

Oscillation Results for Second Order Nonlinear Differential Equations

Jozef Džurina^{1*}, Dáša Lacková^{2†}

¹ Department of Mathematics,
Faculty of Electrical Engineering and Informatics,
Technical University,
B. Němcovej 32, 042 00 Košice, Slovakia

² Department of Applied Mathematics and Business Informatics,
Faculty of Economics,
Technical University,
B. Němcovej 32, 042 00 Košice, Slovakia

Abstract: In this paper, the authors present some new results for the oscillation of the second order nonlinear neutral differential equations of the form

$$\left(r(t)\psi(x(t)) [x(t) + p(t)x(\tau(t))] \right)' + q(t)f(x[\sigma(t)]) = 0.$$

Easily verifiable criteria are obtained that are also new for differential equations without neutral term i.e. for $p(t) \equiv 0$.

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1 Introduction

In this paper we deal with the oscillatory behavior of the solutions of the following neutral differential equation

$$\left(r(t)\psi(x(t)) [x(t) + p(t)x(\tau(t))] \right)' + q(t)f(x[\sigma(t)]) = 0. \quad (E)$$

* jozef.dzurina@tuke.sk

† dasa.lackova@tuke.sk

Throughout this paper we suppose that the following conditions (H1)–(H6) hold.

(H1) $r(t), q(t) \in C([t_0, \infty))$ are positive;

(H2) $p(t) \in C([t_0, \infty))$, $0 \leq p(t) \leq p < 1$;

(H3) $\tau(t) \in C([t_0, \infty))$, $\tau(t) \leq t$, $\lim_{t \rightarrow \infty} \tau(t) = \infty$;

(H4) $\sigma(t) \in C^1([t_0, \infty))$, $\sigma(t) \leq t$, $\lim_{t \rightarrow \infty} \sigma(t) = \infty$, $\sigma'(t) \geq 0$;

(H5) $\psi(u) \in C((-\infty, \infty))$, $0 < m \leq \psi(u) \leq M$;

(H6) $f(u) \in C((-\infty, \infty))$ is nondecreasing, $f \in C^1(\mathbb{R} - \{0\})$ and $uf(u) > 0$ for $u \neq 0$.

The problem of oscillatory behavior of the neutral differential equations is of both theoretical and practical interest. Recently, many results regarding particular cases of (E) have been published (see enclosed references). The authors generalized and extended some known oscillatory results. Notably, Wang & Yu in [14] considered general differential equations and obtained some oscillation criteria for the second order neutral differential equations.

The aim of this paper is to establish some new oscillatory results for (E), which are new even for the corresponding ordinary delay differential equations (i.e. $p(t) \equiv 0$) and improve those results presented in [1]–[14].

By a solution of (E) we mean a function $x \in C^1[T_u, \infty)$, $T_u \geq t_0$, which has the property $r(t)\psi(x(t))[x(t) + p(t)x(\tau(t))] \in C^1[T_u, \infty)$ and satisfies (E) on $[T_u, \infty)$. We consider only those solutions $u(t)$ of (E) which satisfy $\sup\{|u(t)| : t \geq T\} > 0$ for all $T \geq T_u$. We assume that (E) possesses such a solution.

As usual, a solution of (E) is said to be oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$ and (E) is said to be oscillatory if every solution of this equation is oscillatory.

For the sake of convenience, we assume that all functional inequalities, used in this paper, hold eventually, that is, they are satisfied for all sufficiently large t .

2 Oscillation

The following theorems provide sufficient conditions for oscillation of all solutions of (E) with respect to properties of the function $f(u)$.

Theorem 2.1. Assume that $f'(u)$ is nondecreasing in $(-\infty, -t^*)$ and non-increasing in (t^*, ∞) , $t^* \geq 0$. Let

$$R(t) = \int_{t_0}^t \frac{1}{r(s)} ds \rightarrow \infty \quad \text{as } t \rightarrow \infty, \quad (1)$$

$$\int_{t_0}^{\infty} q(s)f(\pm NR[\sigma(s)]) ds = \infty \quad \text{for all } N > 0, \quad (2)$$

$$\int_{t_0}^{\infty} \left(R[\sigma(t)]q(t) - \frac{M\sigma'(t)}{4(1-p)R[\sigma(t)]r[\sigma(t)]f'(\pm KR[\sigma(t)])} \right) dt = \infty, \quad (3)$$

for some $K > 0$. Then (E) is oscillatory.

