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Distinguished geodesics and Jacobi fields on first order jet spaces

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Abstract: In the framework of jet spaces endowed with a non-linear connection, the special curves of these spaces (h-paths, v-paths, stationary curves and geodesics) which extend the corresponding notions from Riemannian geometry are characterized. The main geometric objects and the paths are described and, in the case when the vertical metric is independent of fiber coordinates, the first two variations of energy and the extended Jacobi field equations are derived.

Keywords: jet space, nonlinear connection, Cartan connection, energy, geodesic, stationary curve, path, Jacobi field
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The geometrized framework on first and higher-order osculating spaces was introduced and widely studied by Acad. R.Miron and collaborators ([13]). As a complementary extension of the tangent (first-order osculating) framework, in the last decade, there was developed with significant results the geometric approach on first-order jet spaces ([18], [17], [3]).

1 Basic objects of the geometrized jet framework

Let $\xi = (E = J^1(T, M), \pi, T \times M)$ be the first order jet bundle of mappings $\varphi : T \to M$, where T and M are $\mathcal{C}^\infty$ real differentiable manifolds ($\dim T = m$, $\dim M = n$). The local

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jet coordinates on E will be denoted by

$$(t^\alpha, x^i, y^A)_{(\alpha,i,A)\in I_*} \equiv (y^\mu)_{\mu\in I},$$

where the set of indices I splits as follows

$$I = I_h \cup I_v, \quad I_h = I_{h_1} \cup I_{h_2}, \quad I_v = \overline{m+n+1, m+n+mn}$$

$$I_{h_1} = \overline{1,m}, \quad I_{h_2} = \overline{m+1, m+n}, \quad I_* = I_{h_1} \times I_{h_2} \times I_v.$$

and the indices implicitly take values as described below:

$$\alpha, \beta, \dots \in I_{h_1}; \quad i, j, \dots \in I_{h_2}; \quad A, B, \dots \in I_v; \quad \lambda, \mu, \dots \in I.$$

As well, when appropriate, we identify $A = m + n + n(i - m - 1) + \alpha$ as $A \equiv \binom{i}{\alpha}$ and denote $y^A \equiv x^{\binom{i}{\alpha}} = \frac{\partial x^i}{\partial t^\alpha}$.
We endow E with a *non-linear connection* $N = \{N^A_\mu\}_{\mu\in I_h, A\in I_v}$ which determines the local adapted basis of $\mathcal{X}(E)$, $\mathcal{B} = \{\delta_\alpha, \delta_i, \delta_A\}_{(\alpha,i,A)\in I_*} \equiv \{\delta_\mu\}_{\mu\in I}$, with $\partial_\alpha = \frac{\partial}{\partial t^\alpha}, \partial_i = \frac{\partial}{\partial x^i}$ and

$$\delta_\alpha = \partial_\alpha - N^A_\alpha \delta_A, \quad \delta_i = \partial_i - N^A_i \delta_A, \quad \delta_A = \partial_A = \frac{\partial}{\partial y^A}. \tag{1}$$

The dual basis of $\mathcal{B}$ writes then $\mathcal{B}^* = \{\delta^\alpha, \delta^i, \delta^A\}_{(\alpha,i,A)\in I_*} \equiv \{\delta^\mu\}_{\mu\in I}$, where

$$\delta^\alpha = dt^\alpha, \quad \delta^i = dx^i, \quad \delta^A \equiv \delta y^A = dy^A + N^A_\alpha dt^\alpha + N^A_i dx^i. \tag{2}$$

Any d-linear connection ([4, 6, 17]) $\nabla = \{L^\lambda_{\mu\nu}\}_{\lambda,\mu,\nu\in I}$ on E has its components relative to the adapted basis provided by the relations $\delta^\lambda(\nabla_{\delta_\nu}\delta_\mu) = L^\lambda_{\mu\nu}, \quad \forall \lambda, \mu, \nu \in I$. The coefficients of a linear connection are

$$\nabla \equiv \{L^\lambda_{\mu\nu}\} = \{L^\alpha_{\beta\gamma}, L^\alpha_{\beta k}, L^\alpha_{\beta C}, L^i_{j\gamma}, L^i_{jk}, L^i_{jC}, L^A_{B\gamma}, L^A_{Bk}, L^A_{BC}\}.$$

Among these connections which preserve the two hrizontal and vertical submodules of sections in $\mathcal{X}(E)$, one finds in the presence of a metrical structure on E the *Cartan linear connection*, which is metrical and satisfies the conditions ([18], [17]) $L^i_{j\gamma} = \frac{g^{ik}}{2}\partial_\gamma g_{jk}$, $L^i_{[jk]} = L^i_{[j\binom{k}{\alpha})} = 0$. We shall further consider the case when $h_{\alpha\beta}(t)$ and $g_{ij}(t,x)$ are two non-degenerate N-tensor fields of constant signature on T and M respectively, and hence we may endow E with a semi-Riemannian metric

$$G = \underbrace{h_{\alpha\beta}(t)dt^\alpha \otimes dt^\beta}_{h} + \underbrace{g_{ij}(t,x,y)dx^i \otimes dx^j}_{g} + \underbrace{\tilde{g}_{AB}(t,x,y)\delta y^A \otimes \delta y^B}_{\tilde{g}}, \tag{3}$$

where $\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t)g_{ij}(t,x,y)$.

