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Uncertainty relations expressed by Shannon-like entropies

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Abstract: Besides the well-known Shannon entropy, there is a set of Shannon-like entropies which have applications in statistical and quantum physics. These entropies are functions of certain parameters and converge toward Shannon entropy when these parameters approach the value 1. We describe briefly the most important Shannon-like entropies and present their graphical representations. Their graphs look almost identical, though by superimposing them it appears that they are distinct and characteristic of each Shannon-like entropy. We try to formulate the alternative entropic uncertainty relations by means of the Shannon-like entropies and show that all of them equally well express the uncertainty principle of quantum physics.

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1 Introduction

All types of the Shannon-like (S-L, for short) entropies§, like the Shannon entropy, are based on the notions of probability and uncertainty. Although there is a well-defined mathematical theory of probability, there is no universal agreement about the meaning

§ The Shannon-like entropies are sometimes called the nonstandard [16] or generalized entropies [23].

of probability. Thus, for example, there is the view that probability is an objective property of a system, and another view that it describes a subjective state of belief of a person. Then there is the also the point of view that the probability of an event is the relative frequency of its occurrence in a long or infinite sequence of trials. This latter interpretation is often employed in the mathematical statistics and statistical physics. The probability in everyday life means the degree of ignorance about the outcome of a random trial. Commonly, the probability is interpreted as the degree of the subjective expectation of an outcome of a random trial. Both subjective and statistical probability are "normed" which means that the degree of expectation that an outcome of a random trial occurs, and the degree of the "complementary" expectation, that it does not, always add up to unity.

Intuitively, the *uncertainty* of a random trial is given by the spread of probabilities of its outcomes. The uncertainty of a many component probability distribution is quantitatively given by *one* number H that is a function of all components of a probability distribution, $H(\mathbf{P}) = F(P_1, P_2, ..., P_n)$. This number H satisfies the following requirements¶:

(i) If the probability distribution contains only one component then $H(\mathbf{P}) = 0$. In this case, there is no uncertainty in a random trial because one outcome is realized with certainty.

(ii) The more spread the probability distribution $\mathbf{P}$ is, the larger becomes the value of its uncertainty.

(iii) For a uniform probability distribution $\mathbf{P_u}$, $H(\mathbf{P_u})$ becomes maximal.

An important quantity in the theory of probability is the random variable. A random variable $\tilde{x}$ is a mathematical quantity assuming a set of values with corresponding probabilities. All data necessary for the characterization of a random trial, and the assigned random variable, are usually given by a so-called probabilistic scheme. If $\tilde{x}$ is a discrete random variable then its probability scheme is of the form

S	S_1	S_2	...	S_n
P	$P(x_1)$	$P(x_2)$	...	$P(x_n)$
X	x_1	x_2	...	x_n

$S_1, S_2, ..., S_3$ are the outcomes of a random trial (in quantum physics the *quantum states*), $P(x_1), P(x_2), ..., P(x_n)$ are their probabilities and $x_1, x_2, ..., x_3$ are the values defined on $S_1, S_2, ..., S_n$ (in quantum physics the *eigenvalues*). A probability distribution, $\mathbf{P} \equiv \{P_1, P_2, ..., P_n\}$, is the complete set of probabilities of all individual outcomes of a random trial.

It is well-known that there are several measures of the uncertainty in the theory of probability which can be divided into two classes [29]:

¶ From the mathematical point of view, the probabilistic uncertainty measures map the nonnegative orthant $\mathbf{R}_+^{(\mathbf{n})}$ of the n-dimensional Euclidean space $\mathbf{R}^{(\mathbf{n})}$ into $\mathbf{R}$.

(i) The *moment* measures which give the uncertainty of a random trial by means of the scatter of its values. The moment measures of the uncertainty contain as a rule the values of a random trial as well as the elements of its probability distribution and are often taken as their *central statistical* moments [6].

(ii) The *probabilistic* or *entropic* measures of uncertainty containing in their expressions only components of the probability distribution of a random trial. They determine the sharpness and spreading out of its probability distribution, independent of its actual value of $\tilde{x}$.

$H(\mathbf{P})$ is written as a sum of functions of the individual components of $\mathbf{P}$ (for details see [4] and [15])

$$H(\mathbf{P}) = \sum_{i=1}^{n} f_p(P_i).$$

Functions $f_p(P_n)$ are to be chosen so that their sum satisfies the above requirements arising from an entropic measure of uncertainty. It must result in zero if $P_m = 1$ or 0, and it must be graphically represented by a concave curve. There are several functions of the probability components which satisfy these requirements. The most important are: $f_p^{(1)}(P_m) = -P_m \log P_m$ and $f_p^{(2)}(P_m) = P_m(1-P_m)$. If we take for the uncertainty measure the sum of the functions $f_p^{(1)}$ we have

$$H(\mathbf{P}) = S(\mathbf{P}) = \sum_{i=1}^{n} f_p^{(1)}(P_i) = -\sum_{i=1}^{n} P_i \log P_i. \tag{1}$$

This is the well-known entropic measure of uncertainty called the Shannon entropy [8]. If we take $f_p^{(2)}$ we obtain

$$H_C(\mathbf{P}) = \sum_{i=1}^{n} f_p^{(2)}(P_i) = \sum_{i=1}^{n} P_i(1-P_i) = \sum_{i=1}^{n}(P_i - P_i^2).$$

Since $\sum_{i=1}^{n} P_i = 1$, we have

$$H_C(\mathbf{P}) = 1 - \sum_{i=1}^{n} P_i^2. \tag{2}$$

The quantity $H_C(\mathbf{P})$ is a special case of an entropic uncertainty measure called sometimes the zero-continent entropy [18]. We will deal with it in the next few Sections.

The S-L entropies depend on certain parameters. If these parameters approach 1 then they converge toward the Shannon entropy. In Sec.2 we present some important S-L entropies and describe their mathematical properties. In Sec.3 we depict their graphical representations as a function of components of a probability distribution and the corresponding parameters. In Sec.4 we show how to formulate the uncertainty principle of quantum physics by means of S-L entropies.

2 The Shannon-like entropies

The classical measure of uncertainty, the Shannon entropy $H(\mathbf{P})$, has dominated the literature since it was proposed by Shannon. Recently, due to the desire, mainly in the applied sciences (see e.g. [16], [14] and [13]), to employ entropic measures of uncertainty having properties similar to the Shannon entropy but which are simpler to handle mathematically, the interest in S-L entropies has increased considerably [16]. As a consequence, in the last decades a variety of S-L entropies have been invented.

The central role in the construction of the S-L entropies is played by the expression [29]

$$L(a) = \sum_{i=1}^{n} P_i^a \tag{3}$$

and the function of it $F(L)$ called the basic function. Unlike the Shannon entropy, a S-L entropy does not represent the sum of functions f_p but instead it is a simple function of L. Any basic function must satisfy the condition

$$\frac{dF(L(a \to 1))}{dL} = \pm 1.$$

An important property of $L(a)$ is that its derivative becomes for $a \to 1$ the expression for the Shannon entropy but with the opposite sign

$$\frac{dL(a)}{da} = \sum_{i=1}^{n} P_i \log P_i. \tag{3a}$$

This fact is utilized by the formulation of the S-L entropies. One takes a suitable basic function of $L(a)$, e.g. $F(L(a)) = \log(L(a))$, which becomes for $a \to 1$ zero and divides it by other function $f_d(a)$ which for $a \to 1$ becomes likewise zero, e.g. $f_d(a) = a - 1$. An S-L entropy then represents the ratio

$$H(\mathbf{P}) = \frac{F(L(a))}{f_d(a)}. \tag{4}$$

To evaluate the limit $a \to 1$ of $H(\mathbf{P})$ we must use the L'Hopital rule which yields

$$H(L(a \to 1)) = \frac{\frac{dF(L(a)}{da}}{\frac{df_d(a)}{da}} = -\sum_{i=1}^{n} P_n \log P_n.$$

According to the basic function one obtains different S-L entropies. The most important of them are the following [1]:

(i) The Rényi entropy $H_R(\mathbf{P})$, with the logarithmic basic function, is defined for all real numbers as follows [25]

$$H_R(\mathbf{P}) = \frac{1}{1-\alpha} \log \left\{ \sum_{i=1}^{n} P_i^{\alpha} \right\}, \qquad \alpha \neq 1. \tag{4a}$$

Its 3D-plot for the two-component probability distribution is given in Fig.2.