Proof. Let $K > 0$ be such that (3) holds. Assuming the converse, we admit that (E) has an eventually positive solution $x(t)$. The case, when $x(t) < 0$ can be treated by the same arguments. Set

$$z(t) = x(t) + p(t)x(\tau(t)) . \tag{4}$$

Then $z(t) > 0$, $z(t) \geq x(t)$ and moreover, (E) can be rewritten as

$$\left(r(t)\psi(x(t))z'(t) \right)' + q(t)f\left(x[\sigma(t)]\right) = 0 . \tag{5}$$

Consequently, $\left(r(t)\psi(x(t))z'(t) \right)' < 0$ and taking into account (H1) and (H5), we obtain that either $z'(t) > 0$ or $z'(t) < 0$. It is easy to see that the condition $r(t)\psi(x(t))z'(t) < 0$ together with (1) and (H5) lead to $z(t) \rightarrow -\infty$ as $t \rightarrow \infty$. This contradiction affirms that $z'(t) > 0$ and moreover

$$\begin{aligned} x[\sigma(t)] &= z[\sigma(t)] - p(\sigma(t))x(\tau[\sigma(t)]) \geq z(\sigma(t)) - p[\sigma(t)]z(\tau[\sigma(t)]) \\ &\geq z[\sigma(t)]\left(1 - p[\sigma(t)]\right) \geq p^*z[\sigma(t)] , \end{aligned} \tag{6}$$

where $p^* = 1 - p$. From (H6) we have

$$f\left(x[\sigma(t)]\right) \geq f\left(p^*z[\sigma(t)]\right)$$

and then (5) implies

$$\left(r(t)\psi(x(t))z'(t) \right)' + q(t)f\left(p^*z[\sigma(t)]\right) \leq 0 . \tag{7}$$

Define

$$w(t) = R[\sigma(t)] \frac{r(t)\psi(x(t))z'(t)}{f(p^*z[\sigma(t)])} .$$

Then $w(t) > 0$. Using the fact that $r(t)\psi(x(t))z'(t) \leq Mr[\sigma(t)]z'[\sigma(t)]$, one gets in view of (E)

$$\begin{aligned} w'(t) &\leq \frac{\sigma'(t)}{r[\sigma(t)]} \frac{r(t)\psi(x(t))z'(t)}{f(p^*z[\sigma(t)])} - R[\sigma(t)]q(t) \\ &\quad - R[\sigma(t)] \frac{r(t)\psi(x(t))z'(t)}{f^2(p^*z[\sigma(t)])} f'(p^*z[\sigma(t)])p^*z'[\sigma(t)]\sigma'(t) \\ &\leq \frac{\sigma'(t)}{R[\sigma(t)]r[\sigma(t)]} w(t) - \frac{p^*\sigma'(t)f'(p^*z[\sigma(t)])}{MR[\sigma(t)]r[\sigma(t)]} w^2(t) - R[\sigma(t)]q(t) . \end{aligned}$$

It is easy to verify that

$$\begin{aligned} w'(t) &\leq \frac{M\sigma'(t)}{4p^*R[\sigma(t)]r[\sigma(t)]f'(p^*z[\sigma(t)])} - R[\sigma(t)]q(t) \\ &\quad - \frac{p^*\sigma'(t)f'(p^*z[\sigma(t)])}{MR[\sigma(t)]r[\sigma(t)]} \left[w(t) - \frac{M}{2p^*f'(p^*z[\sigma(t)])} \right]^2 . \end{aligned}$$

Therefore

$$w'(t) \leq \frac{M\sigma'(t)}{4p^*R[\sigma(t)]r[\sigma(t)]f'(p^*z[\sigma(t)])} - R[\sigma(t)]q(t). \quad (8)$$

We shall prove that $r(t)\psi(x(t))z'(t) \rightarrow 0$ as $t \rightarrow \infty$. To show it, we let $r(t)\psi(x(t))z'(t) \rightarrow 2L$ as $t \rightarrow \infty$, $0 < L < \infty$. Then, since $r(t)\psi(x(t))z'(t)$ is decreasing, we see that $r(t)\psi(x(t))z'(t) \geq 2L$. Integrating this inequality from t_1 to $\sigma(t)$, we obtain

$$z[\sigma(t)] \geq z(t_1) + \frac{2L}{M} \left(R[\sigma(t)] - R(t_1) \right) \geq \frac{L}{M} R[\sigma(t)].$$

