Computational details – appendix to the article
"Cartan matrices of selfinjective algebras
of tubular type"

Jerzy Białkowski

1 Reflection sequences of tubular algebras.

In this section we present full lists of reflection sequences of tubular algebras of the types $(3, 3, 3)$ and $(2, 4, 4)$, having a nontrivial rigid automorphism. The lists were generated by a computer program. First, we created the files: tubular333.ps and tubular244.ps (downloadable from *http://www.mat.uni.torun.pl/˜jb/en/research/tubular/*, 80+551 pages, 1.1+15 MB) containing the full list of tubular algebras of the types $(3, 3, 3)$ and $(2, 4, 4)$ respectively. The output generated by the program includes:

- the full list of all tubular algebras of the types $(3, 3, 3)$ and $(2, 4, 4)$ respectively (the program uses the Bongartz-Happel-Vossieck list, [2],[3], of tame concealed algebras and tubular extensions of such algebras in the sense of [4]),
- the reflection equivalence classes of tubular algebras of the type $(3, 3, 3)$ (respectively, of the type $(2, 4, 4)$),
- nontrivial rigid automorphisms of the repetitive algebras from the pairwise nonequivalent reflection classes of tubular algebras of the type $(3, 3, 3)$ (respectively, of the type $(2, 4, 4)$).

The main purpose of presenting those lists is to complete the proof of [1, Proposition 3.3]. This is done by observing that:

- All the sequences, excluding seq 32 for the type $(3, 3, 3)$ and seq 40 and seq 41 (being opposite to each other) for the type $(2, 4, 4)$, contain an algebra which is a tubular extension of some hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_n$ or $\widetilde{\mathbb{D}}_n$.
- The algebra K 19 in the sequence seq 32 for the type $(3, 3, 3)$ is isomorphic to B_3. Likewise, the algebras K 588 and A 582 from the sequences seq 40 and seq 41 for the type $(2, 4, 4)$, are respectively isomorphic to the algebras B_{17} and B_{17}^{op}.
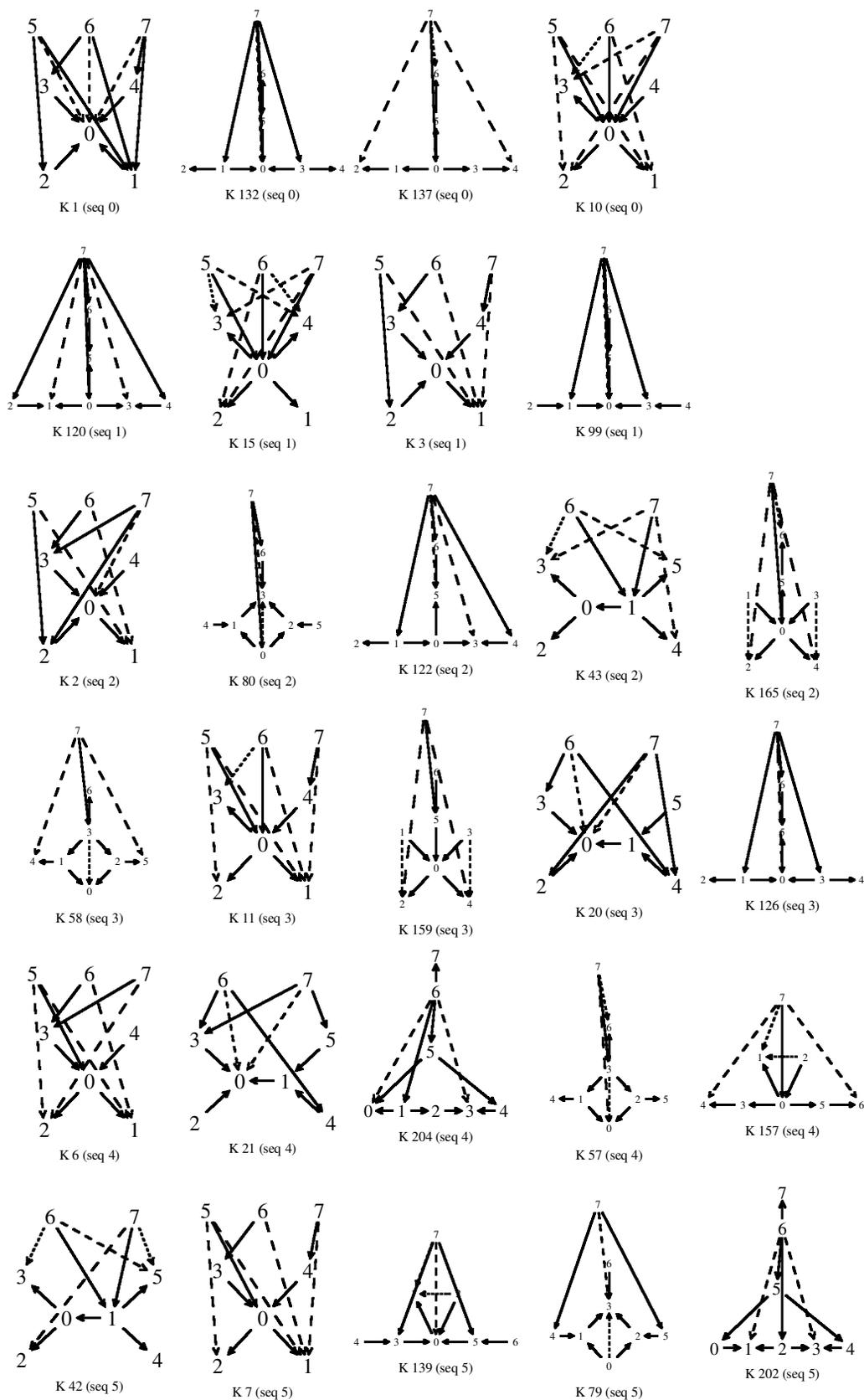
Fig. 1 Reflection sequences of the tubular algebras having a nontrivial rigid authomorphism of type (333).
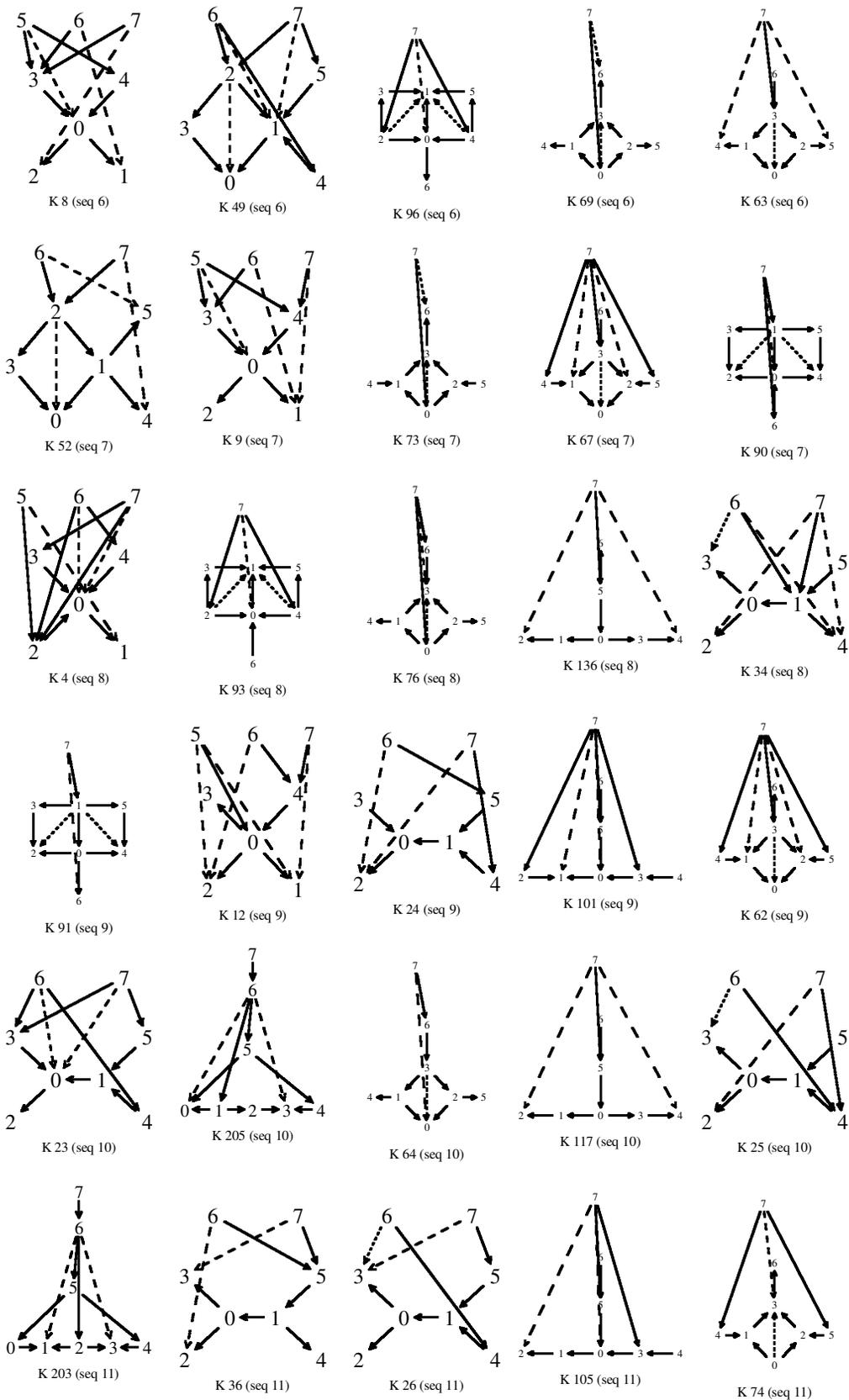
Fig. 2 Reflection sequences of the tubular algebras having a nontrivial rigid authomorphism of type (333).

Fig. 3 Reflection sequences of the tubular algebras having a nontrivial rigid authomorphism of type (333).

Fig. 4 Reflection sequences of the tubular algebras having a nontrivial rigid authomorphism of type (333).

Fig. 5 Reflection sequences of the tubular algebras having a nontrivial rigid authomorphism of type (333).