(ii) The Havrda-Charvat entropy (or β-entropy) with a simple rational basic function, is defined as [10]

$$H_C(\mathbf{P}) = \frac{1}{1-\beta} \left(\sum_{i=1}^{n} P_i^{\beta} - 1 \right), \qquad \beta \in (0, \infty\rangle, \qquad \beta \neq 1. \tag{5}$$

Its 3D-plot for the two-component probability distribution is given in Figs.4.

(iii) The entropy H_T, with the trigonometric basic function, has the form [18]

$$H_T(\mathbf{P}) = \frac{2}{\pi(\gamma-1)} \cos\left(\pi/2 \sum_{i=1}^{n} (P_i^{\gamma})\right) \qquad \gamma \neq 1. \tag{6}$$

Its 3D-plot for the two-component probability distribution is given in Fig.6.

All the S-L entropies listed above converge towards Shannon's entropy if α, β and $\gamma \to 1$. In some instances, it is simpler to compute H_R, H_C or H_T and then recover the corresponding Shannon entropy by taking limits $\alpha, \beta, \gamma \to 1$.

A quick inspection shows that three S-L entropies listed above are all mutually functionally related. For example, each of the Havrda-Charvat entropies can be expressed as a function of Rényi entropy, and vice versa

$$H_C(\mathbf{P}) = \frac{1}{1-\beta} \left(\exp((1-\beta)H_R(\mathbf{P})) - 1\right).$$

There are six properties which are usually considered desirable for a measure of uncertainty defined in terms of probability distributions: (i) symmetry, (ii)expansibility, (iii) subadditivity, (iv) additivity, (v) normalization, and (vi) continuity [1]. The only uncertainty measure which satisfies all these requirements is Shannon's entropy. Each of the other entropies violates at least one of them. As such, the previously mentioned classes of entropies are generalizations of the Shannon entropy in various ways. They are most meaningful for $\alpha, \beta, \gamma > 0$ since they violate the smallest number of the properties in this range of parameters, e. g. Rényi entropies violate only the subaddivity property, Havrda-Charvat entropies violate the additivity property. More details about the properties of each entropies can be found elsewhere (e.g. [1]).

Among the existing S-L entropies, the Havrda and Charvat entropy are perhaps best known and most widely used. This is mainly because Havrda and Charvat entropies have a number of desirable properties which are crucial in many applications. It is more general than Shannon entropy and simpler than Rényi entropy. It depends on a parameter β which is in the interval $\beta \in (0, \infty)$. As such, it represents a family of uncertainty measures which includes the Shannon entropy as a limiting case when $\beta \to 1$. ‖ All the afore mentioned S-L entropies have three important properties:

(i) They assume their maxima for the uniform probability distribution $\mathbf{P}_u$.

(ii) They become zero for the one-component probability distributions.

(iii) They express a measure of the spreading out of a probability distribution. The larger this spread becomes, the smaller values they assume.

The S-L entropies are mathematical quantities which represent the measures of uncertainty of a probability of a statistical system. In physics, the S-L entropies can be

‖ It is noteworthy that two mathematicians, Havrda and Charvat, introduced already in 1967 the $\beta-$ entropy defined as [10] $S_\beta(\mathbf{P}) = \frac{1}{1-\beta}(\sum_{i=1}^{n}(P_i)^\beta - 1)$. This entropy is formally identical with Tsallis entropy reinvented in statistical physics in 1988 [26]. However, one must keep in mind that in the Tsallis entropy $P_i, \quad i = 1, 2, ..., n$ are to be replaced by the relative frequencies of a statistical ensemble. We will not further deal with the application of Tsallis entropy in statistical physics. The reader is referred to [27].

used everywhere where the uncertainty degree of physical probability or statistical systems is needed, *i.e.,* in quantum and statistical physics and in the theory of physical measurement. According to Jaynes theorem, the probabilistic uncertainty of a statistical system is uniquely associated with its *physical* entropy. The S-L entropies are used by the description of the nonextensive statistical systems where the common statistical systems, based on the Boltzmann-Shannon entropy, cannot be applied. For example, using one of the S-L entropies, called the Tsallis entropy, a number of nonextensive systems have been successfully described, e.g. correlated-type anomalous diffusions [22] [3], turbulence in electron plasmas [2], nonlinear dynamical systems [9] etc. (for details see [27]). Moreover, by means of general statistical systems, based on Tsallis' entropy, a class of physical phenomena, e.g. ferromagnetism, multifractals etc. can be successfully described from a unified point of view. In quantum physics the measures of uncertainty of two non-commuting observables are crucial to the formulation of the entropic uncertainty relations (UR) given in the form of the inequalities whose left-hand sides are the sums of uncertainties of two non-commuting observables and their right-hand sides represent the corresponding non-trivial lower bounds. The use of the S-L entropies as measures of the quantum uncertainty may considerably simplify the variational procedure necessary for finding these lower bounds. In the physical theory of measurement, the S-L entropies can appropriately express the degree of 'inaccuracy' of a certain type of physical measurement. Apart from the mentioned applications, the S-L entropies have also been successfully applied in theory of scattering [11] [12].

3 Graphical representation of the Shannon-like entropies

To get an idea of the properties of the S-L entropies we consider a two component probability distribution which makes it possible to show their graphical representation in 3D-plots. A two component probability distribution can be written in the form $\mathbf{P} = \{P, (1-P)\}$ so that the corresponding S-L entropies are functions only of two parameters: P and a. In 3D-plots they can be visually shown as functions of these parameters. To compare the graphical representations of S-L entropies with the Shannon entropy we first depicted in Fig.1 the shape of the Shannon entropy as a function of P.

The Shannon entropy as a function of P is concave and its maximum is at $P = 1/2$. In Fig.2, we plotted the Rényi entropy in 3D-graph to illustrate the dependence on P and α for $\alpha \in [0, 10]$.

We see that as α increases, the shapes of the entropy curves change considerably. An interesting feature of Rényi entropy is that the maxima of its entropy curves do not change with changing α. In order to demonstrate the change of shape of the entropy curves we plotted some of them for the different values of α in Fig.3. The dependencies of the entropy curves on α is clearly visible here.

While the entropy curve for $\alpha = 100$ resembles a triangle, that for $\alpha = 0.3$ lies over the curve assigned to the Shannon entropy.

In Fig.4, we show plots of 3D-graphs for the entropy curves of the Havrda-Charvat entropy as a function of β and P for $\beta \in [0, 10]$.

The maxima of these curves decrease as β increases. The shape of the entropy curves as function of β is shown in Fig.5.

This plot shows how the shape of entropy curves becomes more and more flat as β increases. In Fig.6, the entropy curves of the S-L entropy H_T are represented as functions of P and γ for $\gamma \in [0, 10]$. Again, the shape of these curves become more and more flat with the increase of γ.

Their maxima are plotted against β and γ in Fig.7 (the maxima of H_R are constant and independent of α, Fig.3). The characteristic feature of these shapes is that for $\beta = \gamma = 1$ they have the same value (0.67).

The foregoing figures provide a clear illustration of the dependence of S-L entropies on P and parameters α, β and γ. The graphs of the various S-L entropies look similar, though by superimposing these graphs on one another, we see that they are distinct and characteristic of each S-L entropy. In the next Section we will formulate the well-known uncertainty principle of quantum physics by means of the above described S-L entropies.

4 Uncertainty relations expressed by means of Shannon-like entropies

The fact that two canonically conjugate observables A and B cannot simultaneously have sharp eigenvalues represents the cornerstone of the principle in quantum mechanics and can be quantitatively expressed in different forms, commonly called the uncertainty relations. An uncertainty relation provides an estimate of the minimum uncertainty expected in the outcome of a measurement of an observable, given the uncertainty in the outcome of measurement of another observable. Here, the essential problem is how to mathematically express the uncertainty (imprecision) of observables considered as random variables. In the Heisenberg formulation of the uncertainty relation the standard deviations (variances) of non-commuting observables are taken as the measures of their uncertainties while in the entropic uncertainty relations the entropies of observables are taken as their uncertainties.