The Cartan connection on (E, G) has then the coefficients

$$\overset{c}{\nabla} \equiv \{L^\lambda_{\mu\nu}\} = \{L^\alpha_{\beta\gamma}, 0, 0, L^i_{j\gamma}, L^i_{jk}, L^i_{jC}, L^A_{B\gamma}, L^A_{Bk}, L^A_{BC}\},$$

which are given by

$$L^{\alpha}_{\beta\gamma} = \left|{}^{\alpha}_{\beta\gamma}\right| = \tfrac{1}{2}h^{\alpha\varepsilon}(\delta_{\{\beta}h_{\varepsilon\}\gamma} - \delta_{\varepsilon}h_{\beta\gamma}),\ \ L^{i}_{jk} = \tfrac{1}{2}g^{il}(\delta_{\{k}g_{j\}l} - \delta_{l}g_{jk}),$$

$$L^{i}_{j\gamma} = \tfrac{1}{2}g^{ik}\delta_{\gamma}g_{kj},\ \ L^{i}_{j\binom{k}{\gamma}} = \tfrac{1}{2}g^{il}(\delta_{\binom{\{k}{\gamma}}g_{j\}l} - \delta_{\binom{l}{\gamma}}g_{jk}), \tag{4}$$

$$L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = \delta^{\beta}_{\alpha}L^{i}_{j\gamma} - \delta^{i}_{j}\left|{}^{\beta}_{\alpha\gamma}\right|,\ \ L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = \delta^{\beta}_{\alpha}\left|{}^{i}_{jk}\right|,\ \ L^{\binom{i}{\alpha}}_{\binom{j}{\beta}C} = \delta^{\beta}_{\alpha}L^{i}_{jC},\ \ L^{\alpha}_{\beta j} = L^{\alpha}_{\beta C} = 0.$$

The adapted components of the torsion $\mathcal{T}$ and of the curvature $\mathcal{R}$ of $\overset{c}{\nabla}$ are defined by the relations

$$\delta^{\lambda}(\mathcal{T}(\delta_{\nu},\delta_{\mu})) = T^{\lambda}_{\mu\nu},\ \ \delta^{\lambda}(\mathcal{R}(\delta_{\nu},\delta_{\mu})\delta_{\rho}) = R_{\rho\ \mu\nu}^{\ \lambda},\ \forall\ \lambda,\mu,\nu,\rho \in I.$$

Then the Cartan essential torsion coefficients are, for the case of g dependent on x only ([17], [18, Theorem 4.4])

$$\{T_{\gamma\ \binom{j}{\beta}}^{\ \binom{i}{\alpha}}, T_{k\ \binom{j}{\beta}}^{\ \binom{i}{\alpha}}, T_{\binom{j}{\beta}\ \binom{k}{\gamma}}^{\ \ \binom{i}{\alpha}}, T_{\beta\ j}^{\ i}, T^{i}_{jA}, T_{\beta\ \gamma}^{\ A}, T_{\beta\ j}^{\ A}, T_{i\ j}^{\ A}\}.$$

The five essential and three derived nontrivial sets of curvature N-tensor fields are respectively

$$\{R_{\beta\ \gamma\delta}^{\ \alpha},\ R_{j\ km}^{\ i},\ R_{j\ \gamma\lambda}^{\ i},\ R_{j\ \lambda A}^{\ i},\ R_{j\ CD}^{\ i}\},\ \ \{R_{B\ \gamma\delta}^{\ A}, R_{B\ \lambda k}^{\ A}, R_{B\ \mu C}^{\ A}\},$$

for $\lambda \in I_h,\ \mu \in I$.

We shall investigate especially the ARLS (almost Riemannian Lagrange separated) case where the coefficients g_{ij} depend only on x and g is a Riemannian metric on M; in this case the Cartan connection $\overset{*}{\nabla}$ has just four nontrivial sets of coefficients

$$\overset{*}{\nabla} \equiv \{L^{\lambda}_{\mu\nu}\} = \{L^{\alpha}_{\beta\gamma} = \left|{}^{\alpha}_{\beta\gamma}\right|, 0, 0, 0,\ L^{i}_{jk} = \left|{}^{i}_{jk}\right|\}, 0, L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = -\delta^{i}_{j}\left|{}^{\beta}_{\alpha\gamma}\right|, L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = \delta^{\beta}_{\alpha}\left|{}^{i}_{jk}\right|, 0\}$$

and we have (see the diagram below; [17])

$$T^{i}_{\beta k} = -L^{i}_{\beta k} = 0,\ T^{i}_{jC} = L^{i}_{jC} = 0,\ T^{A}_{BC} = T^{\binom{k}{\gamma}}_{\binom{i}{\alpha}\binom{j}{\beta}} = \delta^{\alpha}_{\gamma}C^{k}_{i\binom{j}{\beta}} - \delta^{\beta}_{\gamma}C^{k}_{j\binom{i}{\alpha}} = 0.$$

