}$ the interior of $\overline{\Omega}$. We say that $\overline{\Omega}$ is a canonical domain if Ω is bounded, convex and there exists a finite family of real valued continuously differentiable maps $\big\{\Phi_i\big\}_{i\in\{1,\dots,q\}}$ such that the following conditions hold

$(\Omega 1)$ $\overline{\Omega} = \bigcap\limits_{i=1}^{q} \Big\{ x \in \mathbb{R}^N : \quad \Phi_i(x) \leq 0 \Big\};$

$(\Omega 2)$ if there exist $x_0 \in \partial\Omega$ and $i \in \{1,\dots,q\}$ such that $\Phi_i(x_0) = 0$,

then $\nabla\Phi_i(x_0) \neq 0$ (see [13]).

Let us note that a compact interval of $\mathbb{R}$ is a canonical domain.
Let $\overline{\Omega}$ be a canonical domain defined by the family $\big\{\Phi_i\big\}_{i\in\{1,\dots,q\}}$ and let us define

$$\alpha(x) = \Big\{ i \in \{1,\dots,q\} : \ \Phi_i(x) = 0 \Big\}, \text{ for any } x \in \partial\Omega$$

and

$$P_{\overline{\Omega}}(x) = \begin{cases} \Big\{ y \in \mathbb{R}^N : \langle \nabla\Phi_i(x), \ y \rangle \leq 0, \forall i \in \alpha(x) \Big\} & \text{if } x \in \partial\Omega \\ \mathbb{R}^N & \text{if } \ x \in \Omega, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^N$.
We recall the notion of the Bouligand contingent cone to $\overline{\Omega}$ at $x \in \overline{\Omega}$

$$T_{\overline{\Omega}}(x) = \left\{ y \in \mathbb{R}^N : \liminf_{\lambda\to 0^+} \frac{\rho(x+\lambda y, \overline{\Omega})}{\lambda} = 0 \right\},$$

where $\rho(z, \overline{\Omega}) = \inf\limits_{v\in\overline{\Omega}} \parallel z - v \parallel$.
We note that $P_{\overline{\Omega}}(x) = T_{\overline{\Omega}}(x)$ for any $x \in \overline{\Omega}$ (see [18], Remark 1.3). Moreover, $T_{\overline{\Omega}}(x)$ is convex and closed for any $x \in \overline{\Omega}$ and $0 \in T_{\overline{\Omega}}(x)$ for any $x \in \overline{\Omega}$.
Let $F : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ be a set valued map. F is said to be lower semicontinuous (l.s.c.) at $(\bar{t}, \bar{x}) \in [0,T] \times \overline{\Omega}$ if, for any open set $A \subseteq \mathbb{R}^N$ such that $A \cap F(\bar{t}, \bar{x}) \neq \emptyset$, there exists a neighbourhood U of $(\bar{t}, \bar{x})$ such that $A \cap F(t,x) \neq \emptyset$ for any $(t,x) \in U \cap ([0,T] \times \overline{\Omega})$.
F is said to be upper semicontinuous (u.s.c.) at $(\bar{t}, \bar{x}) \in [0,T] \times \overline{\Omega}$ if, for any open set $A \subseteq \mathbb{R}^N$ such that $F(\bar{t}, \bar{x}) \subseteq A$, there exists a neighbourhood U of $(\bar{t}, \bar{x})$ such that $F(t,x) \subseteq A$, for any $(t,x) \in U \cap ([0,T] \times \overline{\Omega})$.

Now we give the definition of solution for the impulsive periodic viable problem $(\mathcal{P})$.

Definition 2.1. A solution of problem $(\mathcal{P})$ is a function $x : [0,T] \to \overline{\Omega}$, absolutely continuous in the closed interval $[0, t_1]$ and in the interval $]t_k, t_{k+1}]$ for any $k \in \{1,\dots,p\}$, such that

$x'(t) \in F(t, x(t)) + G(t, x(t))$ a.e. $t \in [0, T] \setminus \{t_1, \ldots, t_p\}$;

$x(t_k^+) = x(t_k) + I_k(x(t_k))$ for any $k \in \{1, \ldots, p\}$;

and $x(0) = x(T)$.