The Shannon entropic uncertainty relation is given as the sum of the Shannon entropies of two non-commuting observables. We take for the S-L entropic uncertainty relations likewise the sums of their S-L entropies. The sum of the Rényi $H_R(\mathbf{P})$, Havrda-Charvat $H_C(\mathbf{P})$ and $H_T(\mathbf{P})$ entropy for two probability distributions $\mathbf{P_1} \equiv \{P_1, P_2, ..., P_n\}$ and $\mathbf{P_2} \equiv \{P'_1, P'_2, ..., P'_n\}$ is

$$H_R(\mathbf{P_1^{(\alpha)}}) + H_R(\mathbf{P_2^{(\alpha)}}) = \frac{1}{1-\alpha} \log \left[\left(\sum_{i=1}^{n} P_i^{(\alpha)} \right) \left(\sum_{i=1}^{n} P_i^{\prime(\alpha)} \right) \right], \tag{7}$$

$$H_C(\mathbf{P_1}^{(\beta)}) + H_C(\mathbf{P_2}'^{(\beta)}) = \frac{1}{1-\beta}\left(\sum_{i=n}^{n} P_i^{(\beta)} - 1 + \sum_{n=1}^{n} P_i'^{(\beta)} - 1\right) = \frac{1}{1-\beta}\Bigg(\sum_{i=n}^{n} P_i^{(\beta)} + \sum_{n=1}^{n} P_i'^{(\beta)} - 2\Bigg) \tag{8}$$

$$\begin{aligned} H_T(\mathbf{P}^{(\gamma)}) + H_T(\mathbf{P}'^{(\gamma)}) &= \frac{2}{\pi(\gamma-1)}\left[\cos\left(\frac{\pi}{2}\sum_{i=1}^{n} P_i^{(\gamma)}\right) + \cos\left(\frac{\pi}{2}\sum_{i=1}^{n} P_i'^{(\gamma)}\right)\right] \\ &= \frac{4}{\pi(\gamma-1)}\left[\cos\left\{\left(\frac{\pi}{4}\right)\left(\sum_{i=n}^{n} P_i^{(\gamma)} + \sum_{n=1}^{n} P_i'^{(\gamma)}\right)\right\}\cos\Bigg\{\left(\frac{\pi}{4}\right) \left(\sum_{i=n}^{n} P_i^{(\gamma)} - \sum_{n=1}^{n} P_i'^{(\gamma)}\right)\Bigg\}\right], \end{aligned} \tag{9}$$

respectively.

It is worth noting that the S-L entropies have especially simple forms for $\alpha = 2$, $\beta = 2$ and $\gamma = 2$. In that case, the expression $L(a)$ is called ‘information energy’ and is given by [28]

$$E_i = \sum_{i=k}^{n} P_k^2. \tag{9a}$$

Inserting the corresponding information energies in the S-L entropies we have

$$\begin{aligned} H_R^{(2)} &= -\log E_i, \\ H_C^{(2)} &= 1 - E_i, \\ H_T^{(2)} &= \frac{2}{\pi}\cos\left(\frac{\pi E_i}{2}\right). \end{aligned} \tag{10}$$

Traditionally, in the mathematical formulation of the uncertainty principle, we consider two Hermitian operators $\mathbf{A}$ and $\mathbf{B}$ which represent two non-commuting observables A and B in a finite N-dimensional Hilbert space. Let $\{|a_i\rangle|\}$ and $\{|b_j\rangle\}$, $i, j = 1, 2, ..., N$, be the corresponding complete sets of normalized eigenvectors. The components of the probability distributions of observables A and B, $P_A = \{p_1, p_2, ..., p_n\}$ and $P_B = \{q_1, q_2, ..., q_n\}$, when the quantum state of the investigated system is described by $|\Phi\rangle$, are given by the equations

$$p_i = |\langle a_i|\Phi\rangle|^2 \qquad q_j = |\langle b_j|\Phi\rangle|^2.$$

The Heisenberg variance uncertainty relation is given by an inequality whose left-hand side represents the product of standard deviations of two non-commuting observables A and B. Usually, we write it in the Robertson form ($\hbar = 1$) [7]

$$\Delta A \ \Delta B \geq (1/2)|\langle\Phi|[\mathbf{A}, \mathbf{B}]|\Phi\rangle|, \tag{10a}$$

where ΔA and ΔB are the standard deviations of A and B and $[\mathbf{A}, \mathbf{B}]$ is their commutator. It has been pointed out that the Robertson uncertainty relation has the following serious shortcoming (see, e.g. [17]). If A and B are two non-commuting observables of a finite N-dimensional Hilbert space then the right-hand side of Robertson uncertainty relation is not a fixed lower bound, but it depends on the state of $|\Phi\rangle$. If one of observables A or B is in its eigenstate then the right-hand side of Robertson uncertainty relation equals zero and no restriction for $\Delta A \Delta B$ is imposed by the Heisenberg UR. Other ways of expressing the uncertainty principle of quantum physics were invented in order to avoid this shortcoming. It has been shown that if we use instead of the product of standard deviations the sum of the entropies of non-commuting observables of the left-hand side of (10a) then the corresponding uncertainty relation does not suffer from the shortcoming mentioned above and better reflects the uncertainty principle of quantum physics than the corresponding Heisenberg one.

The Shannon entropic uncertainty relation has the form of the following inequality

$$S(A) + S(B) \geq S(AB),$$

where $S(A)$ and $S(B)$ are Shannon's entropies of two non-commuting observables A and B. The sum of Shannon entropies is independent of the quantum state bounded by a non-trivial real number $S(AB)$ (see, e.g. [17]).

When formulating the S-L entropic uncertainty relations for two non-commuting observables A and B we proceed on the lines of the familiar Shannon entropic UR, i.e. we take the following inequalities

$$\begin{aligned} H_R^{(\alpha)}(A) + H_R^{(\alpha)}(B) &\geq H_R^{(\alpha)}(AB), \\ H_C^{(\beta)}(A) + H_C^{(\beta)}(B) &\geq H_C^{(\beta)}(AB) \end{aligned} \tag{11}$$

and

$$H_T^{(\gamma)}(A) + H_T^{(\gamma)}(B) \geq H_T^{(\gamma)}(AB), \tag{12}$$

where $H_R^{(\alpha)}(AB), H_C^{(\beta)}(AB)$ and $H_T^{(\gamma)}(AB)$ denote the lower bounds of their left-hand sides. Since, according to the definition of the non-commuting observables, they cannot occur simultaneously in one of their eigenstates, $\mathbf{P}$ and $\mathbf{Q}$ cannot become simultaneously one-component probability distributions, therefore $H_R^{(\alpha)}(A, B)$, $H_C^{(\beta)}(A, B)$ and $H_T^{(\gamma)}(A, B)$ are positive numbers (different from zero). The crucial problem concerning the above uncertainty relations is to find these positive numbers which represent their lower bounds. Their determination does not follow easily and the general treatment of this issue would exceed the scope of this article. Commonly, there are two methods of solving this problem: (i) the determination of these bounds by a variational calculation (ii) their determination by an estimation.

