Central European Science Journals
www.cesj.com

Central European Journal of Mathematics
Central European Science Journals
CEJM 2(1) (2004) 112–122

The minimizing of the Nielsen root classes

Daciberg L. Gonçalves[1] *, Claudemir Aniz[2] †

[1] *Departamento de Matemática - IME-USP, Caixa Postal 66281 - Agência Cidade de São Paulo, 05311-970 - São Paulo - SP – Brasil*

[2] *Universidade Estadual de Mato Grosso do Sul-UEMS, Rua Walter Hubacher s/n, 79750-000/Vila Beatriz Nova Andradina - MS – Brasil*

Received 8 August 2003; accepted 5 November 2003

Abstract: Given a map $f : X \to Y$ and a Nielsen root class, there is a number associated to this root class, which is the minimal number of points among all root classes which are H-related to the given one for all homotopies H of the map f. We show that for maps between closed surfaces it is possible to deform f such that all the Nielsen root classes have cardinality equal to the minimal number if and only if either $NR[f] \leq 1$, or $NR[f] > 1$ and f satisfies the Wecken property. Here $NR[f]$ denotes the Nielsen root number. The condition "f satisfies the Wecken property is known to be equivalent to $|deg(f)| \leq NR[f]/(1-\chi(M_2) - \chi(M_1)/(1-\chi(M_2))$ for maps between closed orientable surfaces. In the case of nonorientable surfaces the condition is $A(f) \leq NR[f]/(1-\chi(M_2) - \chi(M_1)/(1-\chi(M_2))$. Also we construct, for each integer $n \geq 3$, an example of a map $f : K_n \to N$ from an n-dimensionally connected complex of dimension n to an n-dimensional manifold such that we cannot deform f in a way that all the Nielsen root classes reach the minimal number of points at the same time.

Keywords: Nielsen root classes, Absolut degree, Wecken property, closed surfaces, complex
MSC (2000): 55M20, 57M12, 20F99

1 Introduction

Given a map $f : M_1 \to M_2$ consider a Nielsen root class denoted by C. For each homotopy H from f to a map f', there is a well known correspondence between the Nielsen root classes of f and the Nielsen root classes of f'. Brooks in [Br] has defined the notion of

* E-mail: dlgoncal@ime.usp.br
† E-mail: caniz1@bol.com.br

an inessential root class. In the same spirit we can define the concept of the minimal number of a root class (see section 2):

Definition 1.1. Let $MR[f, C]$ be the minimal cardinality among all Nielsen root classes C', where C' is a Nielsen class of f', H-related to C, for H a homotopy between f and f'.

In many relevant cases we have that $MR[f, C]$ is finite. Then the following natural question, related to the question of minimization of number of roots in the homotopy class, arises: Given f, can we deform it to a map f' such that the number of points in each Nielsen root class of f' is the minimal number of roots of the corresponding Nielsen root class? R. Brooks has provided some positive result, see [Br, Theorem 1]. A consequence of his result is: if M_1 is a complex of the same dimension as a manifold N of dimension greater or equal to 3 and all Nielsen classes are inessential, (i.e. the minimal number of roots in the class is zero) then we can deform the map to be root free. This result also gives a motivation for the above more general question and also the similar question for $dim(M_1) > dim(M_2)$, which should be more subtle. In this work we consider the case where $dimM_1 = dimM_2$. We divide into two cases, first when M_1, M_2 are closed surfaces ($dimM_i = 2$), and the second case M_1 is a complex of dimension greater than or equal to three and M_2 a manifold. Here are our main results. For the case of maps between surfaces:

Theorem 1.2. For maps between closed surfaces it is possible to deform f such that all the Nielsen root classes have cardinality equal to the minimal number if and only if either $NR[f] \leq 1$, or $NR[f] > 1$ and f satisfies the Wecken property.

For the case of maps from a complex into a manifold, of the same dimension greater or equal to 3, we observe that if K_n is a manifold as result of [Sc] our question has a positive answer. So, an n-dimensionally connected complex K_n, which is not a manifold, is constructed. Let $\mathbb{P}^n$ be the n-dimensional real projective space.