To obtain our existence results for problem $(\mathcal{P})$ we need the following theorem (see [10], Theorem 3.2):

Theorem 2.2. Let $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, be a canonical domain such that *int* $\overline{\Omega} \neq \emptyset$ and let $H : [0, T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ be a set valued map such that the following conditions hold

$(H1)$ $H(t, x)$ is nonempty, convex, closed a.e. on $[0, T]$, $\forall\, x \in \overline{\Omega}$;

$(H2)$ $H(t, \cdot)$ is u.s.c. in $\overline{\Omega}$, a.e. on $[0, T]$;

$(H3)$ $\exists\ (H_n)_{n \in \mathbb{N}}$, $H_n : [0, T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ such that

$(H3.1)$ $H_n(t, x)$ is nonempty, convex, closed, $\forall\, (t, x) \in [0, T] \times \overline{\Omega}$, $\forall\, n \in \mathbb{N}$;

$(H3.2)$ H_n is l.s.c. in $[0, T] \times \overline{\Omega}$, $\forall\, n \in \mathbb{N}$;

$(H3.3)$ $H_n(t, x) \cap T_{\overline{\Omega}}(x) \neq \emptyset$, $\forall\, (t, x) \in [0, T] \times \overline{\Omega}$, $\forall\, n \in \mathbb{N}$;

$(H3.4)$ $\exists\, \gamma \in L^\infty([0, T])$ such that $\| H_n(t, x) \| \leq \gamma(t)$ a.e. on $[0, T]$, $\forall\, x \in \overline{\Omega}$, $\forall\, n \in \mathbb{N}$;

$(H3.5)$ a.e. $t \in [0, T]$, $\forall\, \epsilon > 0\ \exists\, \bar{n} = \bar{n}(\epsilon, t) \in \mathbb{N}$ such that $H_n(t, x) \subseteq H(t, x) + B(0, \epsilon)$, $\forall n \geq \bar{n}$, $\forall\, x \in \overline{\Omega}$ (where $B(0, \epsilon)$ is the ball with center 0 and radius ϵ);

and let $I_k : \overline{\Omega} \to \mathbb{R}^N$, $k \in \{1, \ldots, p\}$, be an impulse function such that

$(I4)$ I_k is continuous in $\overline{\Omega}$;

$(I5)$ $x + I_k(x) \in \overline{\Omega}$, $\forall\, x \in \overline{\Omega}$.

Then, there exists a solution of the following impulsive periodic viable problem

$$(\mathcal{H}) \begin{cases} x'(t) \in H(t, x(t)) & \text{a.e. } t \in [0, T] \setminus \{t_1, \ldots, t_p\} \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } k \in \{1, \ldots, p\} \\ x(0) = x(T). \end{cases}$$

In [10] we prove Theorem 2.2 using an approximation argument together with a result due to Hristova-Bainov about the existence of a viable periodic solution of impulsive differential equations (see [13]).

3 Existence results for problem $(\mathcal{P})$

In this section we prove our existence results for the impulsive periodic viable problem $(\mathcal{P})$.

Theorem 3.1. Let $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, be a canonical domain such that $int\ \overline{\Omega} \neq \emptyset$ and let $F,\ G : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ be set valued maps such that the following conditions hold

$(F1)$ $\exists\ \theta : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ such that

$(F1.1)$ θ is l.s.c. in $[0,T] \times \overline{\Omega}$;

$(F1.2)$ $\overline{co}\ \theta(t,x) \subseteq F(t,x),\ \ \forall\ (t,x) \in [0,T] \times \overline{\Omega}$;

$(F1.3)$ $\theta(t,x) \subseteq T_{\overline{\Omega}}(x),\ \ \forall\ (t,x) \in [0,T] \times \overline{\Omega}$;

$(G1)$ $G(t,x)$ is nonempty, convex, compact a.e. on $[0,T]$, $\forall\ x \in \overline{\Omega}$;