Integrating (7) from t_1 to ∞ and using the last estimate, we get

$$\begin{aligned} r(t_1)\psi(x(t_1))z'(t_1) &\geq \int_{t_1}^{\infty} q(s)f(p^*z[\sigma(s)])ds \\ &\geq \int_{t_1}^{\infty} q(s)f\left(\frac{p^*L}{M}R[\sigma(s)]\right)ds. \end{aligned}$$

This contradicts (2) and we conclude that $r(t)\psi(x(t))z'(t) \rightarrow 0$ as $t \rightarrow \infty$. Then for every $\lambda > 0$, there exists a $t_2 > t_1$ such that for all $t \geq t_2$

$$r(t)\psi(x(t))z'(t) \leq \frac{\lambda}{2}.$$

In other words

$$r(t)z'(t) \leq \frac{\lambda}{2m}.$$

Dividing both sides by $r(t)$ and then integrating from t_1 to $\sigma(t)$, we get for $\lambda = \frac{mK}{p^*}$

$$z[\sigma(t)] \leq z(t_1) + \frac{\lambda}{2m} \left(R[\sigma(t)] - R(t_1) \right) \leq \frac{K}{p^*} R[\sigma(t)]. \quad (9)$$

We claim that $\lim_{t \rightarrow \infty} z(t) = \infty$. To prove it, assume that $\lim_{t \rightarrow \infty} z(t) = 2c$, $0 < c < \infty$. Thus $z[\sigma(t)] \geq 2c$. Then integrating (7) from t to ∞ we have

$$r(t)\psi(x(t))z'(t) \geq \int_t^{\infty} q(s)f(p^*z[\sigma(s)])ds.$$

Integrating once more from t_1 to ∞ , we get

$$\begin{aligned} 2c &\geq z(t_1) + \int_{t_1}^{\infty} \frac{1}{r(u)\psi(x(u))} \int_u^{\infty} q(s)f(p^*z[\sigma(s)])ds du \\ &\geq \frac{f(p^*c)}{M} \int_{t_1}^{\infty} \frac{1}{r(u)} \int_u^{\infty} q(s)ds du \\ &= \frac{f(p^*c)}{M} \int_{t_1}^{\infty} (R(s) - R(t_1))q(s)ds. \end{aligned}$$

On the other hand, since (3) implies $\int^{\infty} R[\sigma(s)]q(s)ds = \infty$, the previous inequality leads to a contradiction. Hence we conclude that $\lim_{t \rightarrow \infty} z(t) = \infty$. Therefore, in view of (9)

$$t^* < p^*z[\sigma(t)] \leq KR[\sigma(t)].$$

Combining the last inequality with (8), we get

$$w'(t) \leq \frac{M\sigma'(t)}{4p^*R[\sigma(t)]r[\sigma(t)]f'(KR[\sigma(t)])} - R[\sigma(t)]q(t) .$$

Integrating from t_1 to t , one can see that

$$w(t) \leq w(t_1) - \int_{t_1}^t \left(R[\sigma(s)]q(s) - \frac{M\sigma'(s)}{4p^*R[\sigma(s)]r[\sigma(s)]f'(KR[\sigma(s)])} \right) ds .$$

Letting $t \rightarrow \infty$, we get $w(t) \rightarrow -\infty$. This contradiction completes the proof of Theorem 2.1.

Remark. Theorem 2.1 generalizes Theorem 4.4.4 in [4] and Theorem 1 in [14].

Remark. Note that for $f(u) \equiv u$ we can replace assumption (H5) by

$$(\widetilde{H5}) \psi(u) \in C(-\infty, \infty), 0 < \psi(u) \leq M$$

and Theorem 2.1 is still true.

For a particular case of (E), namely for the differential equation

$$(r(t)\psi(x(t)) [x(t)+p(t)x[\tau(t)]]')' + q(t) |x[\sigma(t)]|^{\beta-1} x[\sigma(t)] = 0 , \tag{E_\beta}$$

Theorem 2.1 provides the following corollaries:

Corollary 2.2. Let $0 < \beta < 1$. Assume that (1) holds. If for some $L > 0$

$$\int_{t_1}^\infty \left(R[\sigma(t)]q(t) - \frac{L\sigma'(t)}{R^\beta[\sigma(t)]r[\sigma(t)]} \right) dt = \infty , \tag{10}$$

then (E_β) is oscillatory.