$\overset{c}{\nabla}$ for $h^{*}(t) \otimes g(t,x,y)$ $\qquad$ $\overset{*}{\nabla} := \overset{c}{\nabla}$ for $h^{*}(t) \otimes g(x)$

	h_T	h_M	v		h_T	h_M	v
$h_T h_T$	0	0	$T^{A}_{\beta\gamma}$	$h_T h_T$	0	0	$T^{A}_{\beta\gamma}$
$h_M h_T$	0	$T^{i}_{\beta k}$	$T^{A}_{\beta k}$	$h_M h_T$	0	0	$T^{A}_{\beta k}$
$h_M h_M$	0	0	T^{A}_{jk}	$h_M h_M$	0	0	T^{A}_{jk}
$v h_T$	0	0	$T^{A}_{\beta C}$	$v h_T$	0	0	$T^{A}_{\beta C}$
$v h_M$	0	T^{i}_{jC}	T^{A}_{jC}	$v h_M$	0	0	T^{A}_{jC}
vv	0	0	T^{A}_{BC}	vv	0	0	0

Table 1 The torsions of the Cartan connection.

2 Paths and stationary curves on $J^1(T,M)$

Consider on $\mathcal{JGL}^n = (E, \tilde{g})$ a fixed nonlinear (Cartan-Ehresmann) connection N, and let ∇ be a linear $d-$connection on E; we endow E with the metric G induced by two non-degenerate $d-$tensor fields $h \in T^0_2(T)$ and $g \in T^0_2(M)$. Let $c : J = [a,b] \subset \mathbb{R} \to E$ be a smooth curve, whose image lies in a chart $\tilde{U} \subset E$,

$$c(s) = (t^\alpha(s), x^i(s), y^A(s)) \equiv (y^\mu(s)), \forall t \in J.$$

Definition 2.1.
a) The field $\mathcal{V} = \frac{\delta y^\mu}{\mathrm{d}s}\delta_\mu$ defined on c will be called $d-$*velocity* field of the curve c. The components of $\mathcal{V}$ are explicitely given by

$$\{\mathcal{V}^\mu\}_{\mu \in I} \equiv \left(\dot{t}^\alpha, \dot{x}^i, \frac{\delta y^A}{\mathrm{d}s} = \dot{y}^A + N^A_\beta \dot{t}^\beta + N^A_j \dot{x}^j\right)_{(\alpha,i,A)\in I_*},$$

where we denote by dot the s-derivation. We have also denoted by $\mathcal{A} = \mathcal{A}^\mu \delta_\mu$, where

$$\mathcal{A}^\mu = \frac{\nabla \mathcal{V}^\mu}{ds} \overset{def}{=} \frac{\delta \mathcal{V}^\mu}{ds} + L^\mu_{\nu\rho}\mathcal{V}^\nu\mathcal{V}^\rho,$$

the $d-$*acceleration* on c, which provides the motion of the test-body along c.
b) We shall say that c is a *stationary curve* with respect to ∇ iff $\mathcal{A} = 0$ along the curve.
c) The curve c is called $h-$*curve*, if $\pi_v(\mathcal{V}) = 0$, and $v-$*curve*, if $\pi_h(\mathcal{V}) = 0$, where by π_h and π_v we denoted respectively the $h-$ and $v-$projectors of the canonic splitting induced by N. If a $h-/v-$curve satisfies also the extra condition $\mathcal{A} = 0$, then it is called $h-/v-$*path*, respectively.

Analytically, these curves are described by the following

Theorem 2.2. (Balan [4]) Let $c : J \subset \mathbb{R} \to E$ be a curve. Then the following hold true:
a) c is an $h-$curve iff $\mathcal{V}^A = 0$. The $h-$curve is an $h-$path iff it satisfies

$$\frac{\mathrm{d}\mathcal{V}^\mu}{\mathrm{d}s} + L^\mu_{\nu\rho}\mathcal{V}^\nu\mathcal{V}^\rho = 0,\ \forall \mu \in I_h. \tag{5}$$

b) c is a $v-$curve iff $\mathcal{V}^\mu = 0,\ \forall \mu \in I_h$. The $v-$curve is a $v-$path iff

$$\frac{\delta \mathcal{V}^A}{\mathrm{d}s} + L^A_{BC}\mathcal{V}^B\mathcal{V}^C = 0,\ \forall A \in I_v. \tag{6}$$

It should be mentioned that the implicit sum in the right term of (5) and (6) involves just horizontal/vertical index types, respectively. The particular uniparametric autonomous case of $\mathcal{JGL}^n$ provides the known corresponding paths from the tangent framework for Finslerian, Lagrange and Generalized Lagrange structures (see e.g. [14, 3, 9, 21]).

3 The first variation of energy. Geodesics in $\mathcal{JGL}^n$

We consider now the general case, and define *the geodesics* of $\mathcal{JGL}^n$ as the $\mathcal{C}^\infty$ extremals of the energy $\mathcal{E}$. Let G be a Riemannian metric on E given as in (3), N a nonlinear connection and let $\nabla = \overset{c}{\nabla}$ be the associated Cartan connection on (E, N, G).