$(G2)$ $G(t,\cdot)$ is u.s.c. in $\overline{\Omega}$, a.e. on $[0,T]$;

$(G3)$ $\exists\ (G_n)_{n\in\mathbb{N}},\ G_n : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ such that

$(G3.1)$ $G_n(t,x)$ is nonempty, convex, closed, $\forall\ (t,x) \in [0,T] \times \overline{\Omega},\ \forall\ n \in \mathbb{N}$;

$(G3.2)$ G_n is l.s.c. in $[0,T] \times \overline{\Omega},\ \forall\ n \in \mathbb{N}$;

$(G3.3)$ $G_n(t,x) \cap T_{\overline{\Omega}}(x) \neq \emptyset,\ \forall\ (t,x) \in [0,T] \times \overline{\Omega},\ \forall\ n \in \mathbb{N}$;

$(G3.4)$ $\exists\ \gamma \in L^\infty([0,T])$ such that $\| G_n(t,x) \| \leq \gamma(t)$

a.e. on $[0,T],\ \forall\ x \in \overline{\Omega},\ \forall\ n \in \mathbb{N}$;

$(G3.5)$ a.e. $t \in [0,T],\ \forall\ \epsilon > 0\ \exists\ \bar{n} = \bar{n}(\epsilon,t) \in \mathbb{N}$ such that

$G_n(t,x) \subseteq G(t,x) + B(0,\epsilon),\ \forall n \geq \bar{n},\ \forall\ x \in \overline{\Omega}$;

and let $I_k : \overline{\Omega} \to \mathbb{R}^N$, $k \in \{1,\dots,p\}$, be an impulse function such that

$(I4)$ I_k is continuous in $\overline{\Omega}$;

$(I5)$ $x + I_k(x) \in \overline{\Omega},\ \ \forall\ x \in \overline{\Omega}$.

Then, there exists a solution of the impulsive periodic viable problem $(\mathcal{P})$.

Proof: In order to prove the existence of a solution for problem $(\mathcal{P})$ we will introduce a suitable impulsive periodic viable problem and we will prove that it has a solution by means of Theorem 2.2.

First of all, let us note that the set valued map $\overline{co}\ \theta : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ is l.s.c. in $[0,T] \times \overline{\Omega}$ (see Proposition 2.42 of [14]).

Then, by Michael's Selection Theorem, there exists a continuous selection $h : [0,T] \times \overline{\Omega} \to \mathbb{R}^N$ for the set valued map $\overline{co}\ \theta$, that is

$$h(t,x) \in \overline{co}\ \theta(t,x),\quad \forall (t,x) \in [0,T] \times \overline{\Omega}. \tag{1}$$

Now, let us consider the problem $(\mathcal{H})$ defined on Theorem 2.2 with $H(t,x) = \{h(t,x)\} +$

$G(t,x), \forall (t,x) \in [0,T] \times \overline{\Omega}$ and let us verify that the set valued map H satisfies all the hypotheses of Theorem 2.2.

By $(G1)$, $H(t,x)$ is nonempty, convex and closed a.e. on $[0,T]$, $\forall\ x \in \overline{\Omega}$. Taking into account $(G1)$, $(G2)$ and the continuity of h, we can deduce that $H(t,\cdot)$ is u.s.c. in $\overline{\Omega}$ a.e. $t \in [0,T]$ (see Proposition 2.59 of [14]). Finally, let us consider the sequence of set valued maps $(H_n)_{n\in\mathbb{N}}$, where $H_n : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ is defined in this way:

$$H_n(t,x) = \{h(t,x)\} + G_n(t,x), \forall\ (t,x) \in [0,T] \times \overline{\Omega}, \forall\ n \in \mathbb{N}.$$

By $(G3.2)$ and the continuity of h, we can say that H_n is l.s.c. in $[0,T] \times \overline{\Omega}$, $\forall\ n \in \mathbb{N}$ (see again Proposition 2.59 of [14]). By (1) and $(F1.3)$, we have that

$$h(t,x) \in \overline{co}\ \theta(t,x) \subseteq T_{\overline{\Omega}}(x), \forall\ (t,x) \in [0,T] \times \overline{\Omega}. \tag{2}$$