Theorem 1.3. For each $n \geq 3$ there exist $f : K_n \to \mathbb{P}^n$ such that:
(1) $NR[f] = 2$
(2) The minimal number of a root class is 1
(3) $MR[f, y_0] = \min\{\#(g^{-1}(y_0))|g \text{ homotopic to } f\} > 2$

It follows from [GZ], [BGZ], [GKZ] and [BGKZ1] that if f satisfies the Wecken property then we can minimize all the Nielsen classes at the same time. Also, in light of [BGKZ2], for some pairs of surfaces S_h, S_g all maps satisfy the Wecken property. Therefore a deformation to minimize all the classes at the same time is always possible for such pairs of surfaces. On the other hand, when the target is the torus, it is always possible to decide when the deformation is possible or not.

Nielsen theory has been used to study several different subjects, for example: fixed point theory, root theory, coincidence theory and intersection theory. The type of problem we consider here can be studied in any one of subjects above and the problem is related to the minimal problem in each one of the cases.

This paper is organized into three sections. In section 2 we develop some generalities. Then we show that the minimal number of a Nielsen root class of a map f is independent of the class. This is Proposition 2.3. Also for maps between surfaces, the answer of our question depends only on $\chi(M_1)$, $\chi(M_2)$ and the index $[\pi_1(S_g), f_\#(\pi_1(S_h)]$. This is Theorem 2.5. In section 3 we compute the minimal number of points in a Nielsen root class for the case of maps between surfaces. These are Theorems 3.2 and 3.3. Then we show our main result for maps between surfaces which is Theorem 1.2. In section 4 we construct a family of examples f_n, one for each $n > 2$, of maps from a complex K_n to an n-dimensional manifold N $(n \geq 3)$ which cannot be deformed to a map such that the number of points in each Nielsen root class is minimal. This result is Theorem 4.2.

The authors would like to thank the referee for has careful reading and suggestions, improving the presentation of this work.

2 Generalities about the problem

Let $f : X \to Y$ be an arbitrary map and $y_0 \in Y$ a given base point. We consider roots with respect to this base point, i.e. the preimage, $f^{-1}(y_0)$, of the base point y_0. In this section we recall some basic definitions, and develop some general facts for later use.

Recall that two roots $x_1, x_2 \in f^{-1}(y_0)$ are said to be equivalent if there is a path $\lambda : [0,1] \to X$ such that the class of the loop $f(\lambda)$ is nullhomotopic. This is equivalent to say that for every path λ from x_1 to x_2 the homotopy class of the loop $f(\lambda)$ belongs to the image $f_\#(\pi_1(X, x_0)) \subset \pi_1(Y, y_0)$. The set of equivalence classes under this relation are called the Nielsen root classes of f. Also a homotopy H between two maps f and f' provides a one to one correspondence between the Nielsen root classes of f and the Nielsen root classes of f'. We say that two such classes under this correspondence are H related. See [Ki] for more details.

Following Brooks [Br] we have the definition.

Definition 2.1. A Nielsen root class C of a map f is inessential if it is $H-$related to an empty Nielsen root class, i.e. there is a homotopy H of f such that the class C is $H-$related to an empty Nielsen root class of $f' = H(\ ,1)$. Otherwise it is called essential.

In the same spirit of the definition above, for a Nielsen root class C of f we define:

Definition 2.2. Let $MR[f, C]$ be the minimal cardinality among all Nielsen root classes C', where C' is a Nielsen class of f' H-related to C, for H a homotopy between f and f'.

Our main question is to know when is possible to deform a map f to some map f' with the property that all Nielsen root classes of f' have the minimal number of points. We consider only the case where the target is a manifold.

Under the condition that the target is a manifold, we can show that this number

$MR[f,C]$ is independent of the Nielsen root class. This is relevant to our question.

Proposition 2.3. If Y is a manifold then the number $MR[f,C]$ is independent of the class C.

Proof. We will show that any two root classes are H-related and the result will follow from [Br]. Consider the covering $p : \tilde{Y} \to Y$ which corresponds to the subgroup $f_{\#}(\pi_1(X,x_0)) \subset \pi_1(Y,y_0)$. Let $\tilde{f} : X \to \tilde{Y}$ be a lift of f. Let us fix two points $\tilde{y}_0, \tilde{y}_1$ over y_0. Since $\tilde{Y}$ is a manifold, there is a disk D^n whose interior intersects $p^{-1}(y_0)$ in these two points and a homeomorphism $h : \tilde{Y} \to \tilde{Y}$ with the following properties: a) The map h is the identity in the complement of the interior of D^n, b) There is a homotopy H between the map h and the identity which is relative to the complement of the interior of D^n and c) $h(\tilde{y}_1) = (\tilde{y}_0)$. Suppose that the Nielsen root class C of f corresponds to the preimage of $\tilde{y}_0$ and C_1 is the Nielsen root class which corresponds to the preimage of the point $\tilde{y}_1$. The composite $H \circ (\tilde{f} \times id) : X \times [0,1] \to \tilde{Y}$ is a homotopy between $\tilde{f}$ and $h \circ \tilde{f}$. From [Br, Lemma 1 and Lemma 2], the root classes C and $(h \circ \tilde{f})^{-1}(\tilde{y}_0) = C_1$ are $(p \circ H \circ (\tilde{f} \times id))$-related and the result follows. □