To find the equations of geodesics, we consider a piecewise regular curve $c_\bullet : J = [a, b] \subset \mathbb{R} \to E$, smooth on the intervals $I_r = [s_r, s_{r+1}],\ r = \overline{0, k-1}$, where $a = s_0 < s_1 < \cdots < s_k = b$.

Consider as well a variation of $c_\bullet$ which is piecewise smooth on $I_r, r = \overline{0, k-1}$, given by

$$c : I_\varepsilon = (-\varepsilon, \varepsilon) \times [a, b],\ c(0, s) = c_\bullet(s) \equiv (c^\mu(s))_{\mu \in I},\ \forall s \in [a, b],$$

with fixed ends: $c(u, a) = c_\bullet(a),\ c(u, b) = c_\bullet(b),\ \forall u \in I_\varepsilon$. Denote $\omega(u) = c(u, \ \cdot\ ) : [a, b] \to E, \forall u \in I_\varepsilon$ and $c_s = \frac{\partial c}{\partial s}$, $c_u = \frac{\partial c}{\partial u}$,

$$\mathcal{V} = c_s|_{u=0} = \dot{c}_\bullet = \frac{dc_\bullet}{ds} = \frac{dc_\bullet^\mu}{ds}\partial_\mu = \frac{\delta c_\bullet^\mu}{ds}\delta_\mu;\ \ \mathcal{W} = c_u|_{u=0} = \frac{\delta c^\mu}{du}\ |_{u=0}\ \delta_\mu,$$

and let $\langle \cdot, \cdot \rangle$ be the metric bilinear form $G_{c_\bullet(s)}(\cdot, \cdot)$. The *energy* of the curve $\omega(u)$, is given by

$$\mathcal{E}(u) = \int_a^b \langle c_s, c_s \rangle ds,\ \ u \in I_\varepsilon.$$

Then we have the following

Theorem 3.1. ([7, 22, 23]) *The first variation of the energy is given by*

$$\frac{1}{2}\frac{d\mathcal{E}(u)}{du}\ |_{u=0} = -\sum_{r=1}^{k-1}\langle \mathcal{W}, \Delta_r \mathcal{V}\rangle + \int_a^b (\langle \mathcal{T}(\mathcal{W}, \mathcal{V}), \mathcal{V}\rangle - \langle \mathcal{W}, \nabla_{\mathcal{V}}\mathcal{V}\rangle) ds, \tag{7}$$

where $\Delta_r \mathcal{V} = \lim\limits_{s \searrow s_r} \mathcal{V}(s) - \lim\limits_{s \nearrow s_r} \mathcal{V}(s)$ *and* $\mathcal{T}$ *is the torsion of* ∇.

Proof. Denote $\partial_s = \frac{\partial}{\partial s}$, $\partial_u = \frac{\partial}{\partial u}$, $\nabla_s = \frac{\nabla}{\partial s}$, $\nabla_u = \frac{\nabla}{\partial u}$ and $\Delta_r c_s = \lim\limits_{s \searrow s_r} c_s(u, s) - \lim\limits_{s \nearrow s_r} c_s(u, s)$. Since ∇ is metrical, we have

$$\partial_s \langle c_s, c_u \rangle = \langle \nabla_s c_s, c_u \rangle + \langle c_s, \nabla_s c_u \rangle, \quad \partial_u \langle c_s, c_s \rangle = 2 \langle \nabla_u c_s, c_s \rangle.$$

As well, from $\dot{\partial}_{su} = \dot{\partial}_{us}$ we get $[c_u, c_s] = 0$, and hence $\nabla_u c_s = \mathcal{T}(c_u, c_s) + \nabla_s c_u$ and $\nabla_{\mathcal{W}}\mathcal{V} = \mathcal{T}(\mathcal{W}, \mathcal{V}) + \nabla_{\mathcal{V}}\mathcal{W}$. Then

$$\begin{aligned}
\frac{d\mathcal{E}(u)}{du} &= \tfrac{d}{du}\textstyle\int_a^b \langle c_s, c_s \rangle ds = 2\int_a^b \langle \nabla_u c_s, c_s \rangle ds = 2\int_a^b \langle \mathcal{T}(c_u, c_s) + \nabla_s c_u, c_s \rangle ds \\
&= 2\textstyle\int_a^b \langle \mathcal{T}(c_u, c_s), c_s) \rangle ds + 2\int_a^b (\partial_s \langle c_u, c_s \rangle - \langle c_u, \nabla_s c_s \rangle) ds \\
&= 2\textstyle\int_a^b \langle \mathcal{T}(c_u, c_s), c_s) \rangle ds + 2\displaystyle\sum_{r=0}^{k-1} \langle c_u, c_s \rangle\ |_{s_r}^{s_{r+1}} - 2\int_a^b \langle c_u, \nabla_s c_s \rangle ds \\
&= -2\langle c_u, c_s \rangle|\ |_a^b - 2\sum_{r=1}^{k-1} \langle c_u, \Delta_r c_s \rangle + 2\int_a^b (\langle \mathcal{T}(c_s, c_u), c_s \rangle - \langle c_u, \nabla_s c_s \rangle) ds,
\end{aligned}$$

where the scalar product is induced by the metric G at $c(u,s)$. Then, for $u=0$ replacing $c_s|_{u=0} = \mathcal{V}$, $c_u|_{u=0} = \mathcal{W}$, and using that $\mathcal{W}(a) = \mathcal{W}(b) = 0$, we get the relation (7). □

Remark 3.2.