Moreover, by $(G3.3)$, there exists $y_{n,t,x} \in G_n(t,x) \cap T_{\overline{\Omega}}(x), \forall\ (t,x) \in [0,T] \times \overline{\Omega}, \forall\ n \in \mathbb{N}$. So, by Propositions 5.7 and 5.32 of [14] and by (2), we have that

$$h(t,x) + y_{n,t,x} \in T_{\overline{\Omega}}(x), \forall\ (t,x) \in [0,T] \times \overline{\Omega}, \forall\ n \in \mathbb{N}.$$

Therefore, by definition of H_n, we have

$$H_n(t,x) \cap T_{\overline{\Omega}}(x) \neq \emptyset, \forall\ (t,x) \in [0,T] \times \overline{\Omega}, \forall\ n \in \mathbb{N}.$$

H_n also satisfies conditions $(H3.4)$ and $(H3.5)$, $\forall\ n \in \mathbb{N}$. This follows easily from the continuity of h and from the hypotheses $(G3.4)$ and $(G3.5)$. So, by Theorem 2.2, there exists a solution x for problem $(\mathcal{H})$. Since h is a selection for the set valued map F (see (1) and $(F1.2)$), the function x is also a solution of problem $(\mathcal{P})$.

By Theorem 3.1 we can easily deduce the following

Corollary 3.2. Let $\overline{\Omega} \subset \mathbb{R}^N, N \geq 1$, be a canonical domain such that $int\ \overline{\Omega} \neq \emptyset$ and let $F : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ be a set valued map such that the following conditions hold

$(F1)$ $\exists\ \theta : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ such that

- $(F1.1)$ θ is l.s.c. in $[0,T] \times \overline{\Omega}$;
- $(F1.2)$ $\overline{co}\ \theta(t,x) \subseteq F(t,x), \ \forall\ (t,x) \in [0,T] \times \overline{\Omega}$;
- $(F1.4)$ $\overline{co}\ \theta(t,x) \cap int\ T_{\overline{\Omega}}(x) \neq \emptyset, \ \forall\ (t,x) \in [0,T] \times \overline{\Omega}$.

Moreover, let $G : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ be a set valued map satisfying hypotheses $(G1), (G2)$ and $(G3)$ and let $I_k : \overline{\Omega} \to \mathbb{R}^N$, $k \in \{1,\ldots,p\}$, be an impulse function verifying conditions $(I4)$ and $(I5)$.

Then, there exists a solution of the impulsive periodic viable problem $(\mathcal{P})$.

Proof: Let us consider the set valued map $\psi : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ defined as follows

$$\psi(t,x) = \overline{co}\ \theta(t,x) \cap T_{\overline{\Omega}}(x), \ \forall\ (t,x) \in [0,T] \times \overline{\Omega}.$$

It is enough to prove that ψ verifies hypotheses $(F1.1)$, $(F1.2)$ and $(F1.3)$ of Theorem 3.1. As $T_{\overline{\Omega}}(x)$ is convex and closed for any $x \in \overline{\Omega}$ (see preliminaries), ψ has convex and closed values. Taking into account that $\overline{co}\ \theta(t,x) \subseteq F(t,x)$, $\forall\ (t,x) \in [0,T] \times \overline{\Omega}$, and the definition of ψ, we can say that ψ verifies hypotheses $(F1.2)$ and $(F1.3)$. Because θ satisfies property $(F1.4)$ and the set $\overline{\Omega}$ is nonempty, closed and convex, we have that ψ is l.s.c. in $[0,T] \times \overline{\Omega}$ (see Propositions 2.42, 5.35 and 2.54 of [14]).