Consider two observables A and B with noncommuting Hermitian operators $\hat{A}$ and $\hat{B}$ in an N-dimensional Hilbert space, whose corresponding complete orthonormal sets of eigenvectors $\{|x_j\rangle\}$ and $\{|y_i\rangle\}$ $(i = 1, 2, ..., N)$ are disjointed and have nondegenerate spectra. Let $|\phi\rangle$ be a normalized state vector of N-dimensional Hilbert space, thus we have

$$|\phi\rangle = \sum_i^N a_i |x_i\rangle, \qquad |\phi\rangle = \sum_j^N b_j |y_j\rangle.$$

According to the quantum transformation theory we have the property,

$$
\begin{aligned}
|\phi\rangle &= (\sum_i^N a_i \langle x_i|y_1\rangle)|y_1\rangle + (\sum_i^N a_i \langle x_i|y_2\rangle)|y_2\rangle + ... \\
&= \sum_j^N \sum_i^N a_i \langle x_i|y_j\rangle |y_j\rangle \\
& P_i(A) = |\langle x_i|\phi\rangle|^2 = |a_i|^2, \\
& Q_j(B) = |b_j|^2 = |\langle y_j|\phi\rangle|^2 = |(\sum_i a_i \langle y_i|x_j\rangle)|^2,
\end{aligned} \tag{13}
$$

where $\langle x_i|y_j\rangle \quad i,j = 1,2,3,...,N$ are the elements of the transformation matrix $\mathbf{T}$ between the observables A and B

$$
\mathbf{T} = \begin{pmatrix} \langle x_1|y_1\rangle & \langle x_1|y_2\rangle & ... & \langle x_1|y_n\rangle \\ & & ... & \\ \langle x_n|y_1\rangle & \langle x_n|y_2\rangle & ... & \langle x_n|y_n\rangle \end{pmatrix}.
$$

Accordingly, the sum of Havrda-Charvat entropies of A and B has the form

$$
\frac{1}{1-\beta}\left(\sum_{i=1}^{n} \left|a_i\right|^4 + \sum_{i=1}^{n} \left| \left(\sum_i a_i \langle y_i|x_j\rangle\right) \right|^4 - 2\right).
$$

Given $\mathbf{T}$, the lower bounds of S-L entropic uncertainty relations can be found by variation of corresponding entropic sum over the coefficients $a_1, a_2, ..., a_m$ and thus take the minimum (maximum) variational value as the *exact* lower (upper) bound. Generally, this leads to a series of complicated coupled equations which are often intractable, especially for the sum of Shannon entropies [19]. However, for some simple quantum systems the variational procedure leads quickly to the lower bounds of the corresponding entropies sum. An example of such a quantum system will be described in the next section.

Using an inequality from linear algebra, the lower bounds can also be estimated. For example, according to [5] and [21] the following inequality holds

$$
H_C^{(\beta)}(\mathbf{P}) + H_C^{(\beta)}(\mathbf{Q}) \geq \frac{1}{\beta-1}\left[1 - \left(\frac{2}{1+c}\right)^{2(1-\beta)}\right] + (1-\beta)H_C(\mathbf{P})H_C^{(\beta)}(\mathbf{Q}),
$$

where $c = \sup_{ij}|\langle x_i|y_j\rangle|$. For $\beta \geq 1$ the term $H_C(\mathbf{P})H_C(\mathbf{Q})$ becomes positive and the expression

$$
\frac{1}{\beta-1}\left[1 - \left(\frac{2}{1+c}\right)^{2(1-\beta)}\right]
$$

represents a (weak) lower bound of Havrda-Charvat entropy sum for $\beta \in (1,\infty)$. Of course, by using other more appropriate inequalities from the linear algebra, sharper lower bounds for the entropic sums may be found. Next we write the S-L entropic uncertainty relations for $\alpha = \beta = \gamma = 2$ for a spin-1/2 particle and compare them with the Heisenberg variance uncertainty relation.

5 Particle with spin 1/2

This quantum system instructively illustrates the difference between the S-L uncertainty relations, the Heisenberg variance uncertainty relation and the Shannon entropic uncertainty relation. Consider a quantum system containing a particle with spin $\hbar/2$. We are looking for the different uncertainty relations between the spin components J_x and J_z. According to Eq. (10a), the Heisenberg uncertainty relation for J_x and J_z has the form

$$(\Delta J_x)(\Delta J_z) \geq \frac{\hbar^2}{4} \left| \langle \Phi | [\hat{J}_x, \hat{J}_z] | \Phi \rangle \right| .$$

The state vector of the considered quantum system is a spinor

$$|\Phi\rangle = a_1|z_1\rangle + a_2|z_2\rangle,$$

where

$$a_1 a_1^* + a_2 a_2^* = 1.$$

In order to compare the Heisenberg, the Shannon and the S-L entropic uncertainty relations for the components J_x and J_z, we calculate the product of their standard deviations $U_s(J_x, J_z)$, the sum of their Shannon entropies $S(J_x, J_z)$ as well as sums of their S-L entropies $H_R^{(\alpha)}(J_x, J_z)$, $H_C^{(\beta)}(J_x, J_z)$ and $H_T^{(\gamma)}(J_x, J_z)$. The product of standard deviations of J_x and J_z is [20]

$$U(J_x, J_z) = \langle (\sigma_z - \langle \sigma_z \rangle)^2 \rangle \langle (\sigma_x - \langle \sigma_x \rangle)^2 \rangle.$$

where

$$\hat{\sigma}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we express $U_s(J_x, J_z)$ in terms of a_1, a_2, a_1^* and a_2^* we have

$$U_s(J_x, J_z) = \frac{\hbar^4}{16} \left[1 - (a_1^* a_2 + a_1 a_2^*)^2 \right] \left[1 - (a_1^* a_1 - a_2 a_2^*)^2 \right] . \tag{14}$$

We now introduce the variables r and φ by

$$a_1 = r \exp(i\varphi_1), \qquad a_2 = \left(\sqrt{1 - r^2} \right) \exp(i\varphi_2), \qquad \varphi = \varphi_1 - \varphi_2$$

and calculate the S-L uncertainty relations for spin components J_x and J_z given by the wave functions $|\Phi\rangle_z = a_1|z_1\rangle + a_2|z_2\rangle$.

The probabilistic scheme for J_z and J_x are

J_z	$\|z_1\rangle$	$\|z_2\rangle$
P	$a_1 a_1^*$	$a_2 a_2^*$

and

J_x	$\|x_1\rangle$	$\|x_2\rangle$
P	$2^{-1}(a_1+a_2)(a_1^*+a_2^*)$	$2^{-1}(a_1-a_2)(a_1^*-a_2^*)$

.

We now turn our attention to the S-L entropies with $\alpha = 2, \beta = 2$ and $\gamma = 2$. In terms of the variables r and φ, the probability distributions for J_x and J_z become

$$\mathbf{P_{J_x}} \equiv \left\{r^2, (1-r^2)\right\}$$

and

$$\mathbf{P_{J_z}} \equiv \left\{\frac{1}{2}\left(1+2r\sqrt{1-r^2}\cos\varphi\right), \frac{1}{2}\left(1-2r\sqrt{1-r^2}\cos\varphi\right)\right\}, \tag{15}$$

respectively. Inserting these components in the formula for the information energy, given by Eq.(9a), we obtain

$$E_i^{(J_x)} = r^4 + (1-r^2)^2$$
$$E_i^{(J_z)} = \frac{1}{2} + 2r^2(1-r^2)\cos^2\varphi. \tag{16}$$

With $E_i^{(J_x)}$ and $E_i^{(J_z)}$, the S-L entropic uncertainty relations assume the form

$$H_R^{(2)}(J_z) + H_R^{(2)}(J_x) = H_R^{(2)}(J_x, J_z) = -\log(r^4 + (1-r^2)^2)\Big(\frac{1}{2} \tag{17}$$
$$+2r^2(1-r^2)\cos^2\varphi\Big) \tag{18}$$
$$H_C^{(2)}(J_z) + H_C^{(2)}(J_x) = H_C^{(2)}(J_x, J_z) = 1 - r^4 + (1-r^2)^2 + 1 \tag{19}$$
$$-\left(\frac{1}{2} + 2r^2(1-r^2)\cos^2\varphi\right),$$
$$H_T^{(2)}(J_z) + H_T^{(2)}(J_x) = H_T^{(2)}(J_x, J_z) = \frac{2}{\pi}\cos\left(\frac{\pi}{2}\left(r^4 + (1-r^2)^2\right)\right)$$
$$+\frac{2}{\pi}\cos\left(\frac{\pi}{2}\left(\frac{1}{2} + 2r^2(1-r^2)\cos^2\varphi\right)\right). \tag{20}$$

The expression $H_C^{(2)}(J_x, J_z)$ can be rearranged to a simpler form

$$H_C^{(2)}(J_x, J_z) = \frac{1}{2} + (\sin\varphi)^2 2r^2(1-r^2). \tag{21}$$

Similar rearrangement can be done also for $H_R^{(2)}(J_x, J_z)$ and $H_T^{(2)}(J_x, J_z)$.