Proof. Setting $K = \left(\frac{4(1-p)\beta L}{M} \right)^{\frac{1}{1-\beta}}$ we see that (10) implies (3). On the other hand, it follows from (10) that for some $L > 0$

$$\lim_{t \rightarrow \infty} \left(\int_{t_1}^t R[\sigma(s)]q(s)ds - \frac{L}{1-\beta} R^{1-\beta}[\sigma(t)] \right) = \infty .$$

Hence, for all $N > 0$ and t large enough

$$\int_{t_1}^t R[\sigma(s)]q(s)ds - \frac{L}{1-\beta} R^{1-\beta}[\sigma(t)] > N .$$

This means that

$$\frac{\int_{t_1}^t R[\sigma(s)]q(s)ds}{R^{1-\beta}[\sigma(t)]} > \frac{L}{1-\beta} + \frac{N}{R^{1-\beta}[\sigma(t)]} .$$

Taking limits on both sides and using L'Hospital's rule we obtain

$$\lim_{t \rightarrow \infty} \frac{R^{1+\beta}[\sigma(t)]q(t)r[\sigma(t)]}{\sigma'(t)} > \frac{L}{2}.$$

Therefore for all large t

$$\frac{R^{1+\beta}[\sigma(t)]r[\sigma(t)]q(t)}{\sigma'(t)} > \frac{L}{2}.$$

Obviously,

$$R^\beta[\sigma(t)]q(t) > \frac{L}{2} \frac{d}{dt} \left(\ln R[\sigma(t)] \right).$$

Integrating from t_1 to ∞ we get

$$\int_{t_1}^{\infty} R^\beta[\sigma(t)]q(t)dt = \infty$$

and thus, (2) is satisfied for (E_β) . The assertion of this corollary follows from Theorem 2.1.

Corollary 2.3. Let $0 < \beta < 1$. Assume that (1) holds. Let

$$\liminf_{t \rightarrow \infty} \frac{R^{1+\beta}[\sigma(t)]r[\sigma(t)]q(t)}{\sigma'(t)} > 0. \quad (11)$$

Then (E_β) is oscillatory.

Proof. The reader will easily check that (11) implies (10).

Remark. Theorem 2.1, Corollaries 2.2 and 2.3 are new also for the corresponding delay differential equation

$$(r(t)x'(t))' + q(t)f(x[\sigma(t)]) = 0$$

and generalize Chanturia and Kiguradze's results [3].

Corollary 2.4. Let $\beta = 1$. Assume that (1) holds. If

$$\int^{\infty} \left(R[\sigma(t)]q(t) - \frac{M\sigma'(t)}{4(1-p)R[\sigma(t)]r[\sigma(t)]} \right) dt = \infty, \quad (12)$$

then (E_β) is oscillatory.

Proof. The reader will have no difficulty to show that (12) implies (2) and (3).

Example 2.5. We consider

$$\left(\frac{1}{1+x^2(t)} \left(x(t) + \left(\frac{1}{2} - e^{-t} \right) x[t - |\sin t|] \right) \right)' + \frac{a}{t^2} x[\lambda t] = 0, \quad t \geq 1 \quad (13)$$

where $a > 0$, $0 < \lambda < 1$. We have $\psi(u) = \frac{1}{1+u^2} \leq 1$, $p(t) = \frac{1}{2} - e^{-t} \leq \frac{1}{2}$. It is easy to see that condition (12) reduces to

$$a > \frac{1}{2\lambda} \quad (14)$$

and by Corollary 2.4, Eq.(13) is oscillatory if (14) holds.

Corollary 2.6. Let $\beta = 1$. Assume that (1) holds. Let

$$\liminf_{t \rightarrow \infty} \frac{R^2[\sigma(t)]r[\sigma(t)]q(t)}{\sigma'(t)} > \frac{M}{4(1-p)} .$$

Then (E_β) is oscillatory.

The proof is omitted.

Now we turn our attention to (E) with different properties of the function $f(u)$.

Theorem 2.7. Let (1) hold. Assume that $f'(u)$ is non-increasing in $(-\infty, -t^*)$ and nondecreasing in (t^*, ∞) , $t^* \geq 0$. If for some $K > t^*$

$$\int^\infty \left(R(\sigma(t))q(t) - \frac{M\sigma'(t)}{4(1-p)R[\sigma(t)]r[\sigma(t)]f'(\pm K)} \right) dt = \infty , \tag{15}$$

then (E) is oscillatory.