Remark 3.3. Let us observe that Theorem 3.1 strictly contains Corollary 3.2. Indeed, we can consider the set valued map

$$F(t,x) = \{0\}, \quad \forall\ (t,x) \in [0,1] \times [0,1].$$

4 Applications

In this section we are interested in the following impulsive periodic viable problem

$$(\mathcal{F}) \begin{cases} x'(t) \in F(t,x(t)) & \text{a.e. } \ t \in [0,T] \setminus \{t_1,\ldots,t_p\} \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } \ k \in \{1,\ldots,p\} \\ x(0) = x(T), \end{cases}$$

where $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, is a canonical domain, $F : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ is a set valued map, $0 = t_0 < t_1 < \cdots < t_{p+1} = T$, $I_k : \overline{\Omega} \to \mathbb{R}^N$ is an impulse function for $k \in \{1,\ldots,p\}$, and $x(t^+) = \lim\limits_{s \to t^+} x(s)$.

Remark 4.1. Let us note that by Theorem 3.1 and by Corollary 3.2 we can deduce two existence theorems for the impulsive periodic viable problem $(\mathcal{F})$, by taking $G(t,x) = \{0\}, \forall\ (t,x) \in [0,T] \times \overline{\Omega}$.

If the set valued map F is l.s.c., from Theorem 3.1 and Corollary 3.2 we can deduce the following results about the existence of periodic solutions for the impulsive viable problem $(\mathcal{F})$:

Corollary 4.2. Let $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, be a canonical domain such that *int* $\overline{\Omega} \neq \emptyset$ and let $F : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ be a set valued map such that the following conditions hold

$(F2)$ $F(t,x)$ is nonempty, convex, closed, $\forall\ (t,x) \in [0,T] \times \overline{\Omega}$;

$(F3)$ F is l.s.c. in $[0,T] \times \overline{\Omega}$;

$(F4)$ $F(t,x) \subseteq T_{\overline{\Omega}}(x)$, $\forall\ (t,x) \in [0,T] \times \overline{\Omega}$;

and let $I_k : \overline{\Omega} \to \mathbb{R}^N$, $k \in \{1,\ldots,p\}$, be an impulse function satisfying hypotheses $(I4)$ and $(I5)$.

Then, there exists a solution of the impulsive periodic viable problem $(\mathcal{F})$.

Corollary 4.3. Let $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, be a canonical domain such that *int* $\overline{\Omega} \neq \emptyset$ and let $F : [0,T] \times \overline{\Omega} \to \mathcal{P}(\mathbb{R}^N)$ be a set valued map such that the following conditions hold

$(F2)$ $F(t,x)$ is nonempty, convex, closed, $\forall\ (t,x) \in [0,T] \times \overline{\Omega}$;

$(F3)$ F is l.s.c. in $[0,T] \times \overline{\Omega}$;

$(F5)$ $F(t,x) \cap int\ T_{\overline{\Omega}}(x) \neq \emptyset$, $\forall\ (t,x) \in [0,T] \times \overline{\Omega}$;

and let $I_k : \overline{\Omega} \to \mathbb{R}^N$, $k \in \{1,\dots,p\}$, be an impulse function satisfying hypotheses $(I4)$ and $(I5)$.

Then, there exists a solution of the impulsive periodic viable problem $(\mathcal{F})$.

Remark 4.4. We remark that none of the Corollaries 4.2 and 4.3 cover each other. Indeed, there exist set valued maps verifying hypotheses of Corollary 4.2, but not the conditions of Corollary 4.3. For example, we can consider the set valued map $F : [0,1] \times [0,1] \to \mathcal{P}(\mathbb{R})$ defined as follows

$$F(t,x) = \{0\}, \quad \forall\ (t,x) \in [0,1] \times [0,1].$$

Moreover, there exist set valued maps verifying hypotheses of Corollary 4.3, but not the conditions of Corollary 4.2. For example, let us consider the set valued map $F : [0,1] \times [0,1] \to \mathcal{P}(\mathbb{R})$ defined in this way

$$F(t,x) = [-1,1], \quad \forall\ (t,x) \in [0,1] \times [0,1].$$

Remark 4.5. Let us note that, if F is a continuous single valued map, then Corollary 4.2 contains Corollary 4.3.