Fig. 6 Reflection sequences of the tubular algebras having a nontrivial rigid authomorphism of type (333).
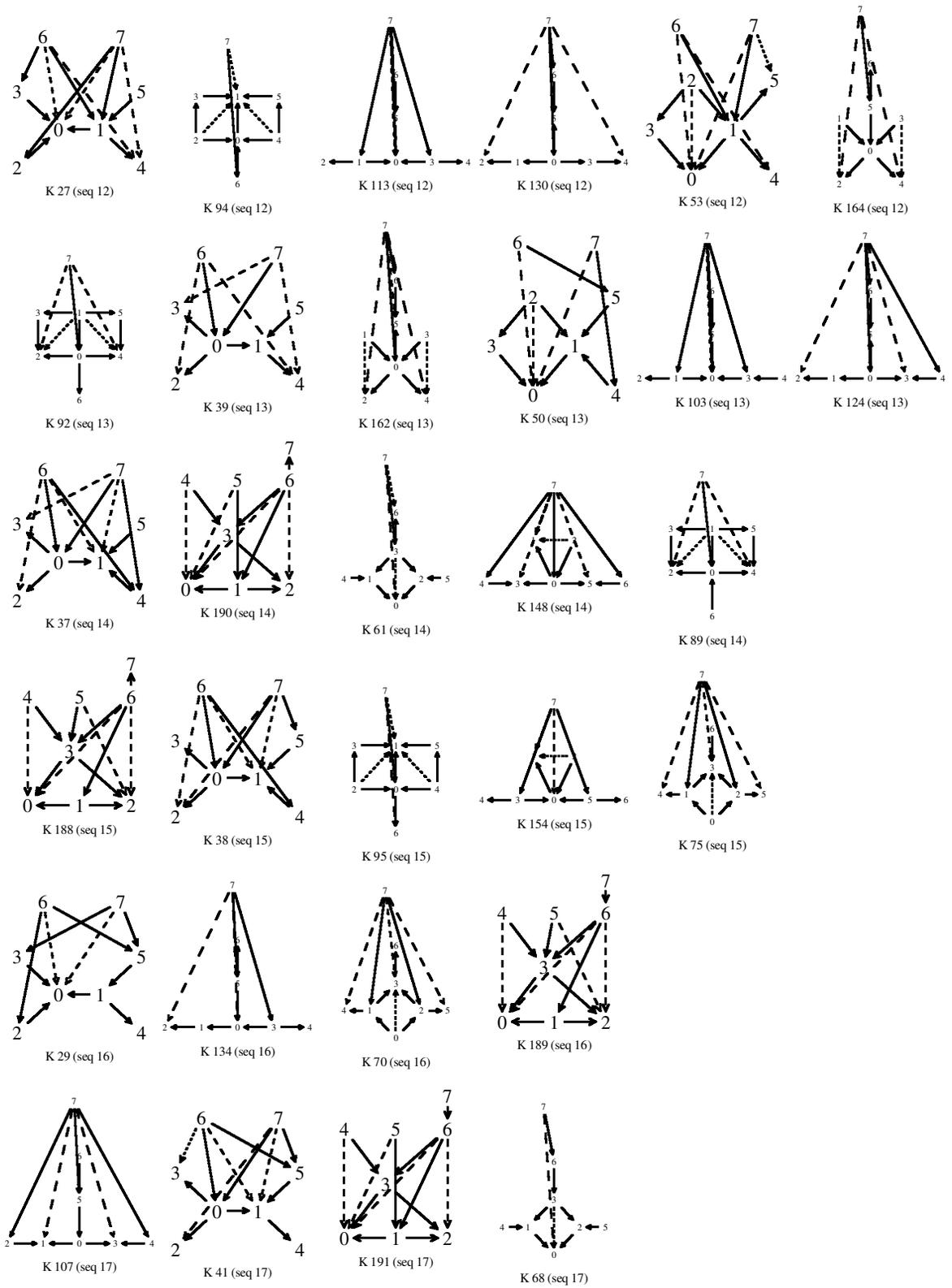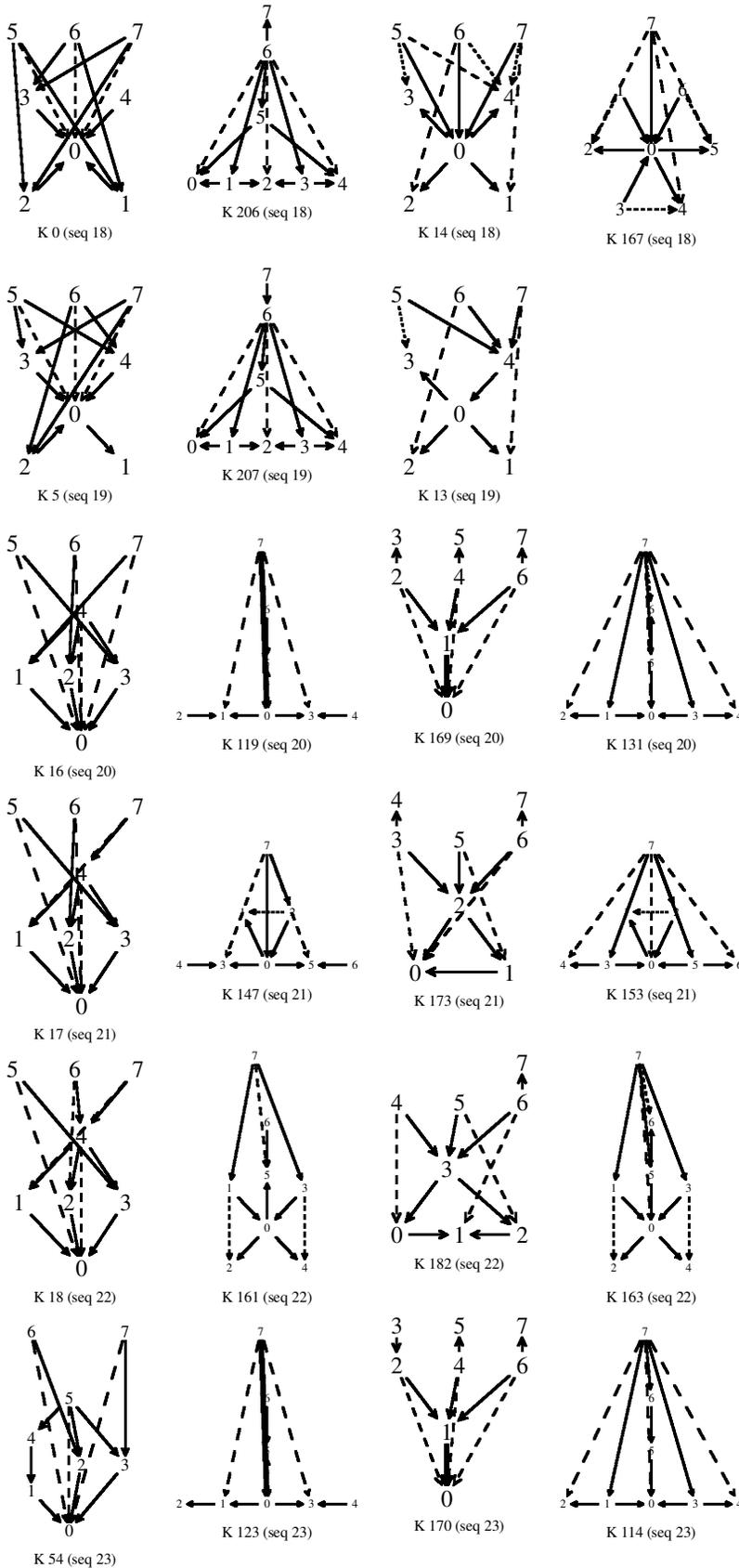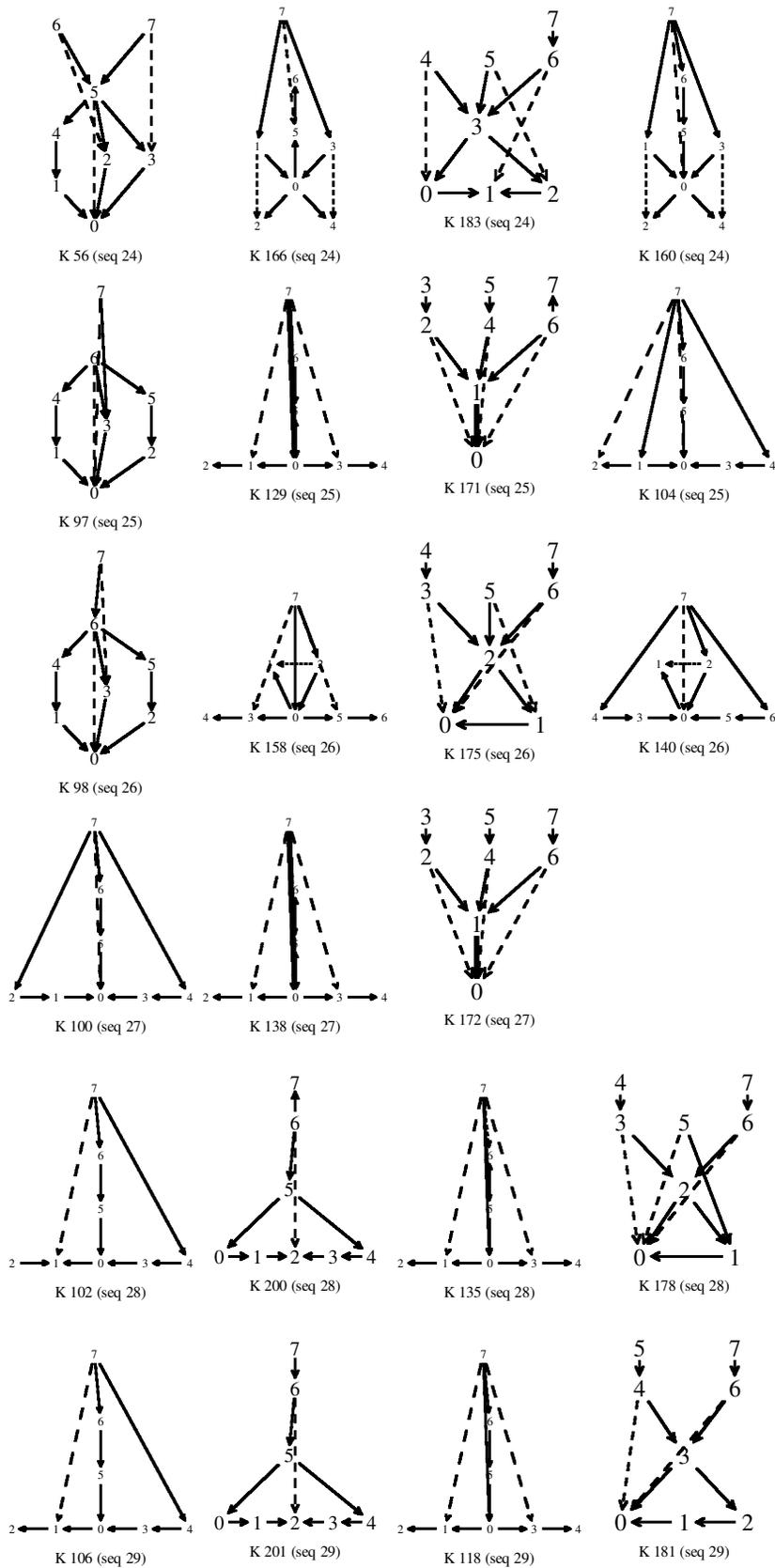
Fig. 7 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).

Fig. 8 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).
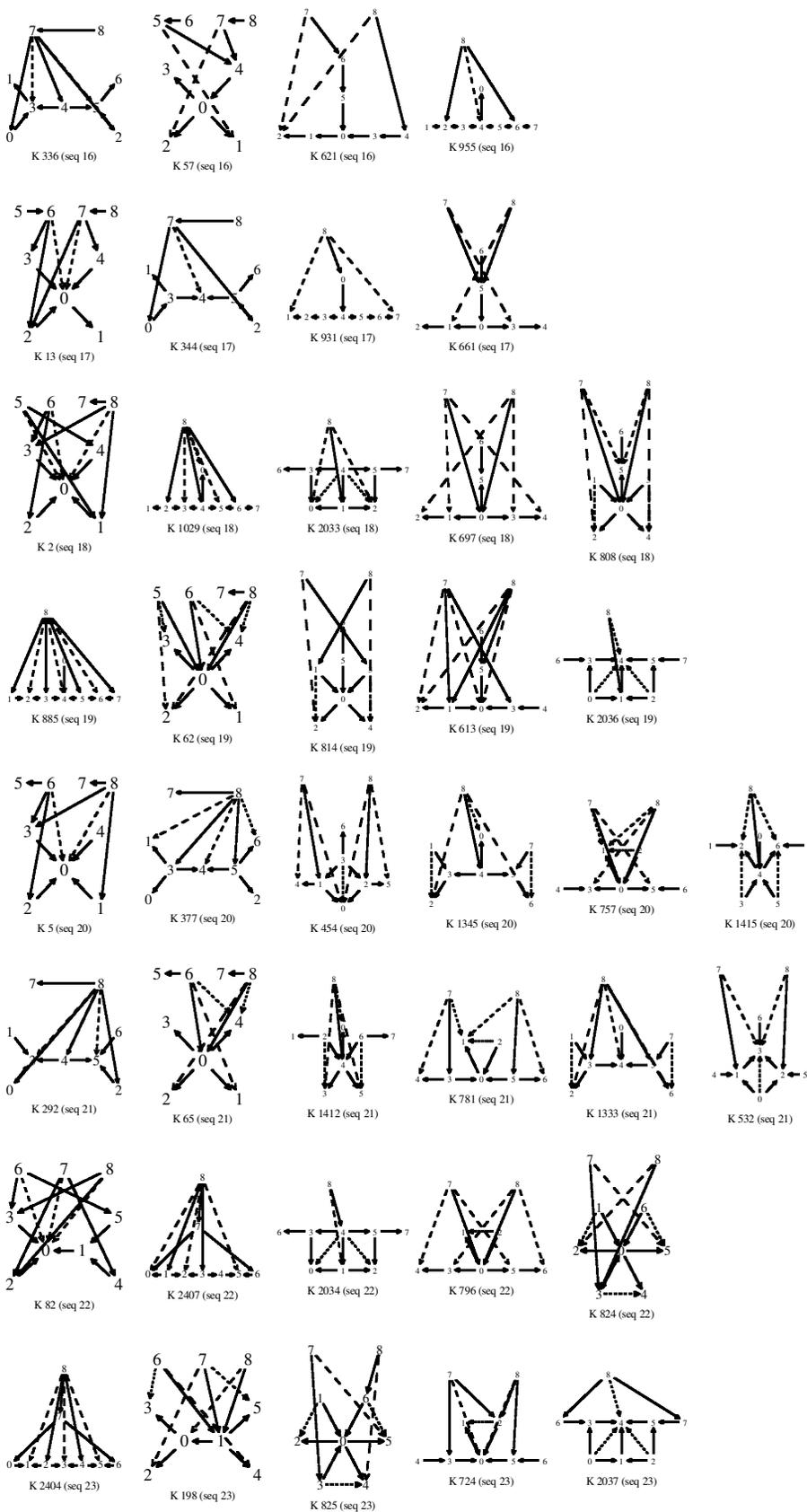
Fig. 9 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).
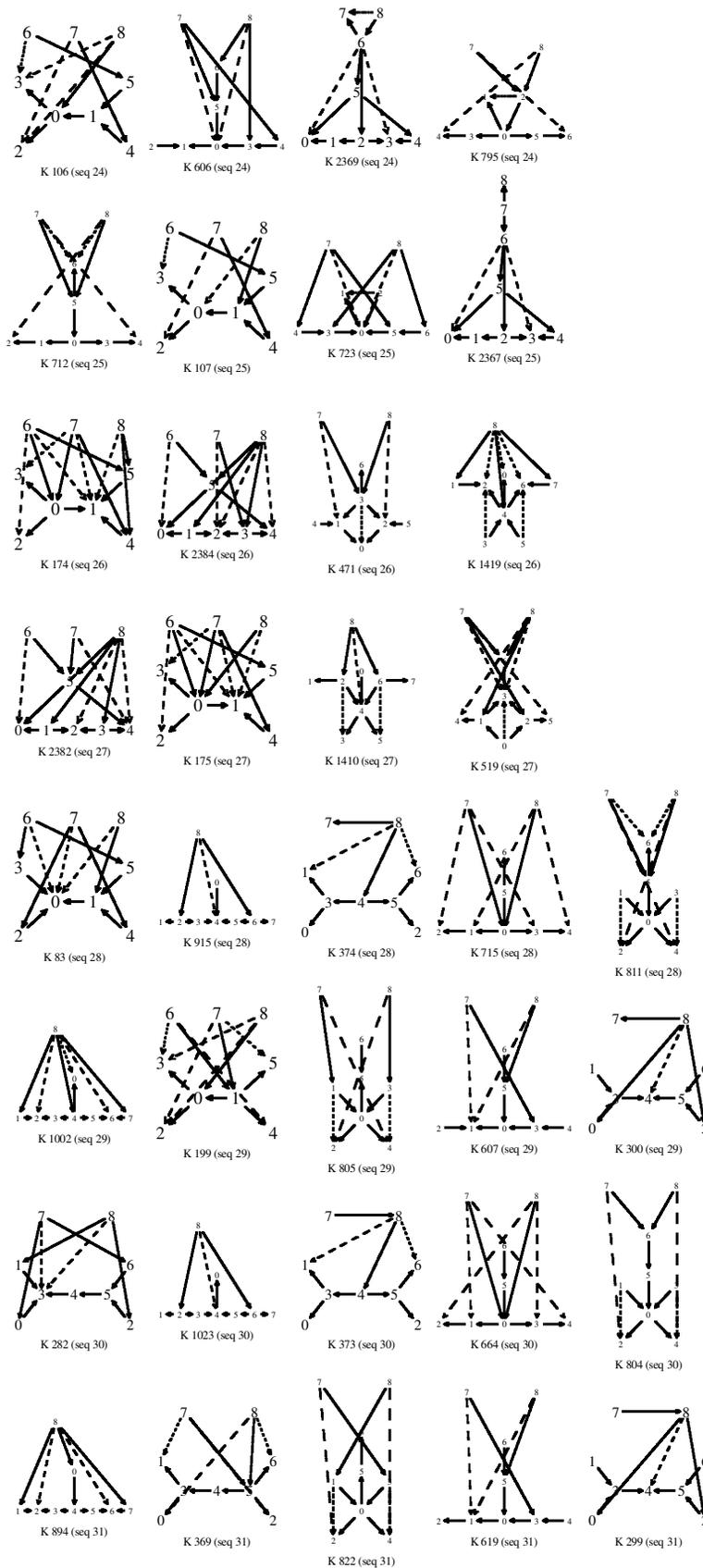
Fig. 10 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).

Fig. 11 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).

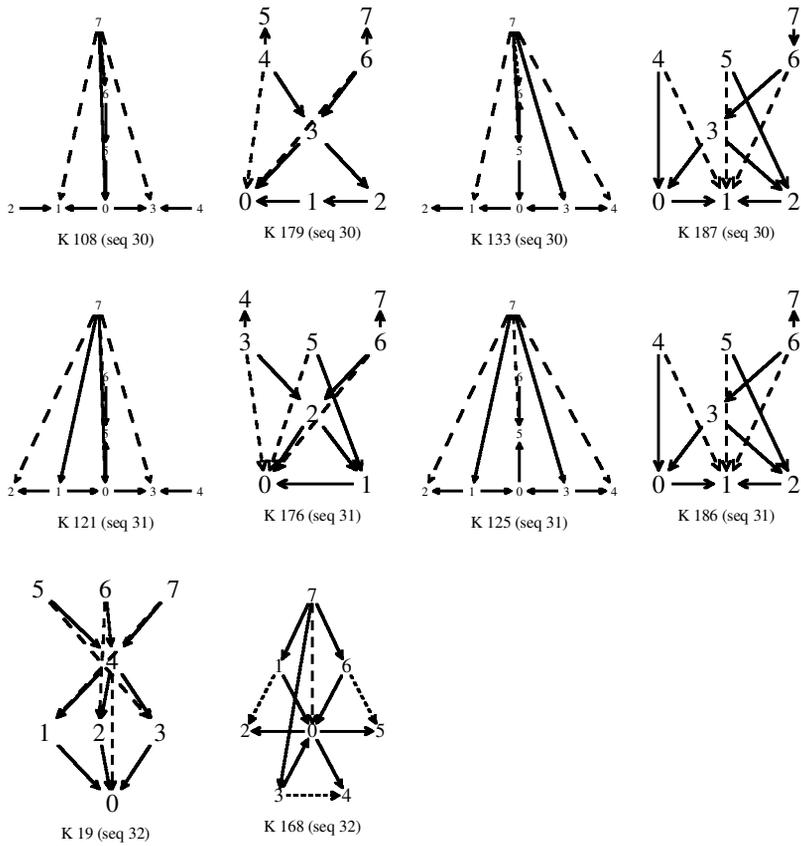**Fig. 12** Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).

Fig. 13 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).
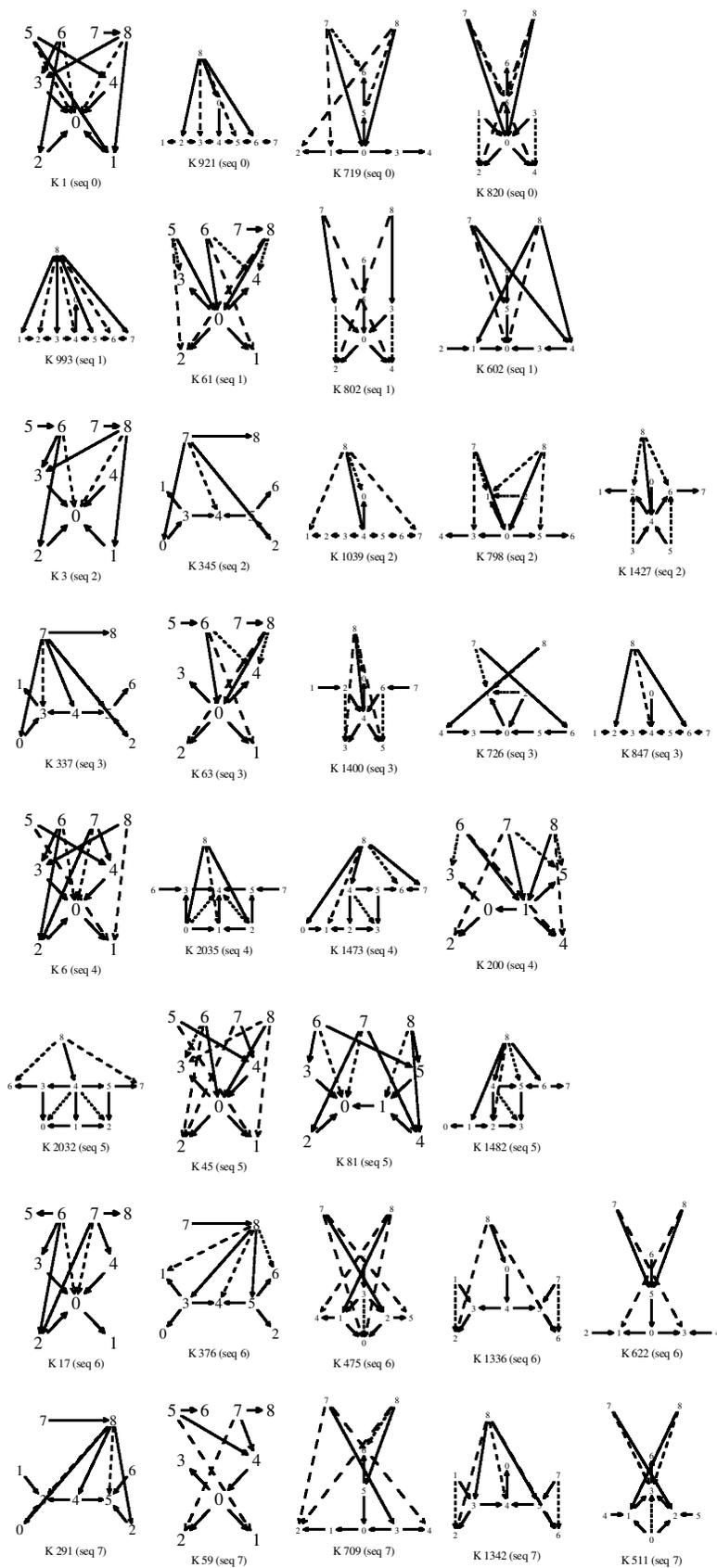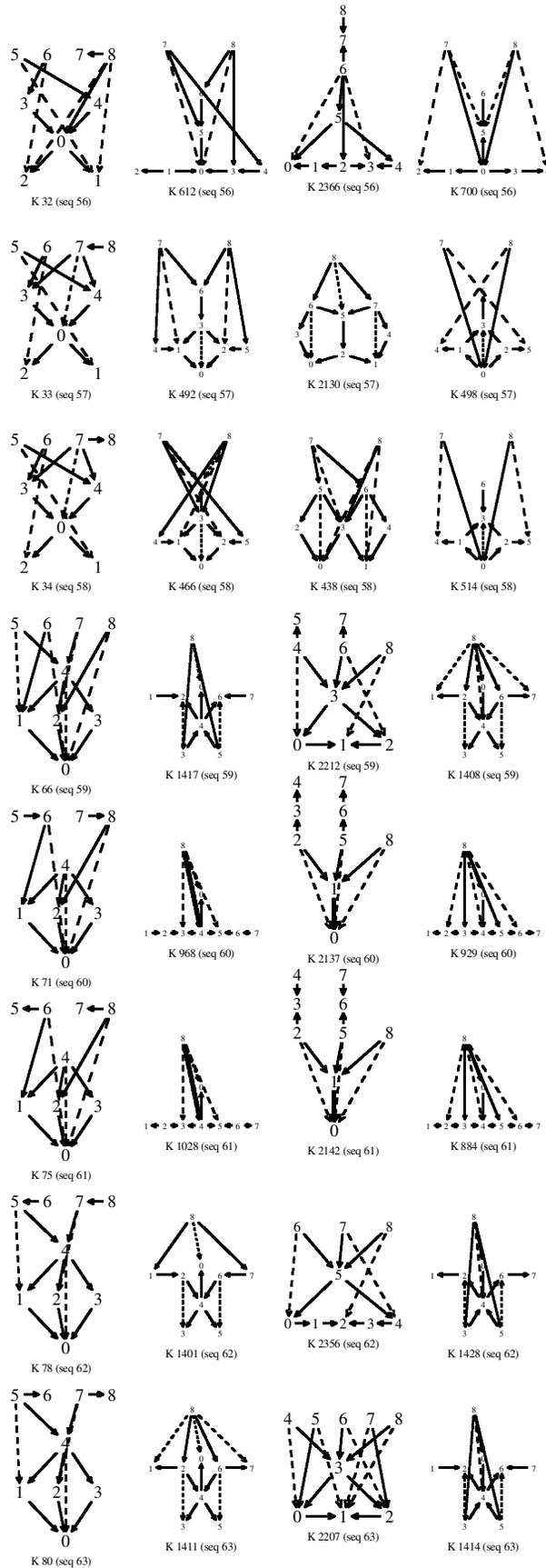
Fig. 14 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).
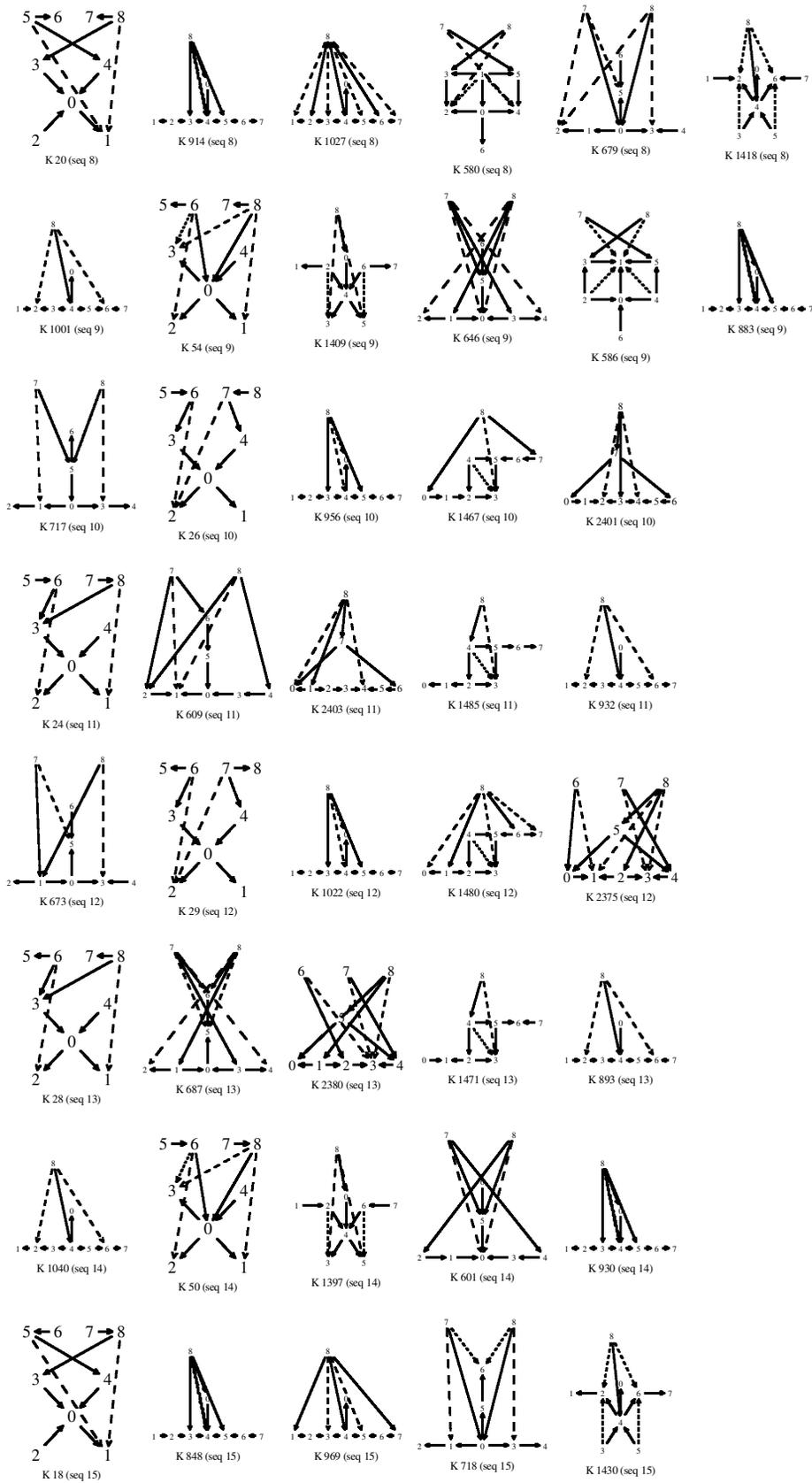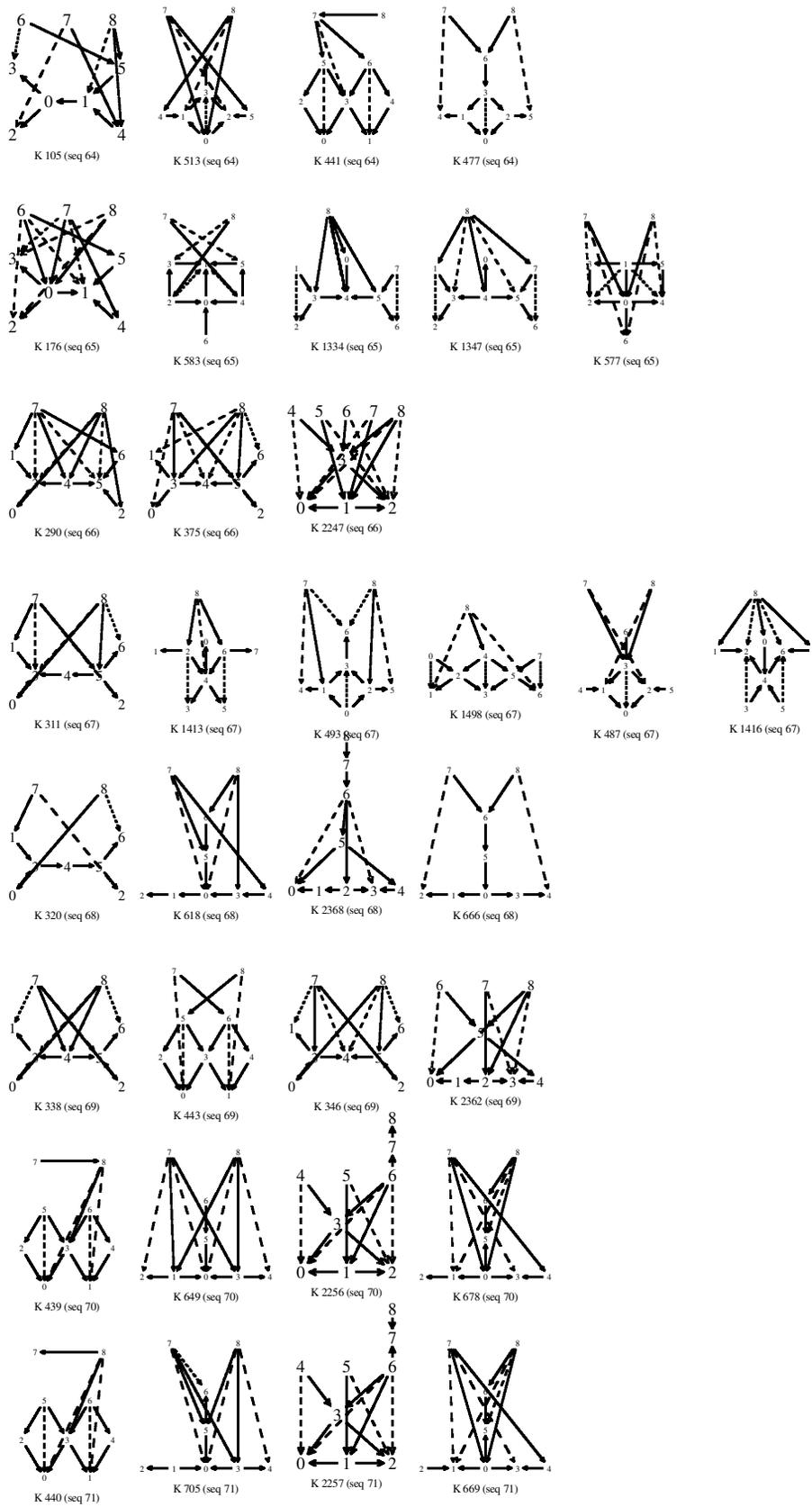
Fig. 15 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).
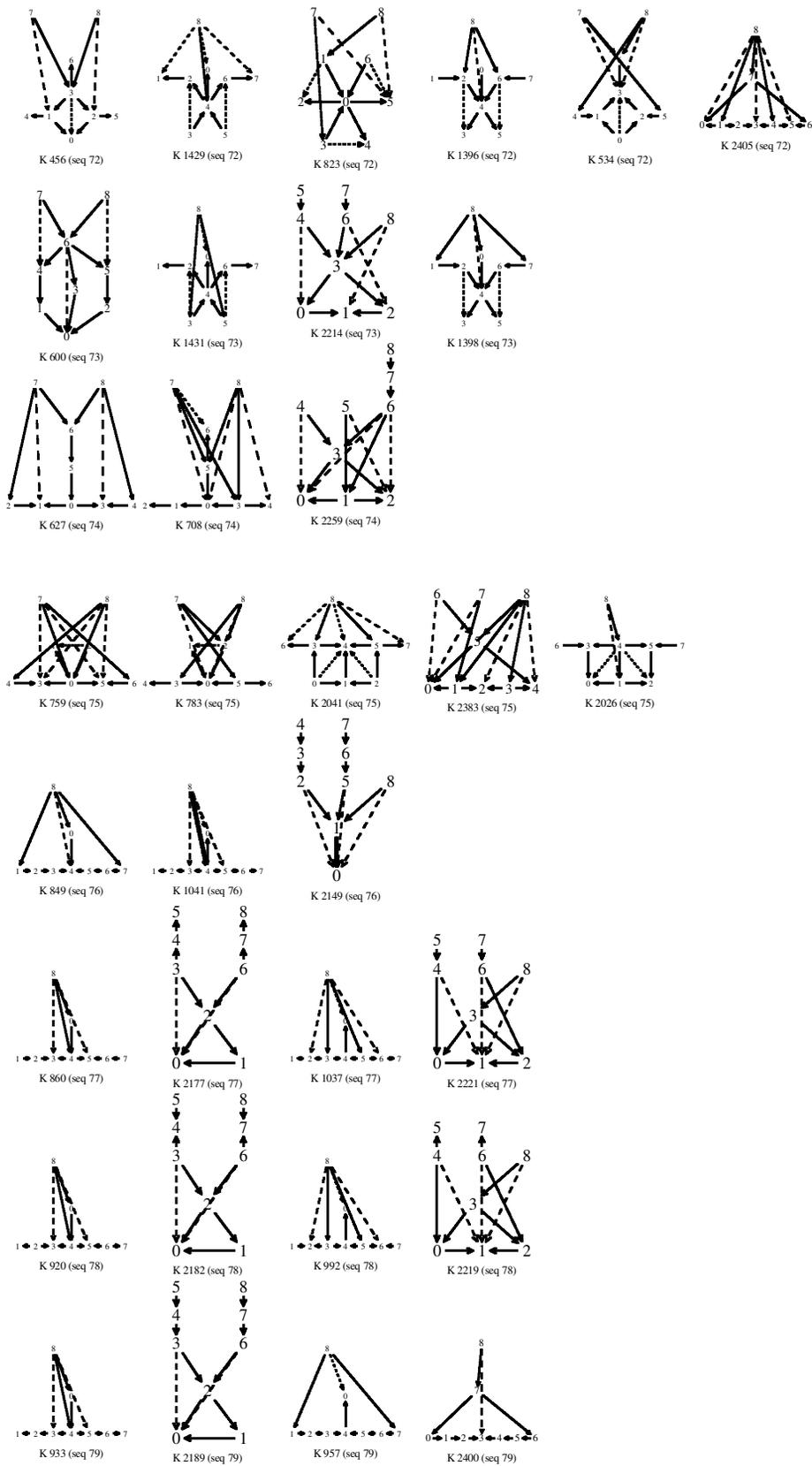
Fig. 16 Reflection sequences of the tubular algebras having a nontrivial of tubular type (244).
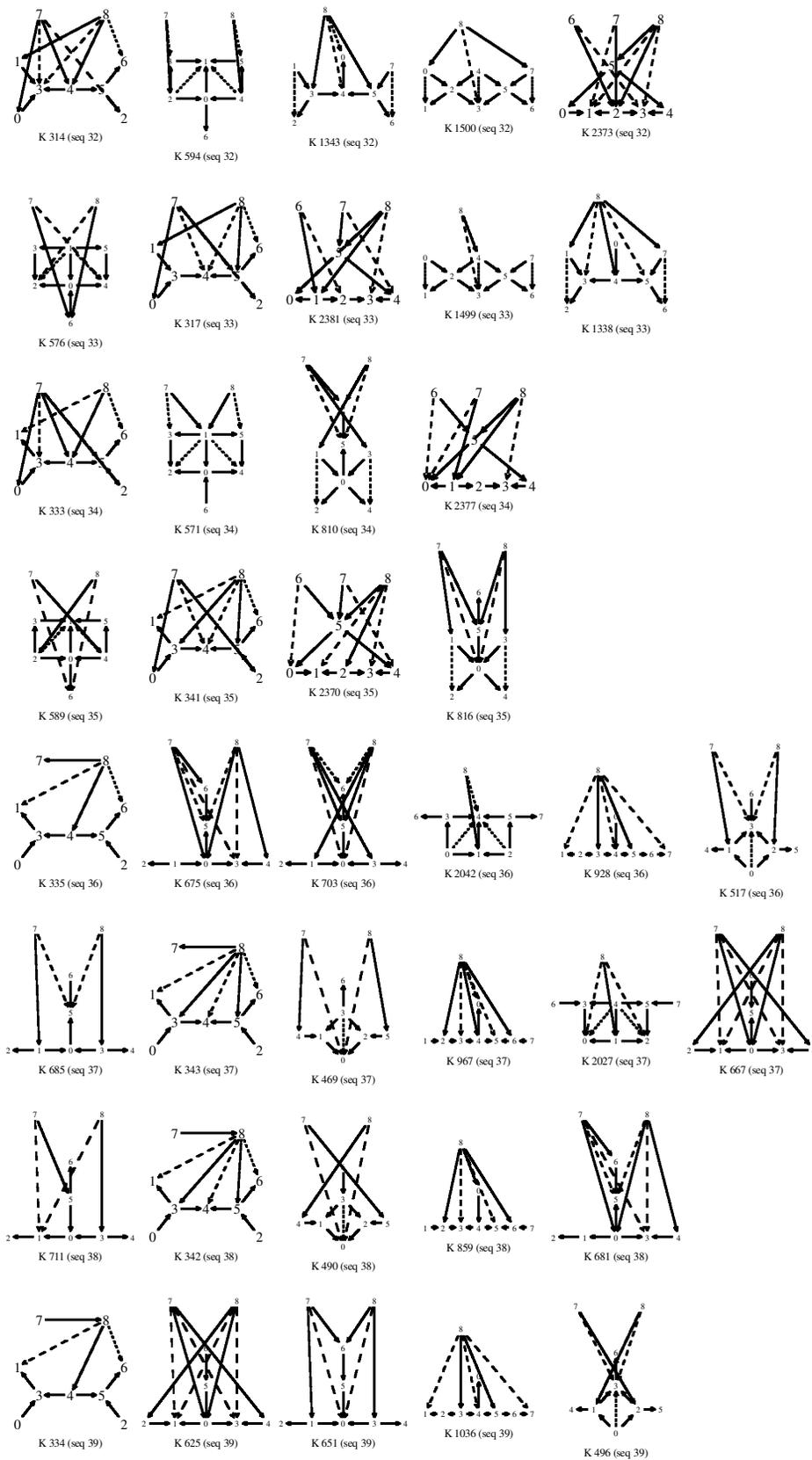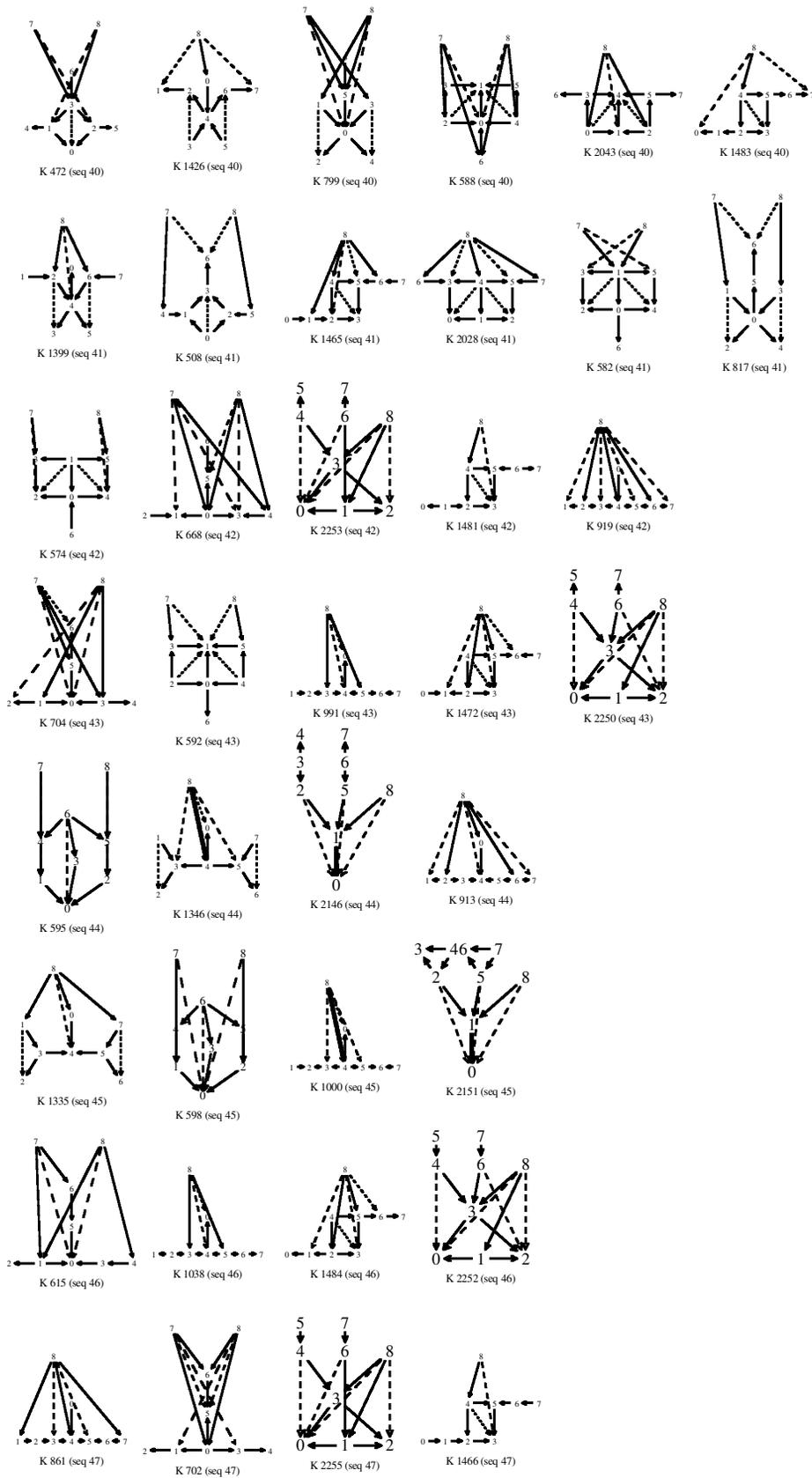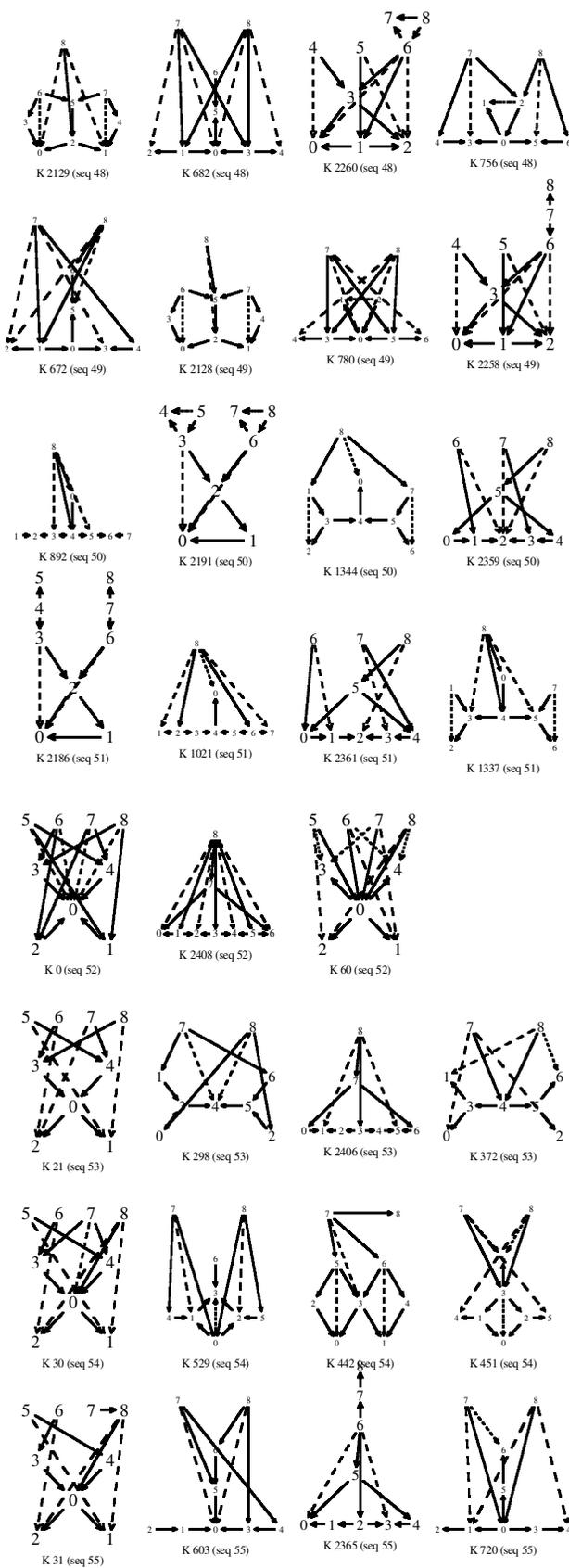
2   Determinants of Cartan matrices.

In this section we present computer programs verifying the conditions (i) and (ii) from the [1, Proposition 4.1]. The programs consist of a set of procedures written in *Maple*. In the sequel, we assume that the files exc2222.txt, exc333.txt, exc244.txt, and exc236.txt contain the Cartan matrices of exceptional tubular algebras. Further, each Cartan matrix of a tubular algebra B is given in the form

$$
\begin{bmatrix}
C_1 & 0 & 0 & \cdots & 0 & 0 \\
C_2 & C_1 & 0 & \cdots & 0 & 0 \\
C_3 & C_2 & C_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
C_{r-1} & C_{r-2} & C_{r-3} & \cdots & C_1 & 0 \\
C_r & C_{r-1} & C_{r-2} & \cdots & C_2 & C_1
\end{bmatrix},
$$

where r is the maximal order of the roots of the Nakayama automorphism $\nu_{\widehat{B}}$. Moreover, the Cartan matrix of the unique exceptional tubular algebra of the type $(2, 3, 6)$ is given as

$$
\begin{bmatrix}
C_1 & 0 \\
C_2 & C_1
\end{bmatrix}.
$$

The above assumptions on the form of Cartan matrices aim at simplifying the procedures used to verify the condition (ii). In the file exc2222.txt we have defined the following table of Cartan matrices of algebras $B_1(\lambda)$, $B_2(\lambda)$, $\lambda \in K \setminus \{0, 1\}$.

$Exceptional2222 :=$

$$
\left[\left[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}
\right] \right]
$$

In the file exc333.txt we have defined the table of Cartan matrices of algebras $B_3, \ldots, B_8$.

$Exceptional333 :=$

$$
\left[\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right],
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right],
\right.
$$

$$
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right],
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1
\end{array}\right],
$$

$$
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1
\end{array}\right],
\left.\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]\right]\right]
$$

In the file exc244.txt we have defined the following table of the Cartan matrices of algebras $B_9, \ldots, B_{13}$.

$$
Exceptional244 := \left[\left[\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1
\end{array}\right], \right.
$$

$$
\left[\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right],
\left[\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1
\end{array}\right],
$$

$$
\left[\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right],
\left.\left[\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}\right]\right]
$$

In the file exc236.txt we have defined a (singleton) table containing the Cartan matrix of the algebra B_{14}.

$$Exceptional236 :=$$

$$\left[\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array} \right] \right]$$

2.1   Condition (i).

All procedures used to verify the condition (i) in [1, Section 4], as defined in the file cond_i.txt, are presented below.

The procedure maketransl constructs the matrix $\Lambda_B^{d_B+1}$ from the matrix $\Lambda_B^{d_B}$, as in the definition of the condition (i) (see [1, Section 4]). The parameter *B1* is equal to the matrix $\Lambda_B^{d_B}$, the parameter *s* equals $d_B + 1$, and the variable *B2* corresponds to the matrix $\Lambda_B^{d_B+1}$.

```
maketransl := proc(B1, s)
local n, l, i, j, k, B2;
      n := coldim(B1);
      l := s*n;
      B2 := matrix(l, l);
      for i to l do
         for j to l do
            B2[i, j] := 0
         end do
      end do;
      for k from 0 to s - 1 do
         for i to n do
            for j to n do
               B2[i + k*n, j + k*n] := B1[i, j]
             end do
         end do
      end do;
      for k to s - 1 do
         for i to n do
            for j to n do
               B2[i + k*n, j + (k - 1)*n] := B1[j, i]
            end do
         end do
      end do;
      B2
   end proc;
```

The procedure comput makes the main computation. The parameters *B1* and *B2* are matrices, and the parameter *differ* is equal to the difference of their respective dimensions. (i.e. the number of rows of *B2* minus the number of rows of *B1*, which is also equal to the number of columns of *B2* minus the number of columns of *B1*).

The behaviour of this procedure depends on the value of the parameter *step*. This

parameter admits values 1, 2 and 3.

(1) For **step** = 1 the procedure compares the determinants of the matrices **B1** and **B2**.

(2) For **step** = 2 the procedure recursively removes in all possible manners the number of **toDel** rows (out of requested **numbDel**), indexed by the numbers equal or greater than **first**. Upon each such sequence of removals the procedure performs a call to itself with **step** set to 1.

(3) For **step** = 3 the procedure recursively removes in all possible manners the number of **toDel** columns (out of requested **numbDel**), indexed by the numbers equal or greater than **first** (for the matrix **B1**) or **first** + **differ** (for the matrix **B2**). Upon each such sequence of removals the procedure calls itself with **step** set to 2.

```
comput := proc(B1, B2, first, toDel, numbDel, differ, step)
local i, n, results;
    if toDel = 0 then
        if step = 1 then results := det(B1)= det(B2)
        else results :=
            comput(B1, B2, 1, numbDel, numbDel, differ, step - 1)
        end if
    elif step = 2 then
        n := rowdim(B1);
        results := true;
        for i from first to n + 1 - toDel do results := results and
            comput(delrows(B1, i .. i),delrows(B2, i .. i), i,
            toDel - 1, numbDel, differ, step)
        end do
    else
        n := coldim(B1);
        results := true;
        for i from first to n + 1 -toDel do results := results and
            comput(delcols(B1, i .. i),
            delcols(B2, i + differ .. i + differ), i, toDel - 1, numbDel,
            differ, step)
        end do
    end if;
    results
end proc
```

Hence, to sum up, the procedure comput checks for equality of all corresponding pairs of minors of the matrices $B1 := \Lambda_B^{d_B}$ and $B2 := \Lambda_B^{d_B+1}$, with $\Lambda_B^{d_B}$ and $\Lambda_B^{d_B+1}$ defined as in [1, Section 4].

The procedure checkalg verifies if the tubular algebra B having the given Cartan matrix A (denoted by $\Lambda_B^{d_B}$ in [1, Section 4]) satisfy the condition (i). As B is of one of the tubular types $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$, the dimension of A is 6, 8, 9 or 10. The procedure checkalg calls the procedure maketransl, to construct a second matrix B (denoted by $\Lambda_B^{d_B+1}$ ibidem) as well as the procedure comput to verify if those matrices have the property described in the condition (i).

```
checkalg := proc(A)
local n, s, B, results, i;
        n := coldim(A);
        if n = 6 then s:= 2
        elif n = 8 then s:= 3
        elif n = 9 then s:= 4
        elif n = 10 then s := 6
        else ERROR('Wrong dimension of the matrix', n)
        end if;
        B := maketransl(A, s+1);
        results := true;
        for i from 0 to n - 1 do
            results := results and comput(A, B, 0, 0, i, n*s, 3)
        end do;
        results
    end proc
```

The procedure **checkalgebras** calls for each element of the table *matrlist* the procedure **checkalg** and declares whether the corresponding tubular algebra satisfies the condition (i) or not. The procedure also displays the information whether all algebras corresponding to the Cartan matrices from *matrlist* satisfy the condition (i).

$checkalgebras :=$ **proc**$(matrlist)$
local $i,\ results$;
  $results :=true$;
  for i **to** $\text{nops}(matrlist)$ **do**
    if not $\text{checkalg}(matrlist[i])$ **then**
      print $('A\ contradiction\ found', matrlist[i])$;
      $results := false$
    else print$('Matrix\ satisfy\ the\ condition',\ matrlist[i])$
    end if
  end do;
  if $results$ **then** print$('All\ matrices\ satisfy\ the\ condition')$ **end if**;
  $results$
end proc

We can use the afore-mentioned procedures in the following way:

```
> with(linalg):
> read "./cond_i.txt":
> read "./exc2222.txt":
> checkalgebras(Exceptional2222);
> read "./exc333.txt":
> checkalgebras(Exceptional333);
> read "./exc244.txt":
> checkalgebras(Exceptional244);
> read "./exc236.txt":
> checkalgebras(Exceptional236);
```

2.2   Condition (ii).

First, we describe the procedures **determ1**, **determ2**, **determ3** defined in the file **cond_ii.txt**. Let B be an exceptional tubular algebra B having no nontrivial rigid automorphisms and let *B1* be the Cartan matrix of B of the form described at the beginning of this section. Further, let *st* be the maximal order of the roots of the Nakayama automorphism $\nu_{\widehat{B}}$. We note that in this case all admissible groups of $\widehat{B}$ are of the form (φ^m), where φ is the (unique nontrivial) root of $\nu_{\widehat{B}}$ of maximal order, and m is a positive integer. The

procedure **determ1** computes the determinants of the Cartan matrices of the algebras $\widehat{B}/(\varphi^m)$ for the appropriate values of m (as specified in the definition of the condition (ii) in [1, Section 4]).

$determ1$:= **proc**($B1$, st)
local d, m, n, i, j, s, $B2$, $B3$, B, $determ$;
  n := coldim($B1$);
  s := n/st;
  if $n = 6$ **then** $d := 2$
  elif $n = 8$ **then** $d := 3$
  elif $n = 9$ **then** $d := 4$
  elif $n = 10$ **then** $d := 6$
  else ERROR('*Wrong dimension of the matrix*', n, eval(n))
  end if;
  $B2$:= transpose($B1$);
  $B3$:= matrix(n, 2*n);
  for i **to** n **do**
    for j **to** n **do**
      $B3[i, j]$:= $B1[j, i]$; $B3[i, n + j]$:= $B1[i, j]$
    end do
  end do;
  for i **to** n **do**
    for j **to** $n + 1$ **do**
      $B3[i, j]$:= $B3[i, j + i$ - 1$]$
    end do;
    for j **from** $n + 2$ **to** 2*n **do**
      $B3[i, j]$:= 0
    end do
  end do;
  $determ$:= $determinants$;
  for m **from** n **by** s **to** $(d + 1)$*n **do**
    B := matrix(m, m);
    for i **to** m **do**
      for j **to** m **do**
        $B[i, j]$:= 0
      end do
    end do;

```
for i to n do
    for j to n do
        B[i, j] := B1[i, j]
    end do
end do;
for i to n do
    for j to n do
        B[i, m - n + j]:= B[i, m - n + j] + B2[i, j]
    end do
end do;
for i from n + 1 to m do
    for j from i - n to i do
        B[i, j] := B3[((i - 1) mod n) + 1, n + 1 + j - i]
    end do
end do;
determ := determ, det(B)
    end do;
print(determ)
end proc
```

The procedure **determ2** should be used for an exceptional tubular algebra B for which there exists a nontrivial root φ of the Nakayama automorphism $\nu_{\widehat{B}}$ such that all positive automorphisms of $\widehat{B}$ are of the form $\varrho\varphi^m$ for some rigid automorphism ϱ of $\widehat{B}$ and some positive integer m. This procedure computes determinants of the Cartan matrices of algebras $\widehat{B}/(\varrho\varphi^m)$ for a given algebra B, a rigid automorphism ϱ and for values of m specified in the definition of the condition (ii) in [1, Section 4]. The parameter **st** should be equal to the order of φ, **B1** should be the Cartan matrix of B in an appropriate form and **tabB2** should be the matrix corresponding to the rigid automorphism φ.

```
determ2 := proc(B1, st, B1r)
local d, m, n, i, j, s, B2, B3, B, determ;
    n := coldim(B1);
    s := n/st;
    if n = 6 then d := 2
    elif n = 8 then d := 3
    elif n = 9 then d := 4
    elif n = 10 then d := 6
    else ERROR('Wrong dimension of the matrix', n,eval(n))
    end if;
```

```
B2 := transpose(multiply(B1r, B1));
B3 := matrix(n, 2*n);
for i to n do
   for j to n do
      B3[i, j] := B1[j, i];
      B3[i, n + j] := B1[i, j]
   end do
end do;
for i to n do
   for j to n + 1 do
      B3[i, j] := B3[i, j + i - 1]
   end do;
   for j from n + 2 to 2*n do
      B3[i, j] := 0
   end do
end do;
determ := determinants;
for m from n by s to (d + 1)*n do
   B := matrix(m, m);
   for i to m do
      for j to m do
         B[i, j] := 0
      end do
   end do;
   for i to n do
      for j to n do
         B[i, j] := B1[i, j]
      end do
   end do;
   for i to n do
      for j to n do
         B[i, m - n + j] := B[i, m - n + j] + B2[i, j]
      end do
   end do;
   for i from n + 1 to m do
      for j from i - n to i do
         B[i, j] := B3[((i - 1) mod n) + 1, n + 1 + j - i]
      end do
   end do;
   determ := determ, det(B)
end do;
print(determ)
end proc
```

The procedure **determ3** should be used only for the exceptional tubular algebras of the tubular type $(2, 3, 6)$. We know that there is only one such algebra B_{14} (up to reflection sequence) and that $\nu_{\widehat{B}_{14}}$ has no nontrivial root. Moreover we know that admissible groups of the automorphisms of $\widehat{B}_{14}$ are of the form (φ_a^{m+1}) or $(\varphi_b \varphi_a^m)$ for the automorphisms φ_a and φ_b described in [1, Section 2] and for some positive integer m. The parameter **B1** should be the Cartan matrix of B_{14}, given in an appropriate form, while the parameter **st** should be equal to 2 (note that $s = 2$ is such that $\varphi_a^s = \varrho\nu_{\widehat{B}_{14}} = \varphi_b^s$ for the (unique) rigid automorphism ϱ of $\widehat{B}_{14}$). The last parameter **determ3** is the table of matrices describing the actions of φ_a^{m+1} or $\varphi_b \varphi_a^m$ (we omit the detailed description of its form). Procedure **determ3** computes the determinants of appropriate algebras of a tubular type.