If we have a map f between surfaces, it follows from Kneser [Kn] (see also [Ep]) that, if $deg(f)$ (degree of f for the orientable case) or $A(f)$ (absolute degree for the nonorientable case) is zero, then the map can be deformed to be root free. So, it suffices to consider the cases where the degree, or the absolute degree, of the map is non zero.

Given X, Y two spaces, consider the group $Homeo(X) \times Homeo(Y)$ with the operation given by $(\psi_1, \phi_1) * (\psi_2, \phi_2) = (\psi_2 \circ \psi_1, \phi_1 \circ \phi_2)$. Define an action of this group on the set Y^X, the function space of all continuous map from X to Y, by $(\psi, \phi).f = \phi \circ f \circ \psi$. We show that our question depends only on the orbits given by this action. Namely:

Lemma 2.4. A map $f : X \to Y$ has the property that it can be deformed to a map f' such that all the root classes have the minimal number of points if and only if the same is true for $\phi \circ f \circ \psi$ where $\psi \in Homeo(X), \phi \in Homeo(Y)$. Furthermore, the minimal number of a Nielsen root class (as defined in Proposition 2.3) is the same for any two maps in the same orbit.

Proof. Let H be a homotopy connecting f to f' such that f' has the property that all the root classes have the minimal number of points. Then we claim that $f' \circ \psi$ also has the same property. For, given a root class C of f', the correspondence $C \to \psi^{-1}(C)$ is a bijection between the root classes of f' and $f' \circ \psi$; so, C, $\psi^{-1}(C)$ have the same cardinality. Also, the homotopy H of f provides a homotopy $H \circ (\psi \times I)$ of $f \circ \psi$, and the H-related classes of f corresponds to the $(H \circ (\psi \times I))-$related classes of $f \circ \psi$ under the homeomorphism ψ. By contradiction, suppose that one Nielsen root class C_2 of $f' \circ \psi$ (which has the cardinality of a root class of f') does not have the minimal number of points. Let us consider a homotopy such that the $H'-$related class to C_2 has less points. But this implies, by the argument above, that one of the classes of a deformation of f'

would have less points than a root class of f', which is a contradiction. Therefore, this implies the Lemma for the maps f and $f \circ \psi$. By a similar argument one can show the Lemma for f and $\phi \circ f$. We leave the details to the reader. Therefore, the result follows. □

For a map $f : N_1 \to N_2$ between surfaces denote by $deg(f)$ the *degree* of f in case both surfaces are orientable, and by $A(f)$ the *absolute degree* of f otherwise (see [BSc] or [Ep]). As an application of Lemma 2.4 and results of Gabai-Kazez [GK1], [GK2] we have:

Theorem 2.5. Let $f, g : N_1 \to N_2$ be two maps between orientable closed surfaces (at least one of them is nonorientable) such that $|deg(f)| = |deg(g)|$ ($A(f) = A(g)$) and $[\pi_1(N_2) : f_\#(\pi_1(N_1))] = [\pi_1(N_2) : g_\#(\pi_1(N_1))]$. The map f has the property that it can be deformed to a map f' such that all the root classes have the minimal number of points if and only if the same is true for g.