Remark 4.6. We observe that Corollary 4.2 and Theorems 3.1 and 3.2 of [10] do not cover each other. Indeed there exist set valued maps verifying the hypotheses of Corollary 4.2, but not the conditions of Theorem 3.2 of [10] (and then of Theorem 3.1 of [10]). For example, we can consider the set valued map $F : [0,1] \times [0,1] \to \mathcal{P}(\mathbb{R})$ defined in this way

$$F(t,x) = \begin{cases} \{0\} & \text{if} \quad t \in [0,1], \quad x = 1 \\ [0,1] & \text{if} \quad t \in [0,1], \quad x \in [0,1[. \end{cases}$$

Moreover, there exist set valued maps verifying the hypotheses of Theorem 3.1 of [10] (and then of Theorem 3.2 of [10]), but not the conditions of Corollary 4.2. For example, we can consider the second set valued map defined as in Remark 4.4. Finally, we note that Corollary 4.3 and Theorems 3.1 and 3.2 of [10] do not cover each other. Indeed, the set valued map $F : [0,1] \times [0,1] \to \mathcal{P}(\mathbb{R})$ defined as follows

$$F(t,x) = \begin{cases} \{-1\} & \text{if} \quad t \in [0,1], \quad x = 1 \\ [-1,1] & \text{if} \quad t \in [0,1], \quad x \in [0,1[\end{cases}$$

verifies the conditions of Corollary 4.3, but not the hypotheses of Theorem 3.2 of [10] (and then of Theorem 3.1 of [10]). Moreover, the first set valued map F defined as in Remark 4.4 satisfies Theorem 3.1 of [10] (and then Theorem 3.2 of [10]), but not the conditions of Corollary 4.3.

In the field of single valued maps from Theorem 3.1 we obtain the following existence result.

Theorem 4.7. Let $\overline{\Omega} \subset \mathbb{R}^N, N \geq 1$, be a canonical domain such that *int* $\overline{\Omega} \neq \emptyset$ and let $f : [0,T] \times \overline{\Omega} \to \mathbb{R}^N$ be a single valued map such that the following conditions hold

$(f3)$ $\exists (f_n)_{n\in\mathbb{N}}, f_n : [0,T] \times \overline{\Omega} \to \mathbb{R}^N$such that

$(f3.2)$ f_n is continuous in $[0,T] \times \overline{\Omega}, \forall\ n \in \mathbb{N}$;

$(f3.3)$ $f_n(t,x) \in T_{\overline{\Omega}}(x),\ \forall\ (t,x) \in [0,T] \times \overline{\Omega},\ \forall\ n \in \mathbb{N}$;

$(f3.4)$ $\exists\ \gamma \in L^\infty([0,T])$ such that $\parallel f_n(t,x) \parallel \leq \gamma(t)$ a.e. on $[0,T],\ \forall\ x \in \overline{\Omega},\ \forall\ n \in \mathbb{N}$;

$(f3.5)$ a.e. $t \in [0,T],\ f_n(t,\cdot) \to f(t,\cdot)$ uniformly on $\overline{\Omega}$ as $n \to \infty$;

and let $I_k : \overline{\Omega} \to \mathbb{R}^N,\ k \in \{1,\ldots,p\}$, be an impulse function satisfying hypotheses $(I4)$ and $(I5)$.
Then there exists a solution of the impulsive periodic viable problem

$$(\mathcal{P}_f)\ \begin{cases} x'(t) = f(t,x(t)) & \text{a.e. } t \in [0,T] \setminus \{t_1,\ldots,t_p\} \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } k \in \{1,\ldots,p\} \\ x(0) = x(T). \end{cases}$$

Remark 4.8. Let us remark that hypotheses $(f3.2)$ and $(f3.5)$ imply only that $f(t,\cdot)$ is continuous in $\overline{\Omega}$ a.e. on $[0,T]$. If the single valued map f is continuous in $[0,T] \times \overline{\Omega}$ and verifies the property $f(t,x) \in T_{\overline{\Omega}}(x),\ \forall\ (t,x) \in [0,T] \times \overline{\Omega}$, then hypothesis $(f3)$ is trivially satisfied.