1. It is known ([12, 11]) that in the case when the metrical connection ∇ is torsionless, then the condition $E(0) = 0$ satisfied for all the variations of $c_\bullet$ as above, implies that $c_\bullet$ is a geodesic of the metric space (i.e., minimizer of both the energy and length functionals, see [12]); in this case the geodesics are shown to be smooth curves, satisfying the condition

$$\nabla_{\mathcal{V}}\mathcal{V} = 0, \text{ where } \mathcal{V} = \dot{c}_\bullet. \tag{8}$$

Hence a natural extension in the jet framework is to define as *stationary curves* (or *d*-geodesics, [7]) of $\mathcal{JGL}^n$ the smooth autoparallel curves of ∇, i.e., which obey (8); these are the autoparallel curves of the Cartan connection. The second name is justified, since in the autonomous case for $m=1$ these project onto (proper) geodesics of M, provided that they are $h-$paths (Anastasiei and Bucătaru [1]).

2. Still, considering the field $\mathcal{F}$ defined by the equation $\langle \mathcal{T}(\mathcal{W},\mathcal{V}),\mathcal{V}\rangle = \langle \mathcal{W},\mathcal{F}\rangle$, the first variation becomes

$$\frac{1}{2}\frac{d\mathcal{E}(u)}{du}\,|_{u=0} = -\sum_{r=1}^{k-1}\langle \mathcal{W},\Delta_r\mathcal{V}\rangle + \int_a^b \langle \mathcal{W},\mathcal{F}-\nabla_{\mathcal{V}}\mathcal{V}\rangle ds, \tag{9}$$

and hence the proper *geodesics* of $\mathcal{JGL}^n$ are the smooth curves which satisfy the equations ([22, 23])

$$\frac{\nabla\mathcal{V}}{ds} = \mathcal{F}, \quad \text{with } \mathcal{F}^\mu = g^{\mu\rho} g_{\lambda\tau}\mathcal{V}^\nu\mathcal{V}^\tau\mathcal{T}^\lambda_{\nu\rho}, \tag{10}$$

where $g_{\lambda\tau} \in \{h_{\alpha\beta}, g_{ij}, \tilde{g}_{AB}\}$. We have

$$\mathcal{F}^\alpha = h^{\beta\alpha} g_{\lambda\tau}\mathcal{V}^\nu\mathcal{V}^\tau\mathcal{T}^\lambda_{\nu\beta} = h^{\beta\alpha}\{h_{\gamma\delta}\mathcal{V}^\nu\mathcal{V}^\delta\mathcal{T}^\gamma_{\nu\beta} + g_{kl}\mathcal{V}^\nu\mathcal{V}^l\mathcal{T}^k_{\nu\beta} + \tilde{g}_{CD}\mathcal{V}^\nu\mathcal{V}^D\mathcal{T}^C_{\nu\beta}\}$$

$$\mathcal{F}^i = g^{ji} g_{\lambda\tau}\mathcal{V}^\nu\mathcal{V}^\tau\mathcal{T}^\lambda_{\nu j} = g^{ji}\{h_{\gamma\delta}\mathcal{V}^\nu\mathcal{V}^\delta\mathcal{T}^\gamma_{\nu j} + g_{kl}\mathcal{V}^\nu\mathcal{V}^l\mathcal{T}^k_{\nu j} + \tilde{g}_{CD}\mathcal{V}^\nu\mathcal{V}^D\mathcal{T}^C_{\nu j}\}$$

$$\mathcal{F}^A = \tilde{g}^{BA} g_{\lambda\tau}\mathcal{V}^\nu\mathcal{V}^\tau\mathcal{T}^\lambda_{\nu B} = \tilde{g}^{BA}\{h_{\gamma\delta}\mathcal{V}^\nu\mathcal{V}^\delta\mathcal{T}^\gamma_{\nu B} + g_{kl}\mathcal{V}^\nu\mathcal{V}^l\mathcal{T}^k_{\nu B} + \tilde{g}_{CD}\mathcal{V}^\nu\mathcal{V}^D\mathcal{T}^C_{\nu B}\}.$$

We note that in the particular case of the Cartan connection $\overset{*}{\nabla}$, the only remainig nonzero terms of the torsion are just $\mathcal{T}^C_{\mu\rho}$.

4 Special geodesics

We subsequently consider the special curves of the $J^1(T,M)$-framework and denote with ∇ the Cartan connection $\overset{c}{\nabla}$.