Proof. First let us observe that if $deg(f) = 0$ then the claim is true since by [Kn] if a map have degree zero it can be deformed to a root free. So let $deg(f) \neq 0$. We consider first the orientable case. Let $f, g : N_1 \to N_2$ be two maps such that $|deg(f)| = |deg(g)|$. Since $deg(f)$ is non zero, it follows the index $[\pi_1(N_2) : f_\#(\pi_1(N_1))] = [\pi_1(N_2) : g_\#(\pi_1(N_1))]$ is finite and denoted by ℓ. Consider the finite coverings N_2', N_2'' of N_2 with base points s_1, s_2, which correspond to the subgroups $f_\#(\pi_1(N_1))$, $g_\#(\pi_1(N_1))$, respectively. These coverings have the same number of sheets ℓ, therefore they are homeomorphic surfaces and let $\phi : N_2' \to N_2''$ be a homeomorphism between them. Consider the lifts $\tilde{f} : N_1 \to N_2'$, $\tilde{g} : N_1 \to N_2''$ (base point preserving) of f, g, respectively, and the map $\phi \circ f$. By Gabai-Kazez [GK1, Corollary 9.4] we have that there is a homeomorphism $\psi : N_1 \to N_1$ such that $\tilde{g} \circ \psi = \phi \circ \tilde{f}$. Therefore $p_2 \circ \tilde{g} \circ \psi = p_2 \circ \phi \circ \tilde{f}$ which implies $g \circ \psi = p_2 \circ \phi \circ \tilde{f}$. By the Lemma 2.4, it follows that if the result is true for g then it is true for $g \circ \psi$, and consequently for $p_2 \circ \phi \circ \tilde{f}$. Now it suffices to compare the latter map with f. For, they have lifts $\phi \circ \tilde{f}$, $\tilde{f}$, respectively. The root classes of f correspond to the preimage of points in $p_2^{-1}(y_0)$. A straightforward argument, similar to the one used in the proof of the Lemma 2.3, shows that if $\tilde{f}$ can be deformed such that the preimage of each point of $p_2^{-1}(y_0)$ contains the same number of points and is the minimal number of a root class, then the same is true for $\phi \circ \tilde{f}$. Then the result follows. The nonorientable case is similar, where the results from Gabai-Kazez in [GK1] are replaced by the one's in [GK2]. □

3 Maps between closed surfaces

In this section we show the results for maps between surfaces. Denote by $NR[f]$ the Nielsen root number of f, which is the number of essential Nielsen root classes. Our main result is:

Theorem 3.1. For maps between closed surfaces it is possible to deform f such that all the Nielsen root classes have cardinality equal to the minimal number if and only if either $NR[f] \leq 1$, or $NR[f] > 1$ and f satisfies the Wecken property.

Proof. The "if part". This is the easy part. Of course, if $NR[f] = 1$ then there is only one root class and there is nothing to be proved. If $NR[f] = 0$ then all classes are inessential and f satisfies the (root) Wecken property. This follows from Kneser [Kn] which shows that f can deformed to a root free map. So, let $NR[f] > 1$ and f satisfies the Wecken property. This means that f can be deformed to a map g which has $NR[f]$ roots. This implies that each Nielsen root classe of g has exactly one point, since all classes are essential. Therefore the result follows.

The "only if part". This is a direct consequence of Theorems 3.2 and 3.3 stated and proved below. □

The result above is based in the calculation of the minimal number of a root class of a map between surfaces. This calculation seems interesting in its own right. We will answer completely this question. First we consider the orientable case.

Theorem 3.2. Let $f : S_h \to S_g$ be a map between orientable surfaces of genus h and g, respectively, $\ell = NR[f]$ and $|deg(f)| = \ell d$. Then, the minimal number of a root class is given by:

a) If $deg(f) = 0$, it is zero.

b) If $deg(f) \neq 0$ and $\ell = 1$, it is $\max\{1, |deg(f)| - (|deg(f)|\chi(S_g) - \chi(S_h))\}$.

c) If $0 < |deg(f)| \leq \ell - \chi(S_h)/(1 - \chi(S_g))$, it is one.

d) If $|deg(f)| > \ell - \chi(S_h)/(1 - \chi(S_g))$, $\ell > 1$ and $(|deg(f)|\chi(S_g) - \chi(S_h)) \geq d - 1$, it is one.

e) If $|deg(f)| > \ell - \chi(S_h)/(1 - \chi(S_g))$, $\ell > 1$ and $0 \leq (|deg(f)|\chi(S_g) - \chi(S_h)) < d - 1$, it is $d - (|deg(f)|\chi(S_g) - \chi(S_h))$.

Proof. First we observe that the Theorem cover all maps. As result of the cases a) and b) we can consider $deg(f) \neq 0$ and $\ell > 1$. The case c) consider all maps such that $0 < |deg(f)| \leq \ell - \chi(S_h)/(1 - \chi(S_g))$ (independent of ℓ). The cases d) and e) consider the remain maps where $\ell > 1$ and $|deg(f)| > \ell - \chi(S_h)/(1 - \chi(S_g))$.

Part a) follows from Kneser [Kn] since the map can be deformed to be root free.

Part b) follows from [BGZ, Theorem 3.3].

Part c) It follows from [GZ, Theorem 3.6] that f satisfies the Wecken property. Therefore, it can be deformed to have exactly ℓ roots, which is the number of root classes. Since all classes are essential it follows that each root classes has exactly one point and the result follows.

Part d) Let us consider a lift $\tilde{f} : S_h \to S_{g'}$ of f, where $p : S_{g'} \to S_g$ is the cover which corresponds to the subgroup $f_{\#}(\pi_1(S_h))$. Because $\ell > 1$, $p^{-1}(y_0) = (y_1, ..., y_\ell)$ has more than one point. Since $|deg(f)|\chi(S_g) - \chi(S_h)) \geq d - 1$ it follows from [BGZ, Theorem 3.3] that the minimal number MR of roots satisfies $\ell < MR \leq (\ell-1)d+1$. Therefore we have $\ell \leq MR-1 \leq (\ell-1)d$. So we can write MR as a sum of ℓ positive numbers where the first summand is 1. At the extreme case $MR-1=(\ell-1)d$ the other summands would be d. These summands define a partition of MR of the form $m_1, ..., m_\ell$ where $0 < m_i \leq d$ and $m_1 = 1$. There is a primitive branched covering of degree d having $(y_1, ..., y_\ell)$ as branching points and over each branching point y_i has m_i points. This follows from [BGZ, Proposition 5.8], for the case where $g > 1$ and, from [BGKZ3, Theorem 1.2] or [BGKZ4, Theorem 3.3], in the case the target is the torus. So, the map which is the composite of the constructed branched covering with the projection p is a map such that the minimal number of points of a root class is one. From Lemma 2.4, the result follows.

Part e) The proof is similar to the previous case and we mention only the point where we have to adapt the argument. In this case as result of the hypothesis we have $(\ell-1)d+1 < MR < \ell d$. So, we can consider a partition of this number as ℓ summands $m_1, ..., m_\ell$ where $m_i = d$ for $i = 2, ..., \ell$ and m_1 the number claimed in the Theorem. Observe that this is a partition which minimize the value of m_1. Then we continue the proof as in the previous case and the result follows. □

Now we move to the case where $f : M_1 \to M_2$ is a map between surfaces not necessarily orientable.

Then we have a result similar to the previous Theorem:

Theorem 3.3. Let $f : M_1 \to M_2$ be a map between two surfaces with nonnegative Euler characteristic, $\ell = NR[f]$ and $A(f) = \ell d$. Then, the minimal number of a root class is given by:
a) If $A(f) = 0$, it is zero.
b) If $A(f) \neq 0$ and $\ell = 1$, it is $\max\{1, A(f) - (A(f)\chi(M_2) - \chi(M_1))\}$.
c) If $0 < A(f) \leq \ell - \chi(M_1)/(1-\chi(M_2))$, it is one.
d) If $A(f) > \ell - \chi(M_1)/(1-\chi(M_2))$, $\ell > 1$ and $(A(f)\chi(M_2) - \chi(M_1)) \geq d-1$, it is one.
e) If $A(f) > \ell - \chi(M_1)/(1-\chi(M_2))$, $\ell > 1$ and $0 \leq (A(f)\chi(M_2) - \chi(M_1)) < d-1$, it is $d - (A(f)\chi(M_2) - \chi(M_1))$.

Proof. Similar to the proof of Theorem 3.2. First we observe that the Theorem cover all maps.

Part a) follows from Kneser [Kn] since the map can be deformed to be root free.

Part b) follows from [BGKZ1, Theorem 1.1].

Part c) It follows from [GKZ,Theorem 4.6] that f satisfies the Wecken property. Therefore, it can be deformed to have exactly ℓ roots, which is the number of root classes. Since all classes are essential it follows that each root classes has exactly one point and the result follows.

Part d) Let us consider a lift $\tilde{f} : M_1 \to M_2'$ of f, where $p : M_2' \to M_2$ is the cover which corresponds to the subgroup $f_\#(\pi_1(M_1))$. Because $\ell > 1$, $p^{-1}(y_0) = (y_1, ..., y_\ell)$ has more than one point. Since $A(f)\chi(M_2) - \chi(M_1)) \geq d - 1$ it follows from [BGKZ1, Theorem 1.1] that the minimal number MR of roots satisfies $\ell < MR \leq (\ell - 1)d + 1$. Therefore we have $\ell \leq MR - 1 \leq (\ell - 1)d$. So we can write MR as a sum of ℓ positive numbers where the first summand is 1. At the extreme case $MR - 1 = (\ell - 1)d$ the other summands would be d. These summands define a partition of MR of the form $m_1, ..., m_\ell$ where $0 < m_i \leq d$ and $m_1 = 1$. There is a primitive branched covering of degree d having $(y_1, ..., y_\ell)$ as branching points and over each branching point y_i has m_i points. If M_2 is orientable, this is the case in the proof of Theorem 3.2. For M_2 nonorientable, the existence of a primitive branched covering follows from [BGKZ1, Theorem 3.3], for the case where M_2 has Euler characteristica negative and, from [BGKZ3, Theorem 1.2] or [BGKZ4, Theorem 3.3], in the case the target is the Klein bottle. So, the map which is the composite of the constructed branched covering with the projection p is a map such that the minimal number of points of a root class is one. From Lemma 2.4, the result follows.

Part e) The proof is similar to the previous case and we mention only the point where we have to adapt the argument. In this case as result of the hypothesis we have $(\ell - 1)d + 1 < MR < \ell d$. So, we can consider a partition of this number as ℓ summands $m_1, ..., m_\ell$ where $m_i = d$ for $i = 2, ..., \ell$ and m_1 the number claimed in the Theorem. Observe that this is a partition which minimize the value of m_1. Then we continue the proof as in the previous case and the result follows. □

3.1 Remark

It would be interesting to explore the consequences of the above results for quadratic equations in free group. See [GZ], [BGZ] and [GKZ].

4 Maps from a complex into a manifold

In this section we construct, for each $n \geq 3$, an n-dimensional complex K_n and a map $f : K_n \to M$ which has the following property: The map f cannot be deformed to a map g such that the number of points of each root class of g is the minimum, as defined in Definition 2.2. The complex K_n has the property that it is n-dimensionally connected, which is a property that holds for a manifold. This type of map was constructed for the first time, at least for a complex K_3 of dimension 3, in [An].

Let us observe that it is not possible to construct such examples if K_n is a manifold, as result of H. Schirmer's work [Sc]. On the otherhand if we relax the condition that K_n is n-dimensionally connected, for example allowing K_n to have cut points (i.e. the link of a vertex is not connected), then one can construct one example (as pointed out by the referee) as follows: Take the wedge of k-copies of S^n and map each of them into P^n by the natural covering map. If k is at least 2 it is not difficult to show that $MR[f, C]$ is 1, the Nielsen root number is 2 but the minimal number of roots is $k+1$.

Denote by $\Delta^n =< v_0, \dots, v_n >$ the n-dimensional simplex. Define $K_n = \cup_{i=1}^{n+1} S_i$, where each S_i is a copy of the sphere S^n, which we regard as the boundary of Δ^{n+1}, i.e. $S_i = \partial < v_{i0}, \dots, v_{i(n+1)} >$. Now we identify the faces of these complexes as follows:

1. $S_1 \cap S_2$

   $v_{10} = v_{20},\ v_{11} = v_{21}, \dots, v_{1(n-1)} = v_{2(n-1)}$
2. $S_2 \cap S_3$

   $v_{21} = v_{31},\ v_{22} = v_{32}, \dots, v_{2n} = v_{3n}$
3. $S_3 \cap S_4$

   $v_{32} = v_{42},\ v_{33} = v_{43}, \dots, v_{3(n+1)} = v_{4(n+1)}$
4. $S_4 \cap S_5$

   $v_{43} = v_{53},\ v_{44} = v_{54}, \dots, v_{4(n+1)} = v_{5(n+1)},\ v_{40} = v_{50}$

   ....

Proposition 4.1. The complex K_n defined as above has the following properties:

(1) Every maximal simplex is n-dimensional.
(2) It is simply connected, i.e. $\pi_1(K_n) = 0$.
(3) It is n-dimensionally connected, i.e. any two n-dimensional simpleces can be joined by a sequence of $n-$simpleces Δ_i such that the intersection of two consecutive ones is a $(n-1)-$simplex.

Proof. Part 1. Given $\sigma \subset K_n$ a p-simplex with $p \leq n-1$, from the definition of K_n it follows that σ belongs to S_i, for some $i = 1, \dots, n+1$. Since S_i is a manifold of dimension n, it implies that $\sigma \subset \tau$, where τ is an n-simplex.

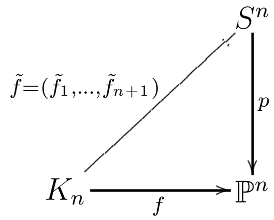
Part 2. Define $K_p = K_{p-1} \cup S_{p+1}$, for $p \leq n$, where $K_0 = S_1$. We show by induction that K_p is simply connected, for all $p \leq n$. The statement is true for $p = 0$, since the sphere S_1 of dimension $n \geq 3$ is simply connected. Suppose by induction hypothesis we have that K_p is simply connected. Since $K_{p+1} = K_p \cup S_{p+2}$, and $K_p \cap S_{p+2}$ is path connected, as result of the identifications given above, we can use the Van Kampen Theorem (see [Ar], pg. 138) to compute $\pi_1(K_{p+1})$. Since the sphere S_{p+2} and K_p are simply connected, the latter one as result of the induction hypothesis, we obtain that K_{p+1} is also simply connected. So, the result follows.

Part 3. Let σ, $\tau \subset K_n$, be two n-simpleces. Without loss of generality we can assume that $\sigma \subset S_k$ and $\tau \subset S_p$ for some k and p such that $1 \leq k \leq p \leq n+1$. For $k = p$ the result is true, since the sphere S_k is a manifold and it has the property that we want to prove. Suppose that $p = k+1$. Since $S_k \cap S_{k+1}$ is a $(n-1)$-simplex $s_{(n-1)}$, let us choose two n-simplexes σ_1, and τ_1, belonging to S_k, and S_{k+1}, respectively, such that $s_{(n-1)}$ is in the boundary of both n-simplexes. Because S_k, S_{k+1} are manifolds we can connect σ to σ_1, and τ_1 to τ, therefore we can connect σ to τ, and the result follows. To finish the proof we can argue by induction on the difference $p-k$. Assuming the result is true for $p - k = l$, let us show for $l+1$. Given σ and τ as above, let γ be an arbitrary n-simplex in S_{k+1}. From above we can connect σ to γ, and by induction hypothesis we can connect γ to τ. Therefore we can connect σ to τ and the result follows. □

Now, we show the main result of this section. Let $\mathbb{P}^n$ be the n-dimensional real projective space.

Theorem 4.2. For each $n \geq 3$ there exists $f : K_n \to \mathbb{P}^n$ such that:
(1) $NR[f] = 2$.
(2) The minimal number of a root class is 1.
(3) $MR[f, y_0] > 2$.

Proof. Two simplicial complexes K, L are homeomorphic if there is a bijection ϕ between the set of the vertices of K and L such that $\{v_1, v_2, \ldots, v_s\}$ is a simplex in K if and only if $\{\phi(v_1), \phi(v_2), \ldots, \phi(v_s)\}$ is a simplex in L (see [Ar], pg 128). Using the fact above, we can construct, for each $i = 2, \ldots, n+1$, homeomorphisms $h_i : S_i \to S_{i-1}$ such that $h_i|S_i \cap S_{i-1}$ is the identity. In order to define our map $f : K_n \to \mathbb{P}^n$ let $\tilde{f}_1 : S_1 \to S^n$ be any homeomorphism from S_1 to the sphere S^n. Define $\tilde{f}_2 = \tilde{f}_1 \circ h_2 : S_2 \to S_1 \to S^n$ and observe that $\tilde{f}_1(x) = \tilde{f}_2(x)$ for $x \in S_1 \cap S_2$. So, from $\tilde{f}_1, \tilde{f}_2$ we obtain a well defined map $S_1 \cup S_2 \to \mathbb{P}^n$ which extends both maps. Inductively define $\tilde{f}_i = f_{i-1} \circ h_i : S_i \to S^n$, which is a homeomorphism with the property $\tilde{f}_i(x) = \tilde{f}_{i-1}(x)$, for all $x \in S_{i-1} \cap S_i$. So define $\tilde{f} : K_n \to S^n$ to be the map such that $\tilde{f}|S_i = \tilde{f}_i$. Let $f = p \circ \tilde{f} : K_n \to \mathbb{P}^n$ be the two-fold covering map.



Part 1. Let $y_0 = f(v_{1(n-1)})$ and $p^{-1}(y_0) = \{\tilde{y}_0, -\tilde{y}_0\}$. Since every map homotopic to $\tilde{f}$ is surjective, it follows that $NR[f] = 2$ (see [Br], lemma 1 and lemma 2) .