Remark 4.9. Theorem 4.7 improves a result due to Hristova - Bainov (see Theorem 2 of [13]). Indeed, in Theorem 4.7 f is not necessarily continuous on $[0,T] \times \overline{\Omega}$ (see Remark 4.8) and we do not require that f is a Lipschitz function with respect to the second variable. Moreover, hypothesis $(f3.3)$ weakens condition 3) in Theorem 2 of [13], being

$$\Big\{\ y \in \mathbb{R}^N :\ \langle\ \nabla\Phi_i(x),\ y\ \rangle < 0,\ \forall\ i \in \alpha(x)\ \Big\} = int\ T_{\overline{\Omega}}(x),$$

$\forall\ x \in \partial\overline{\Omega}, \forall\ t \in [0,T]$.

References

[1] D.D. Bainov, V. Covachev: *Impulsive differential equations with a small parameter*, World Scientific, Series on Advances in Math. for Applied Sciences, 1994.

[2] D.D. Bainov, P.S. Simeonov: *Systems with impulsive effect. Stability, theory and applications*, Ellis Horwood Series in Maths and Appl., Ellis Horwood, Chicester, 1989.

[3] D.D. Bainov, P.S. Simeonov: *Impulsive differential equations. Asymptotic properties of the solutions*, World Scientific, Series on Advances in Math. for Applied Sciences, 1995.

[4] M.Benchohra, A.Boucherif: "Initial value problems for impulsive differential inclusions of first order", *Diff. Eqns. Dyn. Syst.*, Vol. 8, (2000), pp. 51–66.

[5] M. Benchohra, A. Boucherif: "An existence result for first order initial value problems for impulsive differential inclusions in Banach spaces", *Arch. Math.*, Vol. 36, (2000), pp. 159–169.

[6] M. Benchohra, A. Boucherif, J.J. Nieto: "On initial value problems for a class of first order impulsive differential inclusions", *Disc. Math. Diff. Incl. Control Optim.*, Vol. 21, (2001), pp. 159–171.

[7] M. Benchohra, J. Henderson, S.K. Ntouyas: "On a periodic boundary value problem for first order impulsive differential inclusions", *Dyn. Syst. Appl.*, Vol. 10, (2001), pp. 477–488.

[8] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahabi: "Existence results for impulsive lower semicontinuous differential inclusions", *Int. J. Pure Appl. Math.*, Vol. 1, (2002), pp. 431–443.

[9] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahabi: "Existence results for impulsive functional and neutral functional differential inclusions with lower semicontinuous right hand side", *Electron. J. Math. Phys. Sci.*, Vol. 1, (2002), pp.72–91.

[10] T. Cardinali, R. Servadei: "Periodic solutions of nonlinear impulsive differential inclusions with constraints", *Proc. AMS*, Vol. 132, (2004), pp. 2339–2349.

[11] B.C. Dhage, A. Boucherif, S.K. Ntouyas: "On periodic boundary value problems of first-order perturbed impulsive differential inclusions", *Electron. J. Diff. Eqns.*, Vol. 84, (2004), pp. 1–9.

[12] M. Frigon, D. O'Regan: "Existence results for first order impulsive differential equations", *J. Math. Anal. Appl.*, Vol. 193, (1995), pp. 96–113.

[13] S.G. Hristova, D.D. Bainov: "Existence of periodic solutions of nonlinear systems of differential equations with impulse effect", *J. Math. Anal. Appl.*, Vol. 125, (1987), pp. 192–202.

[14] S. Hu, N.S. Papageorgiou: *Handbook of multivalued analysis*, Kluwer, Dordrecht, The Netherlands, 1997.

[15] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov: *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

[16] V.D. Mil'man, A.D.Myshkis: "On the stability of motion in the presence of impulses", *Siberian Math. J.*, Vol. 1, (1960), pp. 233–237.

[17] A.M. Samoilenko, N.A. Perestyuk: "Differential equations with impulse effect", Visca Skola, Kiev, 1987. [in Russian]

[18] P.J. Watson: "Impulsive differential inclusions", *Nonlin. World*, Vol. 4, (1997), pp. 395–402.