It is a straightforward calculation in principle, to determine the wave function at which the minimum or maximum of the entropies of J_x and J_z occurs, and then determine the exact bounds of $H_R^{(2)}(J_x, J_z), H_C^{(2)}(J_x, J_z)$ and $H_T^{(2)}(J_x, J_z)$. The necessary conditions for these extreme values for the corresponding entropies are

$$\frac{\partial H(J_x, J_z)}{\partial r} = 0 \tag{20a}$$
$$\frac{\partial H(J_x, J_z)}{\partial \varphi} = 0. \tag{20b}$$

Inserting Eq.(21) into Eqs.(20a) and (20b) we obtain the following equations which are easy to solve analytically

$$\frac{\partial H_C^{(2)}}{\partial r} = 4r(\sin(\varphi))^2(1-2r^2) = 0 \tag{21a}$$

$$\frac{\partial H_C^{(2)}}{\partial \varphi} = 4r^2(1-r^2)\sin(\varphi)\cos(\varphi) = 0. \tag{21b}$$

A simple calculation yields the value of r and φ for which $H_C^{(2)}(J_x, J_z)$ assumes the extreme values: (i) $r = 1$ or $r = 0$ with arbitrary φ and (ii) $r = 1/\sqrt{2}$ and $\varphi = \pi/2$. With these values, $H_C^{(2)}(J_x, J_z)$ gets its minimum and maximum equal to 1/2 and 1. Hence, $H_C^{(2)}(J_x, J_z)$ is bounded as follows

$$\frac{1}{2} \leq H_C^{(2)}(J_x, J_z) \leq 1.$$

3D-plots of $H_R^{(2)}(A, B), H_C^{(2)}(A, B)$ and $H_T^{(2)}(A, B)$ are shown as functions of r and φ in F igs.10, 11 and 12.

We see that the value of $H(J_x, J_z)$ for all entropies does not fall below certain real non-zero value representing their lower bounds. A remarkable feature of these S-L entropic uncertainty relations is that there is a peak in their graphs in the vicinity of $\varphi = \pi/2$ and $r = 1/\sqrt{2}$. In Fig.10 we see that the determined maximum and minimum of $H_C^{(2)}(A, B)$ is in agreement with those given by its graphical representation.

Now we compare the determination of the upper and lower bounds of S-L entropic uncertainty relations with that of the Shannon entropic and Heisenberg variance uncertainty relations. In our parametrization, $S(J_x, J_z)$ and $U(J_x, J_z)$ gets the form

$$\begin{aligned} S(J_x, J_z) = & -r^2 \ln r^2 - (1-r^2)\ln(1-r^2) - \frac{1}{2}\Big(1 + 2r\sqrt{1-r^2}\cos\varphi\Big) \\ & \ln\left[\frac{1}{2}\Big(1 + 2r\sqrt{1-r^2}\cos\varphi\Big)\right] - \frac{1}{2}\Big(1 - 2r\sqrt{1-r^2}\cos\varphi\Big) \\ & \ln\left[\frac{1}{2}\Big(1 - 2r\sqrt{1-r^2}\cos\varphi\Big)\right] \end{aligned} \tag{22}$$

and

$$U(J_x, J_z) = \frac{1}{16} r^2(1-r^2)\left[1 - 4r^2(1-r^2)\cos^2\varphi\right], \tag{23}$$

respectively. Inserting $S(J_x, J_z)$ into Eqs.(20a) and (20b) we obtain a complicated transcendental equation which we do not present here. These equations are analytically intractable and we have to apply numerical methods for their solutions.

It is easy to verify that the Heisenberg uncertainty relation for $r = 0$ or 1 and for $r = 1/\sqrt{2}$, $U_s(r, \varphi)$ has its minimum and maximum equal to zero and and 1/64, respectively. Therefore, the Heisenberg uncertainty relation is bounded as follows

$$0 \leq U_s(J_z, J_x) \leq \frac{1}{64}.$$

This demonstrates nicely the differences between the Heisenberg variance uncertainty relation, the Shannon entropic uncertainty relation and the S-L uncertainty relations for a pair of the discrete complementary conjugate observables. The relative ease of obtaining the lower and upper bounds of $H_C^{(2)}(J_x, J_z)$ analytically is an advantage of this uncertainty relation over the Shannon entropic one. Figs. 11 and 12 show the Heisenberg and Shannon uncertainty relation as functions of r and φ. The difference between them is clearly seen. From what has been said so far, it follows that:

(i) Besides the well-known Shannon entropy, there is a set of S-L entropies which are generally dependent on certain parameters. These entropies converge to Shannon entropy when these parameters approach 1.

(ii) Considering the two-component probability distribution, the S-L entropies can be shown in 3D-plots as functions of the components of such probability distribution and the corresponding parameters. We have shown that these graphs resemble one another, shearing common properties with the Shannon entropy.

(iii) The sums of the S-L entropies of two non-commuting observables depends on the state of the quantum system. In all S-L entropic uncertainty relations, these sums never fall below certain positive non-zero value.

(iv) It appears that all S-L entropic uncertainty relations express the uncertainty principle of quantum physics equally well.

(v) Due to availability of a number of S-L entropies, a question arises about the criteria for the choice of the proper S-L entropy to be used in a particular case. For this purpose a user has to know the properties which each S-L entropy has and also those which it does not have.

(vi) The choice of the particular S-L entropy for the construction of an actual entropic uncertainty relation is mainly restricted by the degree of the simplicity of the calculation of the upper and lower bands of the corresponding uncertainty relation.

Acknowledgments

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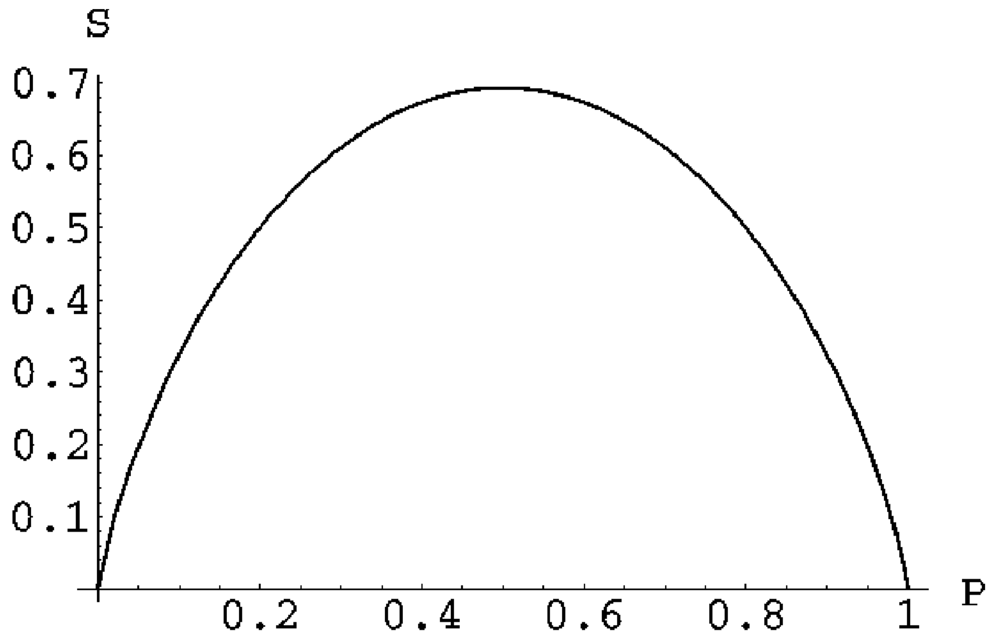

Fig. 1 The Shannon entropy for two-component probability distribution as a function of P.

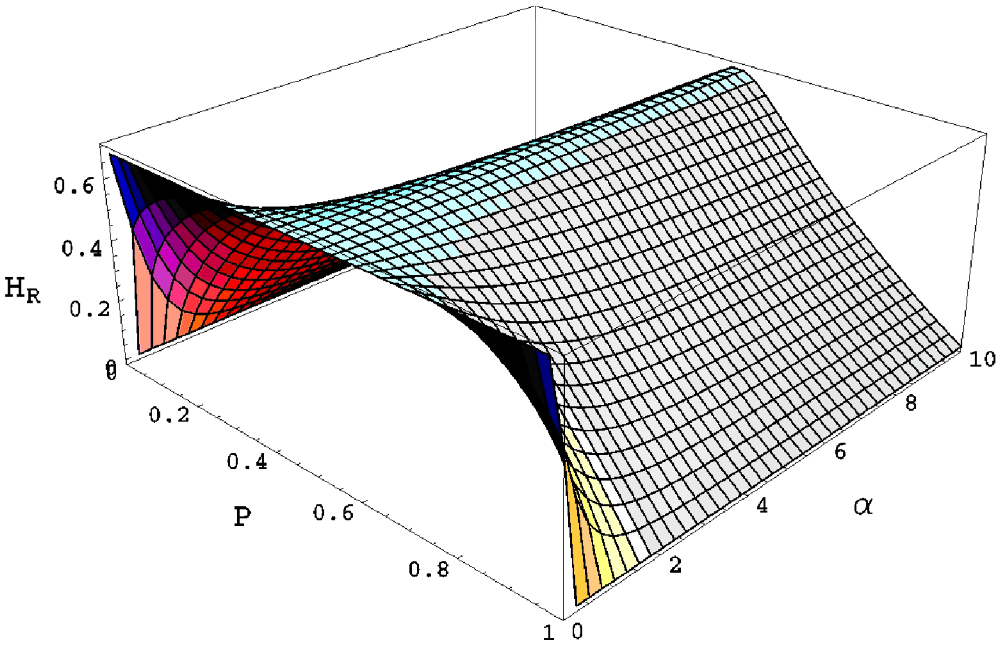

Fig. 2 3D-plot of H_R as function of P and α for $\alpha \in [0, 10]$.

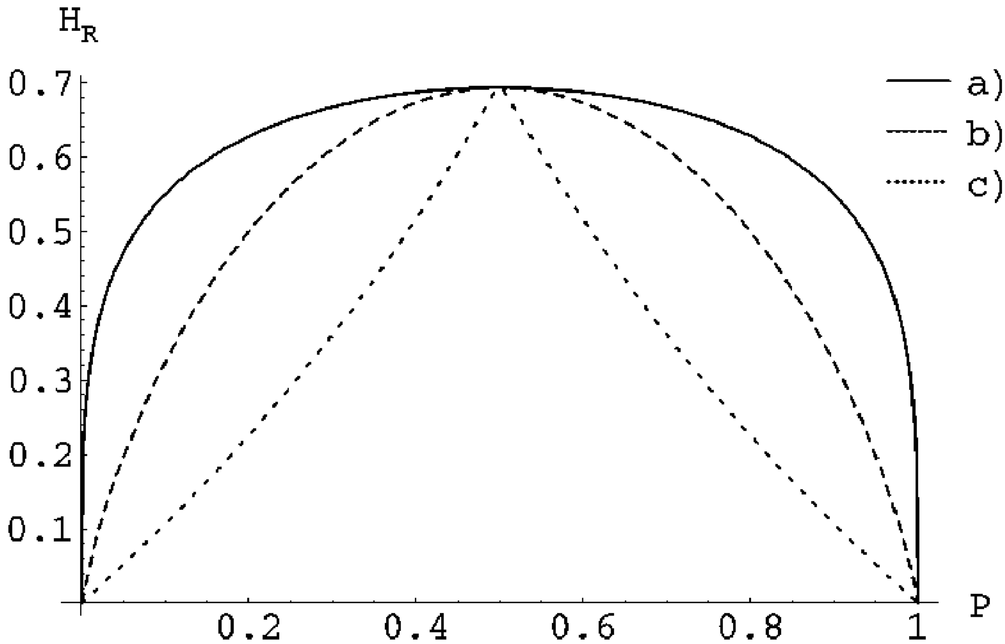

Fig. 3 Graphical representation of the entropy function H_R as a function of P for $\alpha = 0.3$ (curve a), $\alpha = 0.999$ (curve b), and $\alpha = 100$ (curve c).

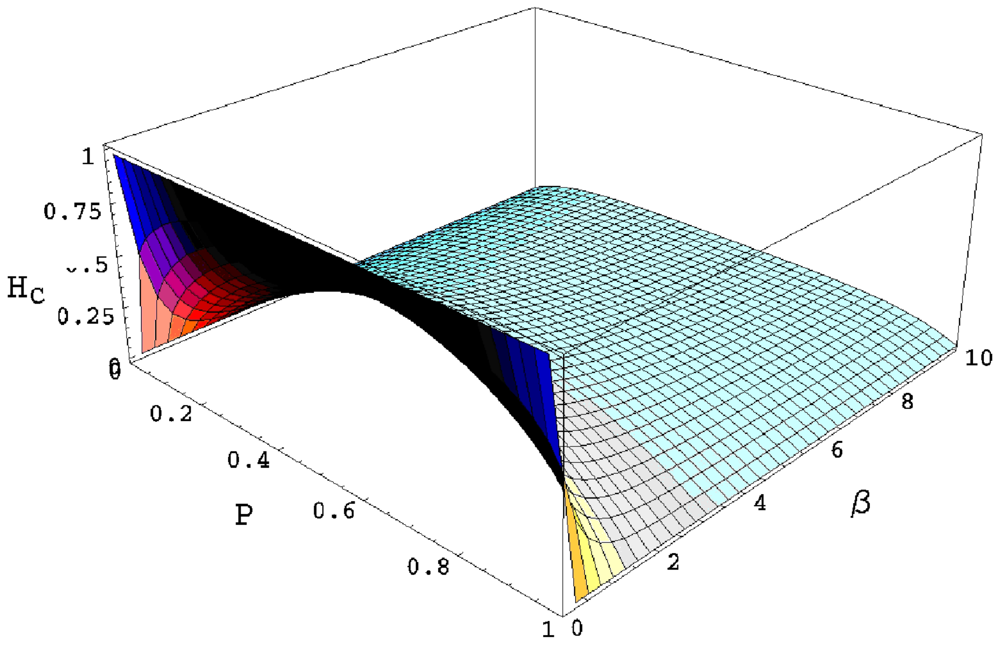

Fig. 4 3D-plot of H_C as function of P and β for $\beta \in [0, 10]$.

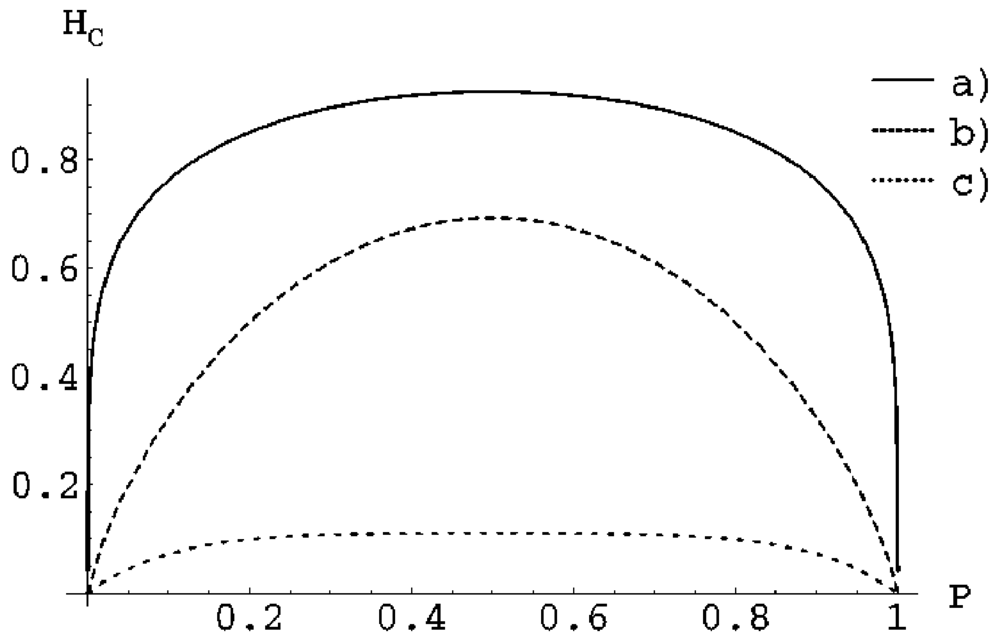

Fig. 5 The shapes of the entropy curves of H_C for $\beta = 0.2$ (curve a) $\beta = 0.999$ (curve b) and $\beta = 10$ (curve c).

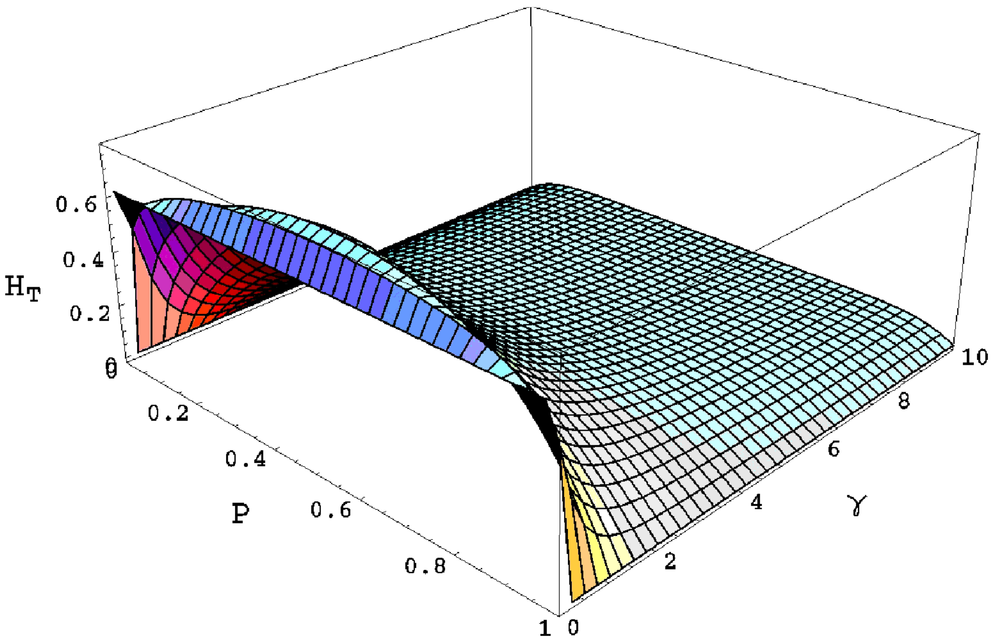

Fig. 6 3D-plot of H_T as function of P and γ for $\gamma \in [0, 10]$.

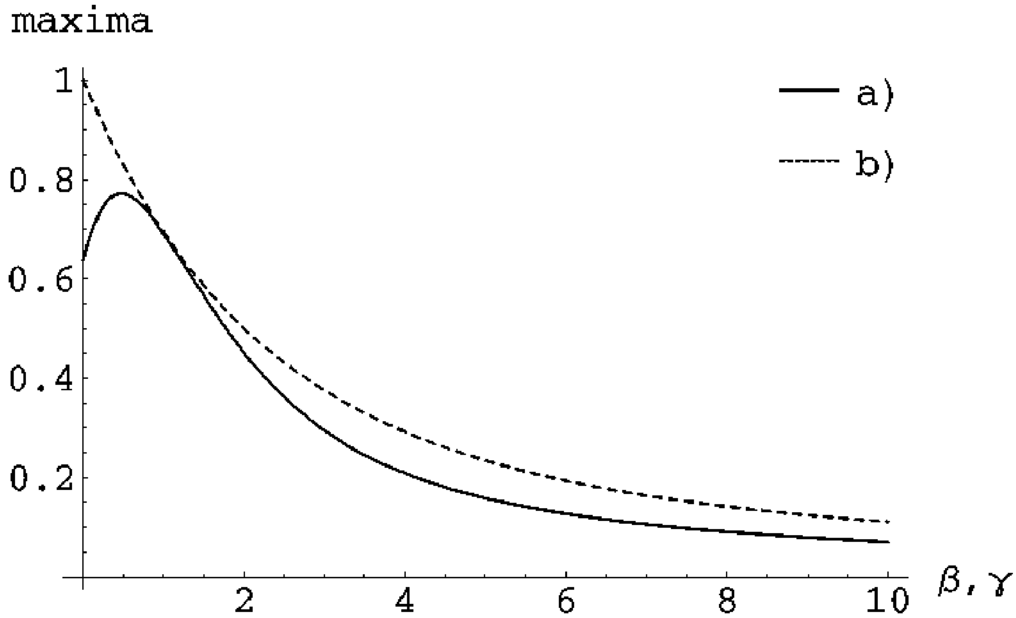

Fig. 7 The maxima of the entropy curves H_C and H_T plotted against parameter β (curve b) and γ (curve a) for $P = 0.5$.

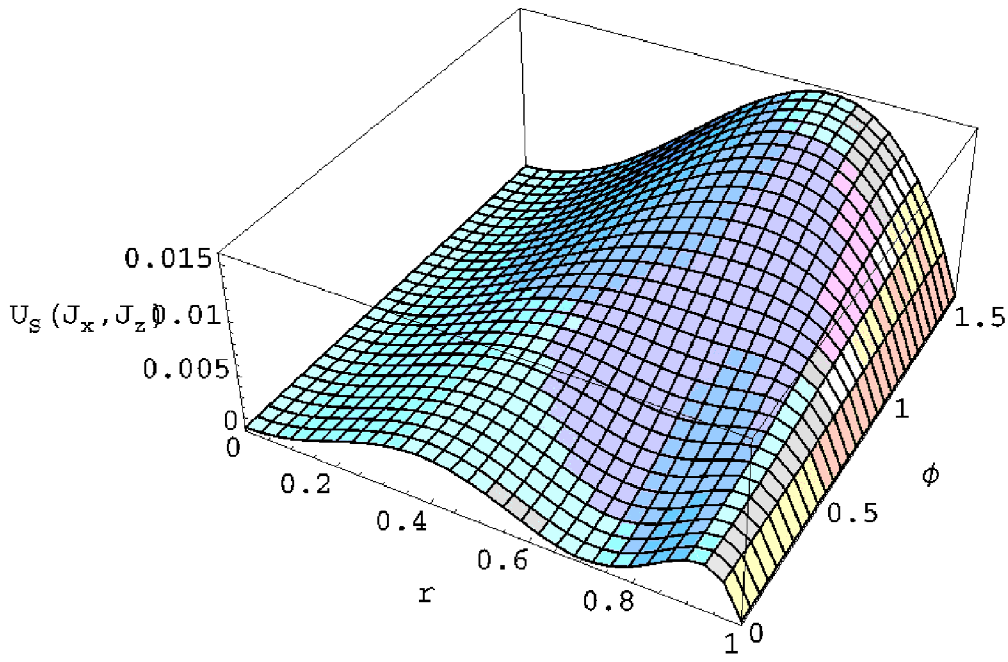

Fig. 8 The Heisenberg variance uncertainty relation for the spin components J_x and J_z as function of r and φ.