I. h_T**-geodesics** ("temporal geodesics"), $x^i = x^i_0$ (=constant). Using $\mathcal{V}^i = \mathcal{V}^A = 0$, these are shown to satisfy:

$$\frac{d\mathcal{V}^\alpha}{ds} + L^\alpha_{\beta\gamma}\mathcal{V}^\beta\mathcal{V}^\gamma = h^{\beta\alpha}h_{\gamma\delta}\mathcal{V}^\varepsilon\mathcal{V}^\delta\mathcal{T}^\gamma_{\varepsilon\beta} \tag{11}$$

$$\mathcal{F}^i = 0 \;\Leftrightarrow\; h_{\gamma\delta}\mathcal{V}^\varepsilon\mathcal{V}^\delta\mathcal{T}^\gamma_{\varepsilon j} = 0 \tag{12}$$

$$\mathcal{F}^A = 0 \;\Leftrightarrow\; h_{\gamma\delta}\mathcal{V}^\varepsilon\mathcal{V}^\delta\mathcal{T}^\gamma_{\varepsilon B} = 0, \tag{13}$$

a system of $m+n+mn$ equations with $m+mn$ unknown components. In particular, for the Cartan connection $\overset{*}{\nabla}$, we have $\mathcal{T}^\gamma_{\varepsilon\beta} = \mathcal{T}^\gamma_{\varepsilon j} = \mathcal{T}^\gamma_{\varepsilon B} = 0$, i.e., the restrictions (12)-(13) are identically satisfied, and the equations (11) rewrite as

$$\frac{d\mathcal{V}^\alpha}{ds} + L^\alpha_{\beta\gamma}\mathcal{V}^\beta\mathcal{V}^\gamma = 0.$$

This proves that the following statements are equivalent:
1) c is an h_T-geodesic;
2) c is an h_T-autoparallel of $\overset{*}{\nabla}$.

Moreover, if h is a Riemannian metric on T, then $L^\alpha_{\beta\gamma} = \left|{}^{\alpha}_{\beta\gamma}\right|$ and consequently the statements 1) and 2) are equivalent also with 3) c is an h_T-curve which projects to a geodesic of T.

II. For h_M**-geodesics** ("spatial geodesics"), using $\mathcal{V}^\alpha = \mathcal{V}^A = 0$ ($\Rightarrow t^\alpha = t^\alpha_0$ - constant), we infer

$$\mathcal{F}^\alpha = 0,\ \frac{D\mathcal{V}^i}{ds} = \mathcal{F}^i,\ \mathcal{F}^A = 0,$$

which rewrite

$$g_{kl}\mathcal{V}^h\mathcal{V}^l\mathcal{T}^k_{h\beta} = 0,\quad g_{kl}\mathcal{V}^h\mathcal{V}^l\mathcal{T}^k_{hB} = 0 \tag{14}$$

$$\frac{d\mathcal{V}^i}{ds} + L^i_{jk}\mathcal{V}^j\mathcal{V}^k = g^{ji}g_{kl}\mathcal{V}^h\mathcal{V}^l\mathcal{T}^k_{hj}. \tag{15}$$

Example 4.1. For $\overset{*}{\nabla}$, we have $\mathcal{T}^k_{h\beta} = \mathcal{T}^k_{hj} = \mathcal{T}^k_{hB} = 0$, and hence, the restrictions (14) are identically satisfied by any curve on $\mathcal{JGL}^n$; this shows that for an h_M-curve, the following are equivalent:
1) c is a geodesic;
2) c is an autoparallel curve of $\overset{*}{\nabla}$;
3) c projects to a geodesic of the Riemannian manifold M.

III. For h**-geodesics**, we have $\mathcal{V}^A = 0$, whence

$$\tfrac{d\mathcal{V}^\alpha}{ds} + L^\alpha_{\beta\gamma}\mathcal{V}^\beta\mathcal{V}^\gamma + L^\alpha_{\beta k}\mathcal{V}^\beta\mathcal{V}^k = \mathcal{F}^\alpha$$

$$\tfrac{d\mathcal{V}^i}{ds} + L^i_{j\gamma}\mathcal{V}^j\mathcal{V}^\gamma + L^i_{jk}\mathcal{V}^j\mathcal{V}^k = \mathcal{F}^i,\quad \mathcal{F}^A = 0.$$

Example 4.2. For $\overset{*}{\nabla}$, the equations above (considering $\mathcal{V}^A = 0$) lead to

$$\tfrac{d\mathcal{V}^\alpha}{ds} + L^\alpha_{\beta\gamma}\mathcal{V}^\beta\mathcal{V}^\gamma + L^\alpha_{\beta k}\mathcal{V}^\beta\mathcal{V}^k = 0$$

$$\tfrac{d\mathcal{V}^i}{ds} + L^i_{j\gamma}\mathcal{V}^j\mathcal{V}^\gamma + L^i_{jk}\mathcal{V}^j\mathcal{V}^k = 0, \quad \mathcal{F}^A = 0.$$

Since $\mathcal{F}^A = \tilde{g}^{AE}\tilde{g}_{CD}\mathcal{V}^\sigma\mathcal{V}^D\mathcal{T}^C_{\sigma E}$, it follows that for horizontal curves the condition $F^A = 0$ is identically satisfied, i.e., c is a horizontal geodesic if c is a horizontal autoparallel curve of $\overset{*}{\nabla}$.