$$determ3 := \mathbf{proc}(B1,\ st,\ tabB2)$$

$\mathbf{local}\ d,\ m,\ n,\ i,\ j,\ s,\ B2,\ B3,\ B,\ determ;$

    $n := \mathrm{coldim}(B1);$

    $s := n/st;$

    $\mathbf{if}\ n = 6\ \mathbf{then}\ d := 2$

    $\mathbf{elif}\ n = 8\ \mathbf{then}\ d := 3$

    $\mathbf{elif}\ n = 9\ \mathbf{then}\ d := 4$

    $\mathbf{elif}\ n = 10\ \mathbf{then}\ d := 6$

    $\mathbf{else}\ \mathrm{ERROR}(\text{'}\textit{Wrong dimension of the matrix}\text{'},\ n,\mathrm{eval}(n))$

    $\mathbf{end\ if};$

    $B3 := \mathrm{matrix}(n,\ 2{*}n);$

    $\mathbf{for}\ i\ \mathbf{to}\ n\ \mathbf{do}$

      $\mathbf{for}\ j\ \mathbf{to}\ n\ \mathbf{do}$

        $B3[i,\ j] := B1[j,\ i];$

        $B3[i,\ n + j] := B1[i,\ j]$

      $\mathbf{end\ do}$

    $\mathbf{end\ do};$

    $\mathbf{for}\ i\ \mathbf{to}\ n\ \mathbf{do}$

      $\mathbf{for}\ j\ \mathbf{to}\ n + 1\ \mathbf{do}$

        $B3[i,\ j] := B3[i,\ j + i\ \text{-}\ 1]$

      $\mathbf{end\ do};$

      $\mathbf{for}\ j\ \mathbf{from}\ n + 2\ \mathbf{to}\ 2{*}n\ \mathbf{do}$

        $B3[i,\ j] := 0$

      $\mathbf{end\ do}$

    $\mathbf{end\ do};$

    $determ := determinants;$

```
for m from n by s to (d + 1)*n do
    B2 := transpose(multiply(tabB2[((m/s - 1) mod
                        nops(tabB2)) + 1], B1));
    B := matrix(m, m);
    for i to m do
        for j to m do
            B[i, j] := 0
        end do
    end do;
    for i to n do
        for j to n do
            B[i, j] := B1[i, j]
        end do
    end do;
    for i to n do
        for j to n do
            B[i, m - n + j] := B[i, m - n + j] + B2[i, j]
        end do
    end do;
    for i from n + 1 to m do
        for j from i - n to i do
            B[i, j] := B3[((i - 1) mod n) + 1, n + 1 + j - i]
        end do
    end do;
    determ := determ, det(B)
end do;
print(determ)
end proc
```

Clearly, using the procedures determ1, determ2, determ3, we may easily check that the tubular algebras $B_1(\lambda)$, $B_2(\lambda)$, $\lambda \in K \setminus \{0,1\}$, $B_3, \ldots, B_{14}$ do satisfy the condition (ii). We present this verification below.

```
> with(linalg):
> read "./cond_ii.txt":
> read "./exc2222.txt":
> aut11:=array(sparse,1..6,1..6):
> aut11[1,1]:=1:   aut11[2,2]:=1:
  aut11[3,3]:=1:   aut11[4,4]:=1:
  aut11[5,6]:=1:   aut11[6,5]:=1:
> aut12:=array(sparse,1..6,1..6):
> aut12[1,1]:=1:   aut12[2,3]:=1:
  aut12[3,2]:=1:   aut12[4,4]:=1:
  aut12[5,5]:=1:   aut12[6,6]:=1:
> aut1i:=array(identity,1..6,1..6):
> aut1r:=array(sparse,1..6,1..6):
> aut1r[1,1]:=1:   aut1r[2,3]:=1:
  aut1r[3,2]:=1:   aut1r[4,4]:=1:
  aut1r[5,6]:=1:   aut1r[6,5]:=1:
> determ2(Exceptional2222[1],2,aut1i);
              determinants,0,16,0,16,0
> determ2(Exceptional2222[1],2,aut1r);
              determinants,0,16,0,16,0
> determ2(Exceptional2222[1],2,aut11);
              determinants,0,0,0,0,0
> determ2(Exceptional2222[1],2,aut12);
              determinants,0,0,0,0,0
>
```

```
[> with(linalg):
[> read "./cond_ii.txt":
[> read "./exc2222.txt":
[> aut21:=array(sparse,1..6,1..6):
[> aut21[1,1]:=1:   aut21[2,2]:=1:
   aut21[3,3]:=1:   aut21[4,4]:=1:
   aut21[5,6]:=1:   aut21[6,5]:=1:
[> aut22:=array(sparse,1..6,1..6):
[> aut22[1,2]:=1:   aut22[2,1]:=1:
   aut22[3,4]:=1:   aut22[4,3]:=1:
   aut22[5,6]:=1:   aut22[6,5]:=1:
[> aut2i:=array(identity,1..6,1..6):
[> aut2r:=array(sparse,1..6,1..6):
[> aut2r[1,1]:=1:   aut2r[2,2]:=1:
   aut2r[3,4]:=1:   aut2r[4,3]:=1:
   aut2r[5,6]:=1:   aut2r[6,5]:=1:
[> determ2(Exceptional2222[2],3,aut2i);
```
$$determinants, 0, 0, 12, 0, 12, 0, 0$$
```
[> determ2(Exceptional2222[2],3,aut2r);
```
$$determinants, 0, 0, 12, 0, 12, 0, 0$$
```
[> determ2(Exceptional2222[2],3,aut21);
```
$$determinants, 0, 0, 0, 0, 0, 0, 0$$
```
[> determ2(Exceptional2222[2],3,aut22);
```
$$determinants, 0, 0, 0, 0, 0, 0, 0$$
```
[>
```

```
[> with(linalg):
[> read "./cond_ii.txt":
[> read "./exc333.txt":
[> aut31:=array(identity,1..8,1..8):
[> aut31:=array(sparse,1..8,1..8):
[> aut31[1,1]:=1:   aut31[2,3]:=1:
   aut31[3,2]:=1:   aut31[4,4]:=1:
   aut31[5,5]:=1:   aut31[6,7]:=1:
   aut31[7,6]:=1:   aut31[8,8]:=1:
[> aut32:=array(sparse,1..8,1..8):
[> aut32[1,1]:=1:   aut32[2,3]:=1:
   aut32[3,4]:=1:   aut32[4,2]:=1:
   aut32[5,5]:=1:   aut32[6,7]:=1:
   aut32[7,8]:=1:   aut32[8,6]:=1:
[> determ2(Exceptional333[1],2,aut3i);
```
$$determinants, 0, 256, 0, 4, 0, 4, 0$$
```
[> determ2(Exceptional333[1],2,aut31);
```
$$determinants, 0, 0, 0, 12, 0, 12, 0$$
```
[> determ2(Exceptional333[1],2,aut32);
```
$$determinants, 0, 16, 0, 16, 0, 16, 0$$
```
[>
```

```
> with(linalg):
> read "./cond_ii.txt":
> read "./exc333.txt":
> read "./exc244.txt":
> determ1(Exceptional333[2],4);
        determinants,0,6,0,6,0,0,12,6,0,6,12,0,0
> determ1(Exceptional333[3],4);
        determinants,0,6,0,6,0,0,12,6,0,6,12,0,0
> determ1(Exceptional333[4],2);
                determinants,0,0,0,12,0,12,0
> determ1(Exceptional333[5],2);
                determinants,0,0,0,12,0,12,0
> determ1(Exceptional333[6],2);
                determinants,0,0,0,12,0,12,0
> determ1(Exceptional244[1],3);
        determinants,0,0,6,0,6,0,0,0,6,0,6,0,0
> determ1(Exceptional244[2],3);
        determinants,0,0,6,0,6,0,0,0,6,0,6,0,0
> determ1(Exceptional244[3],3);
        determinants,0,0,6,0,6,0,0,0,6,0,6,0,0
> determ1(Exceptional244[4],3);
        determinants,0,0,6,0,6,0,0,0,6,0,6,0,0
> determ1(Exceptional244[5],3);
        determinants,0,0,6,0,6,0,0,0,6,0,6,0,0
>
```

```
> with(linalg):
> read "./cond_ii.txt":
> read "./exc236.txt":
> auta:=array(sparse,1..10,1..10):
> auta[1,2]:=1:  auta[2,1]:=1:  auta[3,5]:=1:
  auta[4,4]:=1:  auta[5,3]:=1:  auta[6,6]:=1:
  auta[7,7]:=1:  auta[8,8]:=1:  auta[9,9]:=1:
  auta[10,10]:=1:
> autb:=array(sparse,1..10,1..10):
> autb[1,1]:=1:  autb[2,2]:=1:  autb[3,2]:=1:
  autb[4,4]:=1:  autb[5,5]:=1:  autb[6,7]:=1:
  autb[7,6]:=1:  autb[8,10]:=1:  autb[9,9]:=1:
  autb[10,8]:=1:
> autid:=array(identity,1..10,1..10):
> autr:=array(sparse,1..10,1..10):
> autr[1,2]:=1:  autr[2,1]:=1:  autr[3,5]:=1:
  autr[4,4]:=1:  autr[5,3]:=1:  autr[6,7]:=1:
  autr[7,6]:=1:  autr[8,10]:=1:  autr[9,9]:=1:
  autr[10,8]:=1:
> determ3(Exceptional236[1],2,[auta,autr,autb,autid]);
        determinants,0,1024,0,16,0,16,0,1024,0,16,0,16,0
> determ3(Exceptional236[1],2,[autb,autid,auta,autr]);
        determinants,0,0,0,0,0,0,0,0,0,0,0,0,0,0
>
```

References

[1] J. Białkowski: Cartan matrices of selfinjective algebras of tubular type, *CEJM*, Vol. 2, (2004), pp. 123–142.

[2] K. Bongartz: Critical simply connected algebras, *Manuscr. Math.* Vol. 46, (1984), pp. 117–136.

[3] D. Happel and D. Vossieck: Minimal algebras of infinite representation type with preprojective component, *Manuscr. Math.*, Vol. 42, (1983), pp. 221–243.

[4] C. M. Ringel: "*Tame Algebras and Integral Quadratic Forms*," Lecture Notes in Math., Vol. 1099, Springer, 1984.