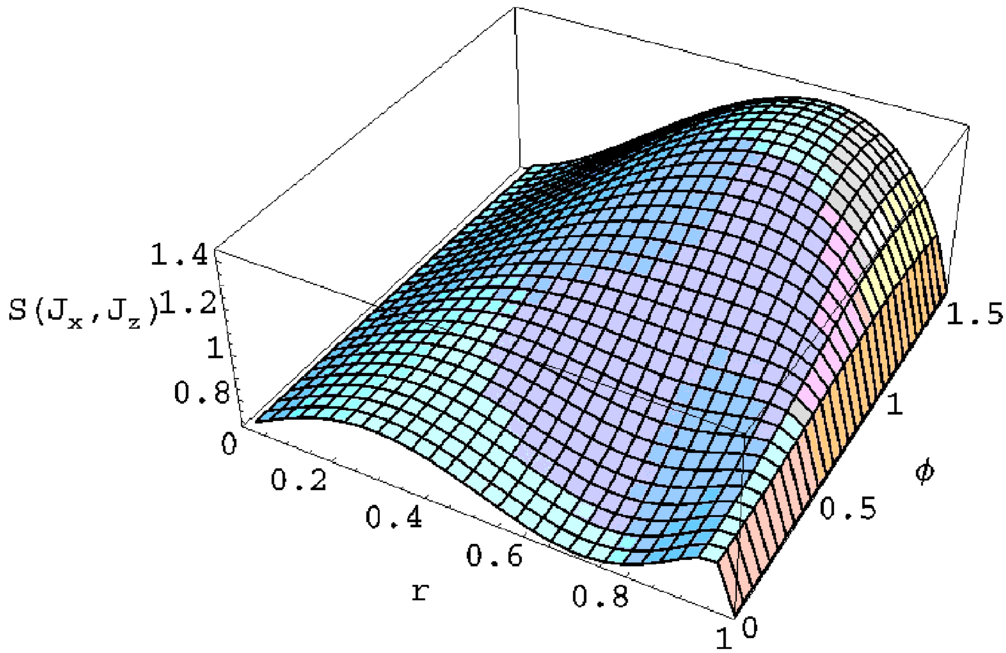

Fig. 9 The Shannon entropic uncertainty relation for spin components J_x and J_z as function of r and φ.

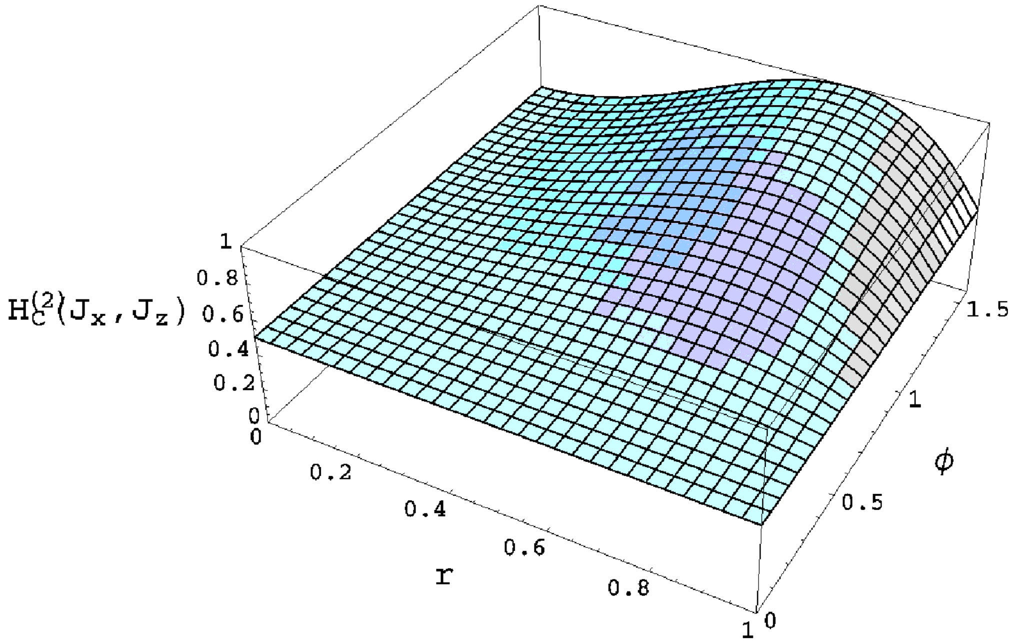

Fig. 10 The uncertainty relation for the spin components J_x and J_z as function of r and φ.

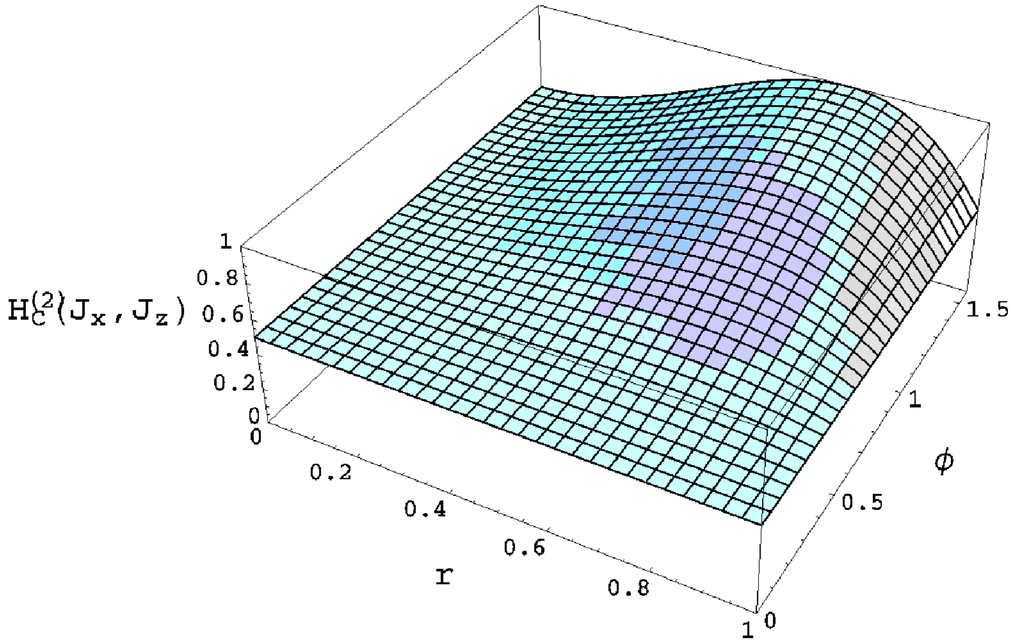

Fig. 11 The uncertainty relation with Rényi entropy for the spin components J_x and J_z as function of r and φ.

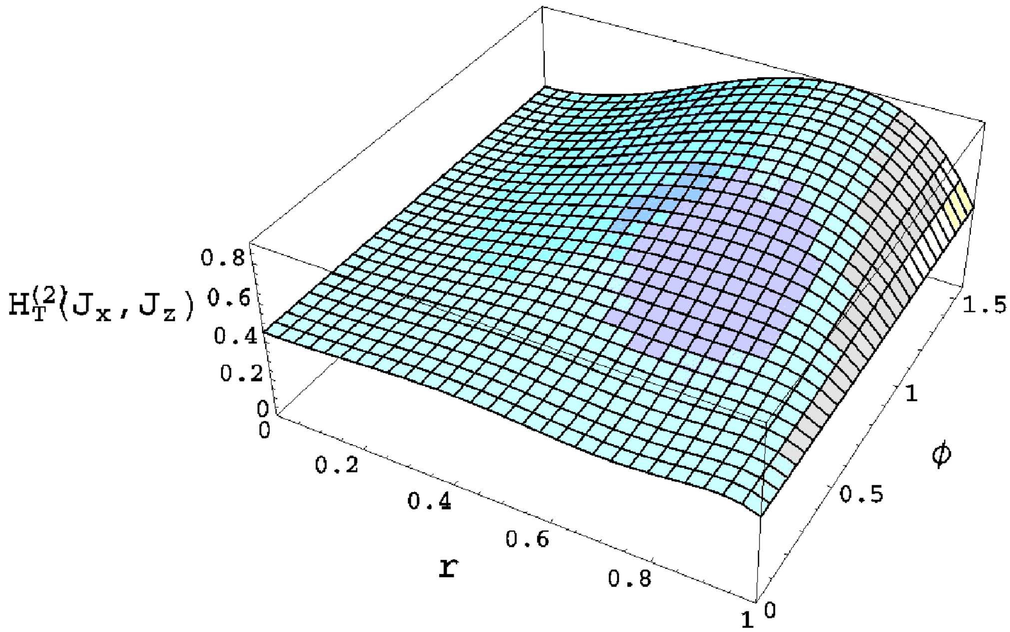

Fig. 12 The uncertainty relation with H_T for the spin components J_x and J_y as function of r and φ.