IV. The v-**geodesics**: ($\mathcal{V}^\alpha = \mathcal{V}^i = 0 \Rightarrow t^\alpha, x^i$ - constant) satisfy the system with the unknown components y^A

$$\mathcal{F}^\alpha = 0, \quad \mathcal{F}^i = 0, \quad \frac{\delta\mathcal{V}^A}{ds} + L^A_{BC}\mathcal{V}^B\mathcal{V}^C = \mathcal{F}^A,$$

which lead to

$$\tilde{g}_{CD}\mathcal{V}^E\mathcal{V}^D\mathcal{T}^C_{E\beta} = 0, \quad \tilde{g}_{CD}\mathcal{V}^E\mathcal{V}^D\mathcal{T}^C_{Ej} = 0 \tag{16}$$

$$\frac{\delta\mathcal{V}^A}{ds} + L^A_{BC}\mathcal{V}^B\mathcal{V}^C = \tilde{g}^{BA}\tilde{g}_{CD}\mathcal{V}^E\mathcal{V}^D\mathcal{T}^C_{EB}. \tag{17}$$

For $\overset{*}{\nabla}$, we have $\mathcal{T}^C_{EB} = 0$, and hence the v-geodesics (in case these exist) are those v-paths of $\overset{*}{\nabla}$ which obey the conditions (16).

5 The second variation of the energy. Jacobi fields

In the study of geodesics, an important tool for locating conjugate points along geodesics and describing *geodesic* variations are the Jacobi fields. We define an alternative to [7] analogous notion for Jacobi fields in the $d-$framework on $\mathcal{JGL}^n$, emerging from the second variation of the energy functional (integral of the square of arc-length, [12, 14]).

Consider $\mathcal{JGL}^n$ endowed with a nonlinear connection and the Cartan connection ∇. Let $c_\bullet$ be a $d-$*geodesic* and $c : I_\varepsilon \times I_\varepsilon \times [a, b]$ a piecewise variation with two parameters of $c_\bullet$, satisfying similar conditions as the variation c in the Theorem above. Denote $\mathcal{W}_i = c_{u_i}|_{(u_1,u_2)=(0,0)}$, $i = \overline{1, 2}$.

Then we have the following

Theorem 5.1. The second variation of the energy (the Hessian) is given by

$$\begin{aligned}\frac{1}{2}\frac{\partial^2\mathcal{E}(u_1, u_2)}{\partial u_1\partial u_2}\ |_{(u_1,u_2)=(0,0)} &= -\sum_{r=1}^{k-1}\langle\mathcal{W}_2, \Delta_r(\mathcal{T}(\mathcal{W}_1, \mathcal{V}) + \nabla_\mathcal{V}\mathcal{W}_1)\rangle \\ &+ \textstyle\int_a^b\langle\mathcal{W}_2, \nabla_{\mathcal{W}_1}\mathcal{F} - \nabla^2_\mathcal{V}\mathcal{W}_1 - \mathcal{R}(\mathcal{W}_1, \mathcal{V})\mathcal{V} - \nabla_\mathcal{V}\mathcal{T}(\mathcal{W}_1, \mathcal{V})\rangle ds.\end{aligned} \tag{18}$$

Proof. We denote $c_{u_i} = \frac{\partial c}{\partial u_i}$, $i = \overline{1,2}$ and $u = (u_1, u_2)$. Then

$$\frac{1}{2}\frac{\partial \mathcal{E}(u_1,u_2)}{\partial u_2} = \int_a^b \langle \mathcal{T}(c_{u_2}, c_s), c_s\rangle ds - \sum_{r=1}^{k-1}\langle c_{u_2}, \Delta_r c_s\rangle - \int_a^b \langle c_{u_2}, \nabla_s c_s\rangle ds,$$

which rewrite, for $\langle \mathcal{T}(\mathcal{W},\mathcal{V})\mathcal{V}\rangle = \langle \mathcal{W}, \mathcal{F}\rangle$

$$\frac{1}{2}\frac{\partial \mathcal{E}(u_1,u_2)}{\partial u_2} = \int_a^b \langle c_{u_2}, \mathcal{F} - \nabla_s c_s\rangle ds - \sum_{r=1}^{k-1}\langle c_{u_2}, \Delta_r c_s\rangle,$$

and hence

$$\frac{1}{2}\frac{\partial^2 \mathcal{E}(u_1,u_2)}{\partial u_1 \partial u_2} = -\sum_{r=1}^{k-1}(\langle \nabla_{u_1} c_{u_2}, \Delta_r c_s\rangle + \langle c_{u_2}, \nabla_{u_1}\Delta_r c_s\rangle) +$$

$$+ \int_a^b (\langle \nabla_{u_1} c_{u_2}, \mathcal{F} - \nabla_s c_s\rangle + \langle c_{u_2}, \nabla_{u_1}\mathcal{F} - \nabla_{u_1}\nabla_s c_s\rangle) ds,$$

where $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}^\mu \delta_\mu$ is the vector field given by the relation $\langle \mathcal{T}(c_{u_2}, c_s), c_s\rangle = \langle \mathcal{F}, c_{u_2}\rangle$, or locally,

$$\tilde{\mathcal{F}}^\mu = g^{\mu\rho} g_{\lambda\tau} \mathcal{T}^\lambda_{\nu\rho} c_s^\nu c_s^\tau \mid_{(u_1,u_2,s)}.$$