Part 2. We have $v_{1(n-1)} \in \tilde{f}^{-1}(\tilde{y}_0)$ or $v_{1(n-1)} \in \tilde{f}^{-1}(-\tilde{y}_0)$. Suppose that $v_{1(n-1)}$ belongs to $\tilde{f}^{-1}(\tilde{y}_0)$, so $\tilde{f}^{-1}(\tilde{y}_0) = \{v_{1(n-1)}\}$. For, $v_{1(n-1)} = \cap_{i=1}^{n+1} S_i$, otherwise we could find a point $v_0 \in K_n$ with the property $\tilde{f}(v_0) = \tilde{y}_0$. But $K_n = \cup_{i=1}^{n+1} S_i$ and so v_0 belong to some S_i and $\tilde{f}(v_0) = \tilde{f}(v_{1(n-1)})$, which contradicts the fact that the restriction of $\tilde{f}$ to S_i is a homeomorphism. Therefore, this class has only one point and it follows that the minimum number of a root class is 1. From Proposition 2.3 this is the case for all the other root classes and the result follows.

Part 3. Since f restricted to each sphere has at least two roots, and the intersection of all the spheres has only one point, we have that $MR[f, y_0] > 2$ and the result follows.
□

Acknowledgements

This work is part of the Projeto-Temático Topologia Algébrica, Geométrica e Diferencial-Fapesp No 2000/05385-8.

References

[An] C. Aniz: *Raízes de funções de um Complexo em uma variedade*, Phd Thesis, São Carlos-SP-Brazil, 2002.

[Ar] M.A. Armstrong: *Basic Topology. Undergraduate Texts in Mathematics*, Springer-Verlag, New York Inc., 1983.

[BGKZ1] S. Bogatyi, D.L. Gonçalves, E.A. Kudryavtseva and H. Zieschang: "Minimal number of roots of surface mappings", *Matem. Zametki*, (to appear).

[BGKZ2] S. Bogatyi, D.L. Gonçalves, E.A. Kudryavtseva and H. Zieschang: "On the Wecken property for the root problem of mappings between surfaces", *Moscow Math. Journal*, (to appear).

[BGKZ3] S. Bogatyi, D.L. Gonçalves, E.A. Kudryavtseva and H. Zieschang: "Realization of primitive branched coverings over surfaces following the Hurwitz approach", *Central Europ. J. of Mathematics*, Vol. 2, (2003), pp. 184–197.

[BGKZ4] S. Bogatyi, D.L. Gonçalves, E.A. Kudryavtseva and H. Zieschang: "Realization of primitive branched coverings over closed surfaces", *Kluwer*, Preprint, (to appear).

[BGZ] S. Bogatyi, D.L. Gonçalves and H. Zieschang: "The minimal number of roots of surface mappings and quadratic equations in free products", *Math. Z.*, Vol. 236, (2001), pp. 419–452.

[Br] R.B.S. Brooks: "On the sharpness of the Δ_2 and Δ_1 Nielsen numbers", *J. Reine Angew. Math.*, Vol. 259, (1973), pp. 101–109.

[BSc] R. Brown and H. Schirmer: "Nielsen root theory and Hopf degree theory", *Pacific J. Math.*, Vol. 198(1), (2001), pp. 49–80.

[Ep] D.B.A. Epstein: "The degree of a map", *Proc. London Math. Soc.*, Vol. 3(16), (1966), pp. 369–383.

[GZ1] D. Gabai and W.H. Kazez: "The classification of maps of surfaces", *Invent. Math.*, Vol. 90, (1987), pp. 219–242.

[GZ2] D. Gabai and W.H. Kazez: "The classification of maps of nonorientable surfaces", *Math. Ann.*, Vol. 281, (1988), pp. 687–702.

[GKZ] D.L. Gonçalves, E.A. Kudryavtseva and H. Zieschang: "Roots of mappings on nonorientable surfaces and equations in free groups", *Manuscr. Math.*, Vol. 107, (2002), pp. 311–341.

[GZ] D.L. Gonçalves and H. Zieschang: "Equations in free groups and coincidence of mappings on surfaces", *Math. Z.*, Vol. 237, (2001), pp. 1–29.

[Kn] H. Kneser: "Die kleinste Bedeckungszahl innerhalb einer Klasse von Flächen abbildungen", *Math. Ann.*, Vol. 103, (1930), pp. 347–358.

[Ki] T.H. Kiang: *The theory of fixed point classes*, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[Sc1] H. Schirmer: " Mindestzahlen von Koinzidenzpunkten", *J. Reine Angew. Math.*, Vol. 194(1-4), (1955), pp. 21–39.