Proof. On the contrary, we assume that $x(t)$ is a positive solution of (E). Arguing as in the proof of Theorem 2.1, we are led to (8). Since $\int^\infty R[\sigma(t)]q(t)dt = \infty$, then similarly as in the proof of Theorem 2.1, it can be shown that $\lim_{t \rightarrow \infty} z(t) = \infty$. Then $p^*z[\sigma(t)] > K$ for any $K > t^*$ and at the same time $f'(p^*z[\sigma(t)]) \geq f'(K)$. Combining the last inequality with (8), we get

$$w'(t) \leq -R(\sigma(t))q(t) + \frac{M\sigma'(t)}{4(1-p)R[\sigma(t)]r[\sigma(t)]f'(K)} .$$

The rest of the proof is similar to the proof of Theorem 2.1 and hence, it is omitted.

Now we apply our previous result to Eq. (E_β) .

Corollary 2.8. Let (1) hold and $\beta > 1$. If for some $L > 0$

$$\int^\infty \left(R(\sigma(t))q(t) - \frac{L\sigma'(t)}{R[\sigma(t)]r[\sigma(t)]} \right) dt = \infty , \tag{16}$$

then (E_β) is oscillatory.

Proof. Direct calculation shows that (15) reduces to (16) for (E_β) .

Corollary 2.9. Let (1) hold and $\beta > 1$. If

$$\liminf_{t \rightarrow \infty} \frac{R^2[\sigma(t)]r[\sigma(t)]q(t)}{\sigma'(t)} > 0 ,$$

then (E_β) is oscillatory.

The proof is trivial and so it is left to the reader.

To relax the monotonicity conditions imposed to $f(u)$ and $f'(u)$, let us consider the following neutral differential equation

$$\left(r(t)\psi(x(t))\left(x(t) + p(t)x[\tau(t)]\right)' \right)' + q(t)h(x[\sigma(t)]) = 0, \quad (\bar{E})$$

subject to conditions (H1) – (H5) and (H7) $h(u) \in C(-\infty, \infty)$, $uh(u) > 0$ for $u \neq 0$.

It is easy to see from the proofs of presented theorems and corollaries that those results can be reformulated for (\bar{E}) employing the following additional condition

$$h(u) \geq Af(u), \quad A > 0$$

to the assumptions of those theorems and corollaries.

Remark. The method presented in this paper can be applied to the more general neutral differential equation of the form

$$\left(r(t)\psi(x(t))\left(x(t) + \sum_{i=1}^n p_i(t)x[\tau_i(t)]\right)' \right)' + \sum_{j=1}^m q_j(t)f_j(x[\sigma_j(t)]) = 0. \quad (E^*)$$

For the sake of simpler exposition, we stated and proved our result for (E). It is obvious how our results can be extended to (E^*) under suitable assumptions on the coefficients and arguments involved.

We finish our paper with the following result, in which we relax the monotonicity condition imposed on $f'(u)$.

Theorem 2.10. Let (1) hold. If

$$\int^{\infty} q(s)ds = \infty, \quad (17)$$

then (E) is oscillatory.

Proof. We assume that $x(t)$ is an eventually positive solution of (E) . Setting $z(t)$ as in (4) and proceeding as in the proof of Theorem 2.1, we conclude that (7) holds. Set

$$w(t) = \frac{r(t)\psi(x(t))z'(t)}{f(p^*z[\sigma(t)])}.$$

Then $w(t) > 0$ and moreover

$$w'(t) \leq -q(t) - \frac{r(t)\psi(x(t))z'(t)f'(p^*z[\sigma(t)])p^*z'[\sigma(t)]\sigma'(t)}{f^2(p^*z[\sigma(t)])} \leq -q(t).$$

An integration of the last inequality yields

$$w(t) \leq w(t_1) - \int_{t_1}^t q(s) ds .$$

Then $w(t) \rightarrow -\infty$ as $t \rightarrow \infty$ and this contradiction finishes the proof.

Example 2.11. We consider

$$\left(r(t)e^{-|x(t)|} (x(t) + px(t - \tau))' \right)' + q(t) \ln \left(1 + x^2[\sigma(t)] \right) \operatorname{sgn} x[\sigma(t)] = 0 ,$$

which is a particular case to (E). By Theorem 2.10, this equation is oscillatory provided that (1) and (17) are satisfied.

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