Then we have $\tilde{\mathcal{F}}(0,0,s) = \mathcal{F}(s)$ and $\Delta_r c_s|_{u=(0,0)} = \Delta_r \mathcal{V}$, $r = \overline{1, k-1}$, since $c_\bullet$ is $\mathcal{C}^1$ on $[a,b]$. As well, $c_\bullet$ being a $d-$geodesic implies $\nabla_s c_s|_{u=(0,0)} = \nabla_\mathcal{V}\mathcal{V} = \mathcal{F}$, and we obtain

$$\frac{1}{2}\frac{\partial^2 \mathcal{E}(u_1,u_2)}{\partial u_1 \partial u_2}\mid_{u=(0,0)} = -\sum_{r=1}^{k-1}\langle \mathcal{W}_2, \Delta_r \nabla_{\mathcal{W}_1}\mathcal{V}\rangle + \int_a^b \langle \mathcal{W}_2, \nabla_{\mathcal{W}_1}\mathcal{F} - \nabla_{\mathcal{W}_1}\nabla_\mathcal{V}\mathcal{V}\rangle ds. \qquad (19)$$

Since $\nabla_{\mathcal{W}_1}\mathcal{V} = \mathcal{T}(\mathcal{W}_1, \mathcal{V}) + \nabla_\mathcal{V}\mathcal{W}_1$ and

$$\nabla_{\mathcal{W}_1}\nabla_\mathcal{V}\mathcal{V} = \mathcal{R}(\mathcal{W}_1,\mathcal{V})\mathcal{V} + \nabla_\mathcal{V}(\underbrace{\mathcal{T}(\mathcal{W}_1,\mathcal{V}) + \nabla_\mathcal{V}\mathcal{W}_1}_{\nabla_{\mathcal{W}_1}\mathcal{V}}) + \nabla_{[\mathcal{W}_1,\mathcal{V}]}\mathcal{V},$$

where the last term cancels on $c_\bullet$, the last term in (19) becomes

$$\int_a^b \langle \mathcal{W}_2, \nabla_{\mathcal{W}_1}\mathcal{F} - \nabla^2_\mathcal{V}\mathcal{W}_1 - \mathcal{R}(\mathcal{W}_1,\mathcal{V})\mathcal{V} - \nabla_\mathcal{V}\mathcal{T}(\mathcal{W}_1,\mathcal{V})\rangle ds,$$

which plugged in (19) leads to (18). □

The theorem suggests the following natural generalization of the concept of Jacobi field for the $d-$framework.

Definition 5.2. A $d-$vector field J on E is called $d-$*Jacobi field* if it satisfies the equation

$$\nabla_J \mathcal{F} = \nabla^2_\mathcal{V} J + \mathcal{R}(J,\mathcal{V})\mathcal{V} + \nabla_\mathcal{V}\mathcal{T}(J,\mathcal{V}). \qquad (20)$$

Locally, this equation rewrites

$$\frac{\nabla^2 J^\mu}{ds^2} + \frac{\nabla \mathcal{T}^\mu}{ds} = \mathcal{R}^\mu + J^\lambda \nabla_{\delta_\lambda}\mathcal{F}^\mu, \mu \in I_h \cup I_v,$$

where we denote $\mathcal{T}^{\mu} = \mathcal{V}^{\lambda} J^{\sigma} \mathcal{T}^{\mu}_{\lambda\sigma}$, $\mathcal{R}^{\mu} = -\mathcal{V}^{\rho} \mathcal{V}^{\lambda} J^{\sigma} \mathcal{R}^{\mu}_{\rho\lambda\sigma}$.

Remarks. For $\nabla_J \mathcal{F} = 0$, the autonomous $\mathcal{JGL}^n$ case for $m = 1$ leads in particular to the extended concept of Jacobi field proposed in the GL^n framework by Anastasiei and Bucătaru ([2]). As well, we note that in the Riemannian case, the $h-part$ of a $d-$Jacobi field coincides with the classical one ([12, 10, 8, 20]).

Let $B(\mathcal{W}_1, \mathcal{W}_2) = \langle \nabla_{\mathcal{V}} \mathcal{W}_1, \mathcal{W}_2 \rangle - \langle \mathcal{W}_1, \nabla_{\mathcal{V}} \mathcal{W}_2 \rangle$; then

$$B(\mathcal{W}_1, \mathcal{W}_2) = -B(\mathcal{W}_2, \mathcal{W}_1), \forall \mathcal{W}_1, \mathcal{W}_2 \in \mathcal{X}(E).$$

Hence being skew-symmetric, B defines a pre-symplectic structure on the set of Jacobi fields along the geodesics of E, as in the classical case ([10, 20]).

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