

On characterization of Poisson and Jacobi structures

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Received 23 October 2002; revised 20 December 2002

Abstract: We characterize Poisson and Jacobi structures by means of complete lifts of the corresponding tensors: the lifts have to be related to canonical structures by morphisms of corresponding vector bundles. Similar results hold for generalized Poisson and Jacobi structures (canonical structures) associated with Lie algebroids and Jacobi algebroids.

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Keywords: Jacobi structures; Poisson structures; Lie algebroids; tangent lifts

MSC (2000): 17B62 17B66 53D10 53D17

1 Introduction

Jacobi brackets are local Lie brackets on the algebra $C^\infty(M)$ of smooth functions on a manifold M . This goes back to the well-known observation by Kirillov [Ki] that in the case of $\mathcal{A} = C^\infty(M)$ every local Lie bracket on \mathcal{A} is of first order (an algebraic version of this fact for arbitrary commutative associative algebra \mathcal{A} has been proved in [Gr]).

Since every skew-symmetric first-order bidifferential operator J on $C^\infty(M)$ is of the form $J = \Lambda + I \wedge \Gamma$, where Λ is a bivector field, Γ is a vector field and I is identity, the corresponding bracket of functions reads

$$\{f, g\}_J = \Lambda(f, g) + f\Gamma(g) - g\Gamma(f). \quad (1)$$

The Jacobi identity for this bracket is usually written in terms of the Schouten-Nijenhuis

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bracket $\llbracket \cdot, \cdot \rrbracket$ as follows:

$$\llbracket \Gamma, \Lambda \rrbracket = 0, \quad \llbracket \Lambda, \Lambda \rrbracket = -2\Gamma \wedge \Lambda. \quad (2)$$

Hence, every Jacobi bracket on $C^\infty(M)$ can be identified with the pair $J = (\Lambda, \Gamma)$ satisfying the above conditions, i.e. with a *Jacobi structure* on M (cf. [Li]). Note that we use the version of the Schouten-Nijenhuis bracket which gives a graded Lie algebra structure on multivector fields and which differs from the classical one by signs. The Jacobi bracket (1) has the following properties:

- (1) $\{a, b\} = -\{b, a\}$ (anticommutativity),
- (2) $\{a, bc\} = \{a, b\}c + b\{a, c\} - \{a, 1\}bc$ (generalized Leibniz rule),
- (3) $\{\{a, b\}, c\} = \{a, \{b, c\}\} - \{b, \{a, c\}\}$ (Jacobi identity),

The generalized Leibniz rule tells that the bracket is a bidifferential operator on $C^\infty(M)$ of first order. In the case when $\Gamma = 0$ (or, equivalently, when the constant function 1 is a central element), we deal with a *Poisson bracket* associated with the bivector field Λ satisfying $\llbracket \Lambda, \Lambda \rrbracket = 0$.

For a smooth manifold M we denote by Λ_M the canonical Poisson tensor on T^*M , which in local Darboux coordinates (x^i, p_j) has the form $\Lambda_M = \partial_{p_j} \wedge \partial_{x^j}$. In [GU] the following characterization of Poisson tensors, in terms of the *complete (tangent) lift* of contravariant tensors $X \mapsto X^c$ from the manifold M to TM , is proved.

Theorem 1.1. A bivector field Λ on a manifold M is Poisson if and only if the tensors Λ_M and $-\Lambda^c$ on T^*M and TM , respectively, are \sharp_Λ -related, where

$$\sharp_\Lambda : T^*M \rightarrow TM, \quad \sharp_\Lambda(\omega_x) = i_{\omega_x}\Lambda(x).$$

So, for a Poisson tensor Λ the map $\sharp_\Lambda : (T^*M, \Lambda_M) \rightarrow (TM, -\Lambda^c)$ is a Poisson map.

The aim of this note is to generalize the above characterization including Jacobi brackets and *canonical structures* associated with *Lie algebroids* and *Jacobi algebroids*.

2 Lie and Jacobi algebroids

A *Lie algebroid* is a vector bundle $\tau : E \rightarrow M$, together with a bracket $\llbracket \cdot, \cdot \rrbracket$ on the $C^\infty(M)$ -module $\text{Sec}(E)$ of smooth sections of E , and a bundle morphism $\rho : E \rightarrow TM$ over the identity on M , called the *anchor* of the Lie algebroid, such that

- (i) the bracket $\llbracket \cdot, \cdot \rrbracket$ is \mathbb{R} -bilinear, alternating, and satisfies the Jacobi identity;
- (ii) $\llbracket X, fY \rrbracket = f\llbracket X, Y \rrbracket + \rho(X)(f)Y$ for all $X, Y \in \text{Sec}(E)$ and all $f \in C^\infty(M)$.

From (i) and (ii) it follows easily

- (iii) $\rho(\llbracket X, Y \rrbracket) = [\rho(X), \rho(Y)]$ for all $X, Y \in \text{Sec}(E)$.

We will often identify sections μ of the dual bundle E^* with linear (along fibres) functions ι_μ on the vector bundle E : $\iota_\mu(X_p) = \langle \mu(p), X_p \rangle$. If Λ is a homogeneous (linear) 2-contravariant tensor field on E , i.e. Λ is homogeneous of degree -1 with respect to the Liouville vector field Δ_E , then $\langle \Lambda, d\iota_\mu \otimes d\iota_\nu \rangle = \{\iota_\mu, \iota_\nu\}_\Lambda$ is again a linear function associated with an element $[\mu, \nu]_\Lambda$. The operation $[\mu, \nu]_\Lambda$ on sections of E^* we call the

bracket induced by Λ . This is the way in which homogeneous Poisson brackets are related to Lie algebroids.

Theorem 2.1. There is a one-one correspondence between Lie algebroid brackets $[\![\cdot, \cdot]\!]_\Lambda$ on the vector bundle E and homogeneous (linear) Poisson structures Λ on the dual bundle E^* determined by

$$\iota_{[\![X,Y]\!]_\Lambda} = \{\iota_X, \iota_Y\}_\Lambda = \Lambda(d\iota_X, d\iota_Y). \quad (3)$$

For a vector bundle E over the base manifold M , let $\mathcal{A}(E) = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}^k(E)$, $\mathcal{A}^k(E) = \text{Sec}(\bigwedge^k E)$, be the exterior algebra of multisections of E . This is a basic geometric model for a graded associative commutative algebra with unity. We will refer to elements of $\Omega^k(E) = \mathcal{A}^k(E^*)$ as to k -forms on E . Here, we identify $\mathcal{A}^0(E) = \Omega^0(E)$ with the algebra $C^\infty(M)$ of smooth functions on the base and $\mathcal{A}^k(E) = \{0\}$ for $k < 0$. Denote by $|X|$ the Grassmann degree of the multisection $X \in \mathcal{A}(E)$.

A Lie algebroid structure on E can be identified with a *graded Poisson bracket* on $\mathcal{A}(E)$ of degree -1 (linear). Such brackets we call *Schouten-Nijenhuis brackets* on $\mathcal{A}(E)$. Recall that a graded Poisson bracket of degree k on a \mathbb{Z} -graded associative commutative algebra $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}^i$ is a graded bilinear map

$$\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$$

of degree k (i.e. $|\{a, b\}| = |a| + |b| + k$) such that

- (1) $\{a, b\} = -(-1)^{(|a|+k)(|b|+k)}\{b, a\}$ (graded anticommutativity),
- (2) $\{a, bc\} = \{a, b\}c + (-1)^{(|a|+k)|b|}b\{a, c\}$ (graded Leibniz rule),
- (3) $\{\{a, b\}, c\} = \{a, \{b, c\}\} - (-1)^{(|a|+k)(|b|+k)}\{b, \{a, c\}\}$ (graded Jacobi identity).

It is obvious that this notion extends naturally to more general gradings in the algebra. For a graded commutative algebra with unity $\mathbf{1}$, a natural generalization of a graded Poisson bracket is *graded Jacobi bracket*. The only difference is that we replace the Leibniz rule by the *generalized Leibniz rule*

$$\{a, bc\} = \{a, b\}c + (-1)^{(|a|+k)|b|}b\{a, c\} - \{a, \mathbf{1}\}bc. \quad (4)$$

Graded Jacobi brackets on $\mathcal{A}(E)$ of degree -1 (linear) we call *Schouten-Jacobi brackets*. An element $X \in \mathcal{A}^2(E)$ is called a *canonical structure* for a Schouten-Nijenhuis or Schouten-Jacobi bracket $[\![\cdot, \cdot]\!]$ if $[\![X, X]\!] = 0$.

As it was already indicated in [KS], Schouten-Nijenhuis brackets are in one-one correspondence with Lie algebroids:

Theorem 2.2. Any Schouten-Nijenhuis bracket $[\![\cdot, \cdot]\!]$ on $\mathcal{A}(E)$ induces a Lie algebroid bracket on $\mathcal{A}^1(E) = \text{Sec}(E)$ with the anchor defined by $\rho(X)(f) = [\![X, f]\!]$. Conversely, any Lie algebroid structure on $\text{Sec}(E)$ gives rise to a Schouten-Nijenhuis bracket on $\mathcal{A}(E)$ for which $\mathcal{A}^1(E) = \text{Sec}(E)$ is a Lie subalgebra and $\rho(X)(f) = [\![X, f]\!]$.

We have the following expression for the Schouten-Nijenhuis bracket:

$$[\![X_1 \wedge \dots \wedge X_m, Y_1 \wedge \dots \wedge Y_n]\!] = \quad (5)$$

$$\sum_{k,l} (-1)^{k+l} \llbracket X_k, Y_l \rrbracket \wedge \dots \wedge \widehat{X_k} \wedge \dots \wedge X_m \wedge Y_1 \wedge \dots \wedge \widehat{Y_l} \wedge \dots \wedge Y_n,$$

where $X_i, Y_j \in \text{Sec}(E)$ and the hat over a symbol means that this is to be omitted.

A Schouten-Nijenhuis bracket induces the well-known generalization of the standard Cartan calculus of differential forms and vector fields [Ma, MX]. The exterior derivative $d : \Omega^k(E) \rightarrow \Omega^{k+1}(E)$ is defined by the standard formula

$$\begin{aligned} d\mu(X_1, \dots, X_{k+1}) &= \sum_i (-1)^{i+1} \llbracket X_i, \mu(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}) \rrbracket \\ &\quad + \sum_{i < j} (-1)^{i+j} \mu(\llbracket X_i, X_j \rrbracket, X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}), \end{aligned} \quad (6)$$

where $X_i \in \text{Sec}(E)$. For $X \in \text{Sec}(E)$, the contraction $i_X : \Omega^p(E) \rightarrow \Omega^{p-1}(E)$ is defined in the standard way and the Lie differential operator \mathcal{L}_X is defined by the graded commutator

$$\mathcal{L}_X = i_X \circ d + d \circ i_X. \quad (7)$$

Since Schouten-Nijenhuis brackets on $\mathcal{A}(E)$ are just Lie algebroid structures on E , by *Jacobi algebroid* structure on E we mean a Schouten-Jacobi bracket on $\mathcal{A}(E)$ (see [GM]). An analogous concept has been introduced in [IM1] under the name of a *generalized Lie algebroid*. Every Schouten-Jacobi bracket on the graded algebra $\mathcal{A}(E)$ of multisections of E turns out to be uniquely determined by a Lie algebroid bracket on a vector bundle E over M and a 1-cocycle $\Phi \in \Omega^1(E)$, $d\Phi = 0$, relative to the Lie algebroid exterior derivative d , namely it is of the form [IM1]

$$\llbracket X, Y \rrbracket_\Phi = \llbracket X, Y \rrbracket + xX \wedge i_\Phi Y - (-1)^x y i_\Phi X \wedge Y, \quad (8)$$

where $\llbracket \cdot, \cdot \rrbracket$ is the Schouten bracket associated with this Lie algebroid and where we use the convention that $x = |X| - 1$ is the shifted degree of X in the graded algebra $\mathcal{A}(E)$. Note that Φ is determined by the Schouten-Jacobi bracket by $i_\Phi X = (-1)^x \llbracket X, \mathbf{1} \rrbracket_\Phi$, so that (4) is satisfied:

$$\llbracket X, Y \wedge Z \rrbracket_\Phi = \llbracket X, Y \rrbracket_\Phi \wedge Z + (-1)^{x(y+1)} Y \wedge \llbracket X, Z \rrbracket_\Phi - \llbracket X, \mathbf{1} \rrbracket_\Phi \wedge Y \wedge Z. \quad (9)$$

We already know that there is one-one correspondence between Lie algebroid structures on E and linear Poisson tensors Λ^{E^*} on E^* . To Jacobi algebroids correspond Jacobi structures $J_\Phi^{E^*}$ on E^* which are homogeneous of degree -1 with respect to the Liouville vector field Δ_{E^*} , namely

$$J_\Phi^{E^*} = \Lambda^{E^*} + \Delta_{E^*} \wedge \Phi^v - I \wedge \Phi^v,$$

where Φ^v is the vertical lift of Φ to a vector field on E^* . The above structure generates a Jacobi bracket which coincides on linear functions with the Poisson bracket associated with Λ^{E^*} .

One can develop a Cartan calculus for Jacobi algebroids similarly to the Lie algebroid case (cf. [IM1]). For a Schouten-Jacobi bracket associated with a 1-cocycle Φ the

definitions of the exterior differential d^Φ and Lie differential $\mathcal{L}^\Phi = d^\Phi \circ i + i \circ d^\Phi$ are formally the same as (6) and (7), respectively. Since, for $X \in \text{Sec}(E)$, $f \in C^\infty(M)$, we have $\llbracket X, f \rrbracket_\Phi = \llbracket X, f \rrbracket + (i_\Phi X)f$, one obtains $d^\Phi \mu = d\mu + \Phi \wedge \mu$. Here $\llbracket \cdot, \cdot \rrbracket$ and d are, respectively, the Schouten-Nijenhuis bracket and the exterior derivative associated with the Lie algebroid.

Example 2.3. A canonical example of a Lie algebroid over M is the tangent bundle TM with the bracket of vector fields. The corresponding complex $(\Omega(TM), d)$ is in this case the standard de Rham complex. A canonical structure for the corresponding Schouten-Nijenhuis bracket is just a standard Poisson tensor.

Example 2.4. A canonical example of a Jacobi algebroid is $(T_1M = TM \oplus \mathbb{R}, (0, 1))$, where T_1M is the Lie algebroid of first-order differential operators on $C^\infty(M)$ with the bracket

$$[(X, f), (Y, g)]_1 = ([X, Y], X(g) - Y(f)), \quad X, Y \in \text{Sec}(TM), \quad f, g \in C^\infty(M),$$

and the 1-cocycle $\Phi = (0, 1)$ is $\Phi((X, f)) = f$. A canonical structure with respect to the corresponding Schouten-Jacobi bracket on the Grassmann algebra $\mathcal{A}(T_1M)$ of first-order polydifferential operators on $C^\infty(M)$ turns out to be a standard Jacobi structure. Indeed, it is easy to see that the Schouten-Jacobi bracket of $A = A_1 + I \wedge A_2 \in \mathcal{A}^{a+1}(T_1M)$ and $B = B_1 + I \wedge B_2 \in \mathcal{A}^{b+1}(T_1M)$ reads

$$\begin{aligned} \llbracket A_1 + I \wedge A_2, B_1 + I \wedge B_2 \rrbracket_1 &= \llbracket A_1, B_1 \rrbracket + (-1)^a I \wedge \llbracket A_1, B_2 \rrbracket + I \wedge \llbracket A_2, B_1 \rrbracket \\ &\quad + aA_1 \wedge B_2 - (-1)^a bA_2 \wedge B_1 + (a - b)I \wedge A_2 \wedge B_2. \end{aligned} \quad (10)$$

Hence, the bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ defined by a bilinear differential operator $\Lambda + I \wedge \Gamma \in \mathcal{A}(T_1M)$ is a Lie bracket (Jacobi bracket on $C^\infty(M)$) if and only if

$$\llbracket \Lambda + I \wedge \Gamma, \Lambda + I \wedge \Gamma \rrbracket_1 = \llbracket \Lambda, \Lambda \rrbracket + 2I \wedge \llbracket \Gamma, \Lambda \rrbracket + 2\Lambda \wedge \Gamma = 0.$$

We recognize the conditions (2) defining a Jacobi structure on M .

There is another approach to Lie algebroids. As it has been shown in [GU1, GU2], a Lie algebroid structure (or the corresponding Schouten-Nijenhuis bracket) is determined by the Lie algebroid lift $X \mapsto X^c$ which associates with $X \in \mathcal{T}(E)$ a contravariant tensor field X^c on E . The complete Lie algebroid and Jacobi algebroid lifts are described as follows.

Theorem 2.5. ([GU1]) For a given Lie algebroid structure on a vector bundle E over M there is a unique *complete lift* of elements $X \in \text{Sec}(E^{\otimes k})$ of the tensor algebra $\mathcal{T}(E) = \bigoplus_k \text{Sec}(E^{\otimes k})$ to linear contravariant tensors $X^c \in \text{Sec}((TE)^{\otimes k})$ on E , such that

- (a) $f^c = \iota_{df}$ for $f \in C^\infty(M)$;
- (b) $X^c(\iota_\mu) = \iota_{\mathcal{L}_X \mu}$ for $X \in \text{Sec}(E)$, $\mu \in \text{Sec}(E^*)$;
- (c) $(X \otimes Y)^c = X^c \otimes Y^v + X^v \otimes Y^c$, where $X \mapsto X^v$ is the standard vertical lift of tensors from $\mathcal{T}(E)$ to tensors from $\mathcal{T}(TE)$, i.e. the complete lift is a derivation with respect to the vertical lift.

This complete lift restricted to skew-symmetric tensors is a homomorphism of the corresponding Schouten-Nijenhuis brackets:

$$[[X, Y]]^c = [[X^c, Y^c]]. \quad (11)$$

Moreover,

$$[[X, Y]]^v = [[X^c, Y^v]]. \quad (12)$$

Corollary 2.6. If $P \in \mathcal{A}^2(E)$ is a canonical structure for the Schouten bracket, i.e. $[[P, P]] = 0$, then P^c is a homogeneous Poisson structure on E . The corresponding Poisson bracket determines the Lie algebroid bracket

$$[[\alpha, \beta]]_P = i_{\#_P(\alpha)} d\beta - i_{\#_P(\beta)} d\alpha + d(P(\alpha, \beta)) \quad (13)$$

on E^* .

Remark. For the canonical Lie algebroid $E = TM$, the above complete lift gives the better-known *tangent lift* of multivector fields on M to multivector fields on TM (cf. [IY, GU]). In this case the complete lift is an injective operator, so Λ is a Poisson tensor on M if and only if Λ^c is a Poisson tensor on TM . The complete Lie algebroid lift of just sections of E , i.e. the formula (b), was already indicated in [MX1].

Let us see how these lifts look like in local coordinates. Let (x^a) be a local coordinate system on M and let e_1, \dots, e_n be a basis of local sections of E . We denote by e^{*1}, \dots, e^{*n} the dual basis of local sections of E^* and by (x^a, y^i) (resp. (x^a, ξ_i)) the corresponding coordinate system on E (resp. E^*), i.e., $\iota_{e_i} = \xi_i$ and $\iota_{e^{*i}} = y^i$. The vertical lift is given by

$$(c_{i_1, \dots, i_k} e_{i_1} \otimes \dots \otimes e_{i_k})^v = c_{i_1, \dots, i_k} \partial_{y^{i_1}} \otimes \dots \otimes \partial_{y^{i_k}}.$$

If for the Lie algebroid bracket we have $[e_i, e_j] = c_{ij}^k e_k$ and if the anchor sends e_i to $d_i^a \partial_{x^a}$, then

$$\Lambda^{E^*} = \frac{1}{2} c_{ij}^k \xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + d_i^a \partial_{\xi_i} \wedge \partial_{x^a}. \quad (14)$$

Moreover,

$$f^c = \frac{\partial f}{\partial x^a} d_j^a y^j \quad (15)$$

and

$$(X^i e_i)^c = X^i d_i^a \partial_{x^a} + (X^i c_{ji}^k + \frac{\partial X^k}{\partial x^a} d_j^a) y^j \partial_{y^k}. \quad (16)$$

It follows that, for $P = \frac{1}{2} P^{ij} e_i \wedge e_j$, we have

$$P^c = P^{ij} d_j^a \partial_{y^i} \wedge \partial_{x^a} + (P^{kj} c_{lk}^i + \frac{1}{2} \frac{\partial P^{ij}}{\partial x^a} d_l^a) y^l \partial_{y^i} \wedge \partial_{y^j}. \quad (17)$$

There is an analog of the Lie algebroid complete lift for Jacobi algebroids which will represent the Schouten-Jacobi bracket on $\mathcal{A}(E)$ in the Schouten-Jacobi bracket of first-order polydifferential operators on E . Here by *polydifferential operators* we understand

skew-symmetric multidifferential operators. Let $[\![\cdot, \cdot]\!]_\Phi$ be the Schouten-Jacobi bracket on $\mathcal{A}(E)$ associated with a Lie algebroid structure on E and a 1-cocycle Φ .

Definition. ([GM]) The *complete Jacobi lift* of an element $X \in \mathcal{T}^k(E)$ is the multidifferential operator of first order on E , i.e. an element of $\text{Sec}((T_1E)^{\otimes k})$, defined by

$$\widehat{X}_\Phi = X^c - (k-1)\iota_\Phi X^v + i_{I \otimes d(\iota_\Phi)} X^v, \quad (18)$$

where X^c is the complete Lie algebroid lift, X^v is the vertical lift and $i_{I \otimes d(\iota_\Phi)}$ is the derivation acting on the tensor algebra of contravariant tensor fields which vanishes on functions and satisfies $i_{I \otimes d(\iota_\Phi)} X = X(\iota_\Phi)I$ on vector fields. The derivation property yields

$$i_{I \otimes d(\iota_\Phi)}(X_1^v \otimes \cdots \otimes X_k^v) = \sum_i \langle X_i, \Phi \rangle X_1^v \otimes \cdots \otimes X_{i-1}^v \otimes I \otimes X_{i+1}^v \otimes \cdots \otimes X_k^v.$$

for $X_1, \dots, X_k \in \text{Sec}(E)$.

Theorem 2.7. ([GM]) The complete Jacobi lift has the following properties:

- (a) $\widehat{f}_\Phi = \iota_{d\Phi} f$ for $f \in C^\infty(M)$;
- (b) $\widehat{X}_\Phi = X^c + (i_\Phi X)^v I$ for $X \in \text{Sec}(E)$;
- (c) $\widehat{(X \otimes Y)}_\Phi = \widehat{X}_\Phi \otimes Y^v + X^v \otimes \widehat{Y}_\Phi - \iota_\Phi(X^v \otimes Y^v)$;
- (d) For skew-symmetric tensors X and Y ,

$$[\![\widehat{X}_\Phi, \widehat{Y}_\Phi]\!]_1 = ([\![X, Y]\!]_\Phi)_\Phi^\wedge,$$

where $[\![\cdot, \cdot]\!]_1$ is the Schouten-Jacobi bracket of first-order polydifferential operators;

- (e) For skew-symmetric X and Y

$$[\![\widehat{X}_\Phi, Y^v]\!]_1 = ([\![X, Y]\!]_\Phi)^v;$$

We remark that in [GM] only skew-symmetric tensors have been considered, but the extension to arbitrary tensors is straightforward.

Definition. ([GM]) The *complete Poisson lift* of an element $X \in \mathcal{T}^k(E)$ is the contravariant tensor $\widehat{X}_\Phi^c \in \text{Sec}((TE)^{\otimes k})$, defined by

$$\widehat{X}_\Phi^c = X^c - (k-1)\iota_\Phi X^v + i_{\Delta_E \otimes d(\iota_\Phi)} X^v, \quad (19)$$

where Δ_E is the Liouville vector field on the vector bundle E and X^c is the complete Lie algebroid lift, X^v is the vertical lift and $i_{\Delta_E \otimes d(\iota_\Phi)}$ is the derivation acting on the tensor algebra of contravariant tensor fields which vanishes on functions and satisfies $i_{\Delta_E \otimes d(\iota_\Phi)} X = X(\iota_\Phi)\Delta_E$ on vector fields.

Theorem 2.8. ([GM]) The Poisson lift has the following properties:

- (a) $\widehat{f}_\Phi^c = \iota_{d\Phi} f$ for $f \in C^\infty(M)$;
- (b) $\widehat{X}_\Phi^c(\iota_\mu) = \iota_{\mathcal{L}_X^\Phi \mu}$ for $X \in \text{Sec}(E)$, $\mu \in \text{Sec}(E^*)$;

- (c) $(\widehat{X \otimes Y})_{\Phi}^c = \widehat{X}_{\Phi}^c \otimes Y^v + X^v \otimes \widehat{Y}_{\Phi}^c - \iota_{\Phi}(X^v \otimes Y^v);$
 (d) For skew-symmetric X and Y

$$[\widehat{X}_{\Phi}^c, \widehat{Y}_{\Phi}^c] = ([X, Y]_{\Phi})_{\Phi}^{\wedge c}.$$

Corollary 2.9. If $P \in \mathcal{A}^2(E)$ is a canonical structure for the Schouten-Jacobi bracket, i.e. $\llbracket P, P \rrbracket_{\Phi} = \llbracket P, P \rrbracket + 2P \wedge i_{\Phi}P = 0$, then \widehat{P}_{Φ} (resp. \widehat{P}_{Φ}^c) is a homogeneous Jacobi (resp. homogeneous Poisson) structure on E . The corresponding Jacobi and Poisson brackets coincide on linear functions and determine the Lie algebroid bracket

$$[\alpha, \beta]_P = i_{\#P(\alpha)}d^{\Phi}\beta - i_{\#P(\beta)}d^{\Phi}\alpha + d^{\Phi}(P(\alpha, \beta)) \quad (20)$$

on E^* .

3 Characterization of Poisson tensors.

Theorem 1.1 of the Introduction can be generalized in the following way. Let us remark first that any two-contravariant tensor Λ (which is not assumed to be skew-symmetric) defines a bracket $[\cdot, \cdot]_{\Lambda}$ on 1-forms on M by

$$[\mu, \nu]_{\Lambda} = i_{\sharp_{\Lambda}(\mu)}d\nu - i_{\sharp_{\Lambda}(\nu)}d\mu + d\langle \Lambda, \mu \otimes \nu \rangle, \quad (21)$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between contravariant and covariant tensors.

Theorem 3.1. For a two-contravariant tensor Λ on a manifold M the following are equivalent:

- (i) Λ is a Poisson tensor;
- (ii) \sharp_{Λ} induces a homomorphism of $[\cdot, \cdot]_{\Lambda}$ into the bracket of vector fields:

$$\sharp_{\Lambda}([\mu, \nu]_{\Lambda}) = [\sharp_{\Lambda}(\mu), \sharp_{\Lambda}(\nu)]; \quad (22)$$

- (iii) The canonical Poisson tensor Λ_M and the negative of the complete lift $-\Lambda^c$ are \sharp_{Λ} -related;
- (iv) There is a vector bundle morphism $F : T^*M \rightarrow TM$ over the identity on M such that the canonical Poisson tensor Λ_M and the negative of the complete lift $-\Lambda^c$ are F -related;
- (v) The morphism \sharp_{Λ} relates Λ_M with the complete lift of a 2-contravariant tensor Λ_1 .
- (vi) There is a vector bundle morphism $F : T^*M \rightarrow TM$ over the identity on M such that

$$F([\mu, \nu]_{\Lambda}) = [F(\mu), F(\nu)]. \quad (23)$$

- (vii) There is a 2-contravariant tensor Λ_1 on M such that

$$\sharp_{\Lambda}([\mu, \nu]_{\Lambda_1}) = [\sharp_{\Lambda}(\mu), \sharp_{\Lambda}(\nu)]; \quad (24)$$

Proof. The implication (i) \Rightarrow (ii) is a well-known fact (cf. e.g. [KSM]).

Assume now (ii). To show (iii) one has to prove that the brackets on functions $\{\cdot, \cdot\}_{\Lambda_M}$ and $\{\cdot, \cdot\}_{\Lambda^c}$ induced by tensors Λ_M and Λ^c by contractions with differentials of functions are \sharp_Λ -related, i.e.

$$-\{f, g\}_{\Lambda^c} \circ \sharp_\Lambda = \{f \circ \sharp_\Lambda, g \circ \sharp_\Lambda\}_{\Lambda_M} \quad (25)$$

for all $f, g \in C^\infty(TM)$. Due to the Leibniz rule, it is sufficient to check (25) for linear functions, i.e. for functions of the form ι_μ , where μ is a 1-form and $\iota_\mu(v_x) = \langle \mu(x), v_x \rangle$. It is well known (see [Co, GU]) that the brackets induced by Λ and its complete lift are related by

$$\{\iota_\mu, \iota_\nu\}_{\Lambda^c} = \iota_{[\mu, \nu]_\Lambda}. \quad (26)$$

It is also known (cf. (3)) that

$$\iota_{[X, Y]} = \{\iota_X, \iota_Y\}_{\Lambda_M}$$

for vector fields X, Y on M . Since $\iota_\mu \circ \sharp_\Lambda = -\iota_{\sharp_\Lambda(\mu)}$, we get

$$\begin{aligned} -\{\iota_\mu, \iota_\nu\}_{\Lambda^c} \circ \sharp_\Lambda &= -\iota_{[\mu, \nu]_\Lambda} \circ \sharp_\Lambda = \iota_{\sharp_\Lambda([\mu, \nu]_\Lambda)} \\ &= \iota_{[\sharp_\Lambda(\mu), \sharp_\Lambda(\nu)]} = \{\iota_{\sharp_\Lambda(\mu)}, \iota_{\sharp_\Lambda(\nu)}\}_{\Lambda_M} \\ &= \{\iota_\mu \circ \sharp_\Lambda, \iota_\nu \circ \sharp_\Lambda\}_{\Lambda_M} \end{aligned} \quad (27)$$

which proves (ii) \Rightarrow (iii). In fact, (27) proves equivalence of (ii) and (iii).

Replacing in (27) the mapping \sharp_Λ by a vector bundle morphism $F : T^*M \rightarrow TM$, we get equivalence of (iv) and (vi). Similarly, (v) is equivalent to (vii). The implication (iii) \Rightarrow (iv) is obvious, so let us show (iv) \Rightarrow (i). Assume that F relates Λ_M and Λ^c . We will show that this implies that Λ is skew-symmetric and $F = \sharp_\Lambda$. Since the assertion is local over M we can use coordinates (x^a) in M and the adapted coordinate systems (x^a, p_i) in T^*M and (x^a, \dot{x}^j) in TM . Writing $\Lambda = \Lambda^{ij} \partial_{x^i} \otimes \partial_{x^j}$ and $F(x^a, p_i) = (x^a, F^{ij} p_i)$, we get

$$F_*(\partial_{p_i} \wedge \partial_{x^i}) = F^{ij} p_s \frac{\partial F^{sk}}{\partial x^i} \partial_{\dot{x}^j} \wedge \partial_{\dot{x}^k} - F^{ij} \partial_{x^i} \wedge \partial_{\dot{x}^j}. \quad (28)$$

Since

$$\Lambda^c = \frac{\partial \Lambda^{ij}}{\partial x^k} \dot{x}^k \partial_{\dot{x}^i} \otimes \partial_{\dot{x}^j} + \Lambda^{ij} (\partial_{x^i} \otimes \partial_{\dot{x}^j} + \partial_{\dot{x}^i} \otimes \partial_{x^j}), \quad (29)$$

comparing the vertical-horizontal parts we get $\Lambda^{ij} = F^{ij} = -F^{ji}$, i.e. Λ is skew-symmetric and $F = \sharp_\Lambda$. Going backwards with (27) we get (ii). But for skew tensors we have (cf. [KSM])

$$\frac{1}{2} \llbracket \Lambda, \Lambda \rrbracket (\mu, \nu, \gamma) = \langle \sharp_\Lambda([\mu, \nu]_\Lambda) - [\sharp_\Lambda(\mu), \sharp_\Lambda(\nu)], \gamma \rangle, \quad (30)$$

where $\llbracket \cdot, \cdot \rrbracket$ is the Schouten-Nijenhuis bracket, so that $\llbracket \Lambda, \Lambda \rrbracket = 0$, i.e. Λ is a Poisson tensor.

Finally, (v) is equivalent to (iii), since (iii) \Rightarrow (v) trivially and exchanging the role of F and Λ in (28) and (29) we see that, as above, $\Lambda^{ij} = F^{ij}$, so that any tensor whose complete lift is \sharp_Λ -related to Λ_M equals $-\Lambda$. ■

A similar characterization is valid for any Lie algebroid. Let us consider a vector bundle E over M with a Lie algebroid bracket $[\![\cdot, \cdot]\!]$ instead of the canonical Lie algebroid TM of vector fields (cf. [Ma, KSM, GU0]). The multivector fields are now replaced by multisections $\mathcal{A}(E) = \oplus_k \mathcal{A}^k(E)$, $\mathcal{A}^k(E) = \text{Sec}(\bigwedge^k E)$, of E and the standard Schouten-Nijenhuis bracket with its Lie algebroid counterpart. A Lie algebroid Poisson tensor (*canonical structure*) is then a skew-symmetric $\Lambda \in \mathcal{A}^2(E)$ satisfying $[\![\Lambda, \Lambda]\!] = 0$. Such a structure gives a *triangular Lie bialgebroid* in the sense of [MX]. We have the exterior derivative d on multisections of the dual bundle E^* (we will refer to them as to "exterior forms"). For any $\Lambda \in \text{Sec}(E \otimes E)$ the formula (21) defines a bracket on "1-forms". We have an analog of the complete lift (cf. [GU1, GU2])

$$\text{Sec}(E^{\otimes k}) \ni \Lambda \mapsto \Lambda^c \in \text{Sec}((TE)^{\otimes k})$$

of the tensor algebra of sections of E into contravariant tensors on the total space E . The Lie algebroid bracket corresponds to a linear Poisson tensor Λ^{E^*} on E^* (which is just Λ_M in the case $E = TM$) by (3). Since the tensor Λ^{E^*} may be strongly degenerate, linear maps $F : E^* \rightarrow E$ do not determine the related tensors uniquely, so we cannot have the full analog of Theorem 3.1. However, since for skew-symmetric tensors the formula (30) remains valid [KSM], a part of Theorem 3.1 can be proved in the same way, *mutatis mutandis*, in the general Lie algebroid case. Thus we get the Lie algebroid version of Theorem 1.1 (cf. [GU1]).

Theorem 3.2. For any bisection $\Lambda \in \mathcal{A}^2(E)$ of a Lie algebroid E the following are equivalent:

- (i) Λ is a canonical structure, i.e. $[\![\Lambda, \Lambda]\!] = 0$;
- (ii) \sharp_Λ induces a homomorphism of $[\cdot, \cdot]_\Lambda$ into the Lie algebroid bracket:

$$\sharp_\Lambda([\mu, \nu]_\Lambda) = [\sharp_\Lambda(\mu), \sharp_\Lambda(\nu)]; \quad (31)$$

- (iii) The canonical Poisson tensor Λ_M and the negative of the complete lift $-\Lambda^c$ are \sharp_Λ -related.

4 Jacobi algebroids and characterization of Jacobi structures

We have introduced in Section 2 Jacobi and Poisson complete lifts related to Jacobi algebroids. For a standard Jacobi structure $J = (\Lambda, \Gamma)$ on M we will denote these lifts of J by \hat{J} and \hat{J}^c , respectively. The Jacobi lift \hat{J} is the Jacobi structure on $E = TM \oplus \mathbb{R}$ given by [GM]

$$\hat{J} = (\Lambda^c - t\Lambda^v + \partial_t \wedge (\Gamma^c - t\Gamma^v), \Gamma^v), \quad (32)$$

where Λ^v and Γ^v are the vertical tangent lifts of Λ and Γ , respectively, and t is the standard linear coordinate in \mathbb{R} . We consider here tangent lifts as tensors on $TM \oplus \mathbb{R} = TM \times \mathbb{R}$ instead on TM . The linear Jacobi structure (32) has been already considered by Iglesias and Marrero [IM].

Similarly, the Poisson lift \widehat{J}^c is the linear Poisson tensor on $TM \oplus \mathbb{R}$ given by [GM]

$$\widehat{J}^c = \Lambda^c - t\Lambda^v + \partial_t \wedge \Gamma^c + \Delta_{TM} \wedge \Gamma^v, \quad (33)$$

where Δ_{TM} is the Liouville (Euler) vector field on the vector bundle TM . This is exactly the linear Poisson tensor corresponding to the Lie algebroid structure on $T^*M \oplus \mathbb{R}$ induced by J and discovered first in [KSB]:

$$\begin{aligned} [(\alpha, f), (\beta, g)]_J &= (\mathcal{L}_{\sharp_\Lambda(\alpha)}\beta - \mathcal{L}_{\sharp_\Lambda(\beta)}\alpha - d\langle \Lambda, \alpha \wedge \beta \rangle + f\mathcal{L}_\Gamma\beta - g\mathcal{L}_\Gamma\alpha - i_\Gamma(\alpha \wedge \beta), \\ &\quad \langle \Lambda, \beta \wedge \alpha \rangle + \sharp_\Lambda(\alpha)(g) - \sharp_\Lambda(\beta)(f) + f\Gamma(g) - g\Gamma(f)), \end{aligned} \quad (34)$$

Of course, these lifts and an analog of the bracket (34) are well-defined for any first-order bidifferential operator

$$J = \Lambda + I \otimes \Gamma_1 + \Gamma_2 \otimes I + \alpha I \otimes I, \quad (35)$$

where Λ is a 2-contravariant tensor, Γ_1, Γ_2 are vector fields, and α is a function on M . The associated bracket acts on functions on M by

$$\{f, g\}_J = \langle \Lambda, df \otimes dg \rangle + f\Gamma_1(g) + g\Gamma_2(f) + \alpha fg,$$

The Jacobi lift of J is the first-order bidifferential operator on $TM \oplus \mathbb{R}$ given by

$$\begin{aligned} \widehat{J} &= \Lambda^c - t\Lambda^v + \partial_t \otimes (\Gamma_1^c - t\Gamma_1^v) + (\Gamma_2^c - t\Gamma_2^v) \otimes \partial_t + (\alpha^c - t\alpha^v)\partial_t \otimes \partial_t \\ &\quad + I \otimes (\Gamma_1^v + \alpha^v\partial_t) + (\Gamma_2^v + \alpha^v\partial_t) \otimes I \end{aligned}$$

and the Poisson lift is the 2-contravariant tensor field

$$\begin{aligned} \widehat{J}^c &= \Lambda^c - t\Lambda^v + \partial_t \otimes (\Gamma_1^c - t\Gamma_1^v) + (\Gamma_2^c - t\Gamma_2^v) \otimes \partial_t + (\alpha^c - t\alpha^v)\partial_t \otimes \partial_t \\ &\quad + \Delta_E \otimes (\Gamma_1^v + \alpha^v\partial_t) + (\Gamma_2^v + \alpha^v\partial_t) \otimes \Delta_E \\ &= \Lambda^c - t\Lambda^v + \partial_t \otimes \Gamma_1^c + \Gamma_2^c \otimes \partial_t + (\alpha^c + t\alpha^v)\partial_t \otimes \partial_t \\ &\quad + \Delta_{TM} \otimes \Gamma_1^v + \Gamma_2^v \otimes \Delta_{TM}. \end{aligned}$$

The mapping $\sharp_J : E^* = T^*M \oplus \mathbb{R} \rightarrow E = TM \oplus \mathbb{R}$ reads

$$\sharp_J(\omega_x, \lambda) = (\sharp_\Lambda(\omega_x) + \lambda\Gamma_1(x), \Gamma_2(x)(\omega_x) + \alpha(x)\lambda).$$

Note that any morphism from the vector bundle $E^* = T^*M \oplus \mathbb{R}$ into $E = TM \oplus \mathbb{R}$ over the identity on M is of this form.

The bidifferential operators \widehat{J} and \widehat{J}^c define brackets $\{\cdot, \cdot\}_{\widehat{J}}$ and $\{\cdot, \cdot\}_{\widehat{J}^c}$, respectively, on functions on $TM \oplus \mathbb{R}$. These brackets coincide on linear functions which close on a subalgebra with respect to them, so that they define the bracket $[\cdot, \cdot]_J$ on sections of $T^*M \oplus \mathbb{R}$ (which coincides with the bracket (34) for skew-symmetric operators) by

$$\{\iota_{(\mu, f)}, \iota_{(\nu, g)}\}_{\widehat{J}} = \{\iota_{(\mu, f)}, \iota_{(\nu, g)}\}_{\widehat{J}^c} = \iota_{[(\mu, f), (\nu, g)]_J},$$

where μ, ν are 1-forms, f, g are functions on M , and $\iota_{(\mu, f)} = \iota_\mu + tf^v$. Here we identify $T^*M \oplus \mathbb{R}$ with $T^*M \times \mathbb{R}$ and use the linear coordinate λ in \mathbb{R} . For the similar identification of $TM \oplus \mathbb{R}$ we use the coordinate t of \mathbb{R} in $TM \times \mathbb{R}$, since both \mathbb{R} 's play dual roles.

We have two canonical structures on the vector bundle $E^* = T^*M \oplus \mathbb{R} \simeq T^*M \times \mathbb{R}$. One is the Jacobi structure (bracket)

$$J_M = \Lambda_M + \Delta_{T^*M} \wedge \partial_\lambda + \partial_\lambda \wedge I \quad (36)$$

and the other is the Poisson structure Λ_M regarded as the product of Λ_M on T^*M with the trivial structure on \mathbb{R} . These brackets coincide on linear functions which close on a subalgebra with respect to both brackets, so that they define a Lie algebroid structure on the dual bundle $E = TM \oplus \mathbb{R}$. This is the Lie algebroid of first-order differential operators with the bracket

$$[(X, f), (Y, g)]_1 = ([X, Y], (X(g) - Y(f))),$$

where X, Y are vector fields and f, g are functions on M .

Theorem 4.1. For a first-order bidifferential operator J the following are equivalent:

- (J1) J is a Jacobi bracket;
- (J2) The canonical Jacobi bracket J_M and $-\hat{J}$ are \sharp_J -related;
- (J3) There is a first-order bidifferential operator J_1 such that J_M and $-\hat{J}$ are \sharp_{J_1} -related;
- (J4) There is a first-order bidifferential operator J_1 such that J_M and $-\hat{J}_1$ are \sharp_J -related;
- (J5) The contravariant tensors Λ_M and $-\hat{J}^c$ are \sharp_J -related;
- (J6) There is a first-order bidifferential operator J_1 such that Λ_M and $-\hat{J}^c$ are \sharp_{J_1} -related;
- (J7) There is a first-order bidifferential operator J_1 such that Λ_M and $-\hat{J}_1^c$ are \sharp_J -related;
- (J8) For any 1-forms μ, ν and functions f, g on M

$$\sharp_J([(\mu, f), (\nu, g)]_J) = [\sharp_J(\mu, f), \sharp_J(\nu, g)]_1.$$

- (J9) There is a first-order bidifferential operator J_1 such that

$$\sharp_{J_1}([(\mu, f), (\nu, g)]_J) = [\sharp_{J_1}(\mu, f), \sharp_{J_1}(\nu, g)]_1.$$

- (J10) There is a first-order bidifferential operator J_1 such that

$$\sharp_J([(\mu, f), (\nu, g)]_{J_1}) = [\sharp_J(\mu, f), \sharp_J(\nu, g)]_1.$$

Before proving this theorem we introduce some notation and prove a lemma. For a first-order bidifferential operator J as in (35), the *poissonization* of J is the tensor field on $M \times \mathbb{R}$ of the form

$$P_J = e^{-s}(\Lambda + \partial_s \otimes \Gamma_1 + \Gamma_2 \otimes \partial_s + \alpha \partial_s \otimes \partial_s), \quad (37)$$

where s is the coordinate on \mathbb{R} . Identifying $T^*(M \times \mathbb{R})$ with $T^*M \times T^*\mathbb{R}$ (with coordinates (s, λ) in $T^*\mathbb{R}$) and $T(M \times \mathbb{R})$ with $TM \times T\mathbb{R}$ (with coordinates (s, t) in $T\mathbb{R}$) we can write

$$\begin{aligned} \Lambda_{M \times \mathbb{R}} &= \Lambda_M + \partial_\lambda \wedge \partial_s, \\ P_J^c &= e^{-s}(\Lambda^c - t\Lambda^v + \partial_t \otimes (\Gamma_1^c - t\Gamma_1^v) + (\Gamma_2^c - t\Gamma_2^v) \otimes \partial_t \\ &\quad + \partial_s \otimes (\Gamma_1^v + \alpha^v \partial_t) + (\Gamma_2^v + \alpha^v \partial_t) \otimes \partial_s + (\alpha^c - t\alpha^v) \partial_t \otimes \partial_t, \\ \sharp_{P_J}(\omega_x, \lambda_s) &= e^{-s}(\sharp_\Lambda(\omega_x) + \lambda_s \Gamma_1(x), \Gamma_2(x)(\omega_x) + \lambda_s \alpha(x)). \end{aligned}$$

In local coordinates $x = (x^l)$ on M and adapted local coordinates (x, p) on T^*M and (x, \dot{x}) on TM we have

$$(x^l, s, \dot{x}^i, t) \circ \sharp_{P_J} = (x^l, s, e^{-s}(\Lambda^{ki}p_k + \lambda\Gamma_1^i), e^{-s}(\Gamma_2^k p_k + \lambda\alpha))$$

for $\Lambda = \Lambda^{ij}\partial_{x^i} \otimes \partial_{x^j}$, $\Gamma_u = \Gamma_u^k \partial_{x^k}$, $u = 1, 2$. It is well known that P_J is a Poisson tensor if and only if J is a Jacobi structure [Li, GL]. In view of Theorem 3.2, we can conclude that J is a Jacobi structure if and only if $\Lambda_{M \times \mathbb{R}}$ and P_J^c are related by the map $\sharp_{P_J} : T^*(M \times \mathbb{R}) \rightarrow T(M \times \mathbb{R})$. Since $T(M \times \mathbb{R}) \simeq E \times \mathbb{R}$ and $T^*(M \times \mathbb{R}) \simeq E^* \times \mathbb{R}$, we can consider the bundles $E = TM \oplus \mathbb{R}$ and $E^* = T^*M \oplus \mathbb{R}$ as submanifolds of $T(M \times \mathbb{R})$ and $T^*(M \times \mathbb{R})$, respectively, given by the equation $s = 0$.

For any function $\phi \in C^\infty(E)$ we denote by $\check{\phi}$ the function on $T(M \times \mathbb{R}) = E \times \mathbb{R}$ given by $\check{\phi}(v_x, s) = e^s \phi(v_x)$. Similarly, for any function $\varphi \in C^\infty(E^*)$ we denote by $\tilde{\varphi}$ the function on $T^*(M \times \mathbb{R}) = E^* \times \mathbb{R}$ given by $\tilde{\varphi}(u_x, s) = e^s \varphi(u_x)$.

It is a matter of easy calculations to prove the following.

Lemma 4.2.

- (a) The maps $\phi \mapsto \check{\phi}$ and $\phi \mapsto \tilde{\phi}$ are injective.
- (b) For any first-order bidifferential operator J ,

$$\check{\phi} \circ \sharp_{P_J} = (\phi \circ \sharp_J)^\sim.$$

- (c) For any $\phi, \psi \in C^\infty(E)$,

$$\{\check{\phi}, \check{\psi}\}_{P_J^c} = (\{\phi, \psi\}_J)^\sim.$$

- (d) For any $\phi, \psi \in C^\infty(E^*)$,

$$\{\tilde{\phi}, \tilde{\psi}\}_{\Lambda_{M \times \mathbb{R}}} = (\{\phi, \psi\}_{J_M})^\sim.$$

- (e) For linear $\phi, \psi \in C^\infty(E)$,

$$\{\check{\phi}, \check{\psi}\}_{P_J^c} = (\{\phi, \psi\}_{J^c})^\sim.$$

- (f) For linear $\phi, \psi \in C^\infty(E^*)$,

$$\{\tilde{\phi}, \tilde{\psi}\}_{\Lambda_{M \times \mathbb{R}}} = (\{\phi, \psi\}_{\Lambda_M})^\sim.$$

Proof of Theorem 4.1. Due to the above Lemma the following identities are valid for arbitrary $\phi, \psi \in C^\infty(E)$ and arbitrary first-order bidifferential operators J, J_1 :

$$\begin{aligned} (\{\phi, \psi\}_J \circ \sharp_{J_1})^\sim &= (\{\phi, \psi\}_J)^\sim \circ \sharp_{P_{J_1}} = \{\check{\phi}, \check{\psi}\}_{P_J^c} \circ \sharp_{P_{J_1}} \\ (\{\phi \circ \sharp_{J_1}, \psi \circ \sharp_{J_1}\}_{J_M})^\sim &= \{(\phi \circ \sharp_{J_1})^\sim, (\psi \circ \sharp_{J_1})^\sim\}_{\Lambda_{M \times \mathbb{R}}} = \{\check{\phi} \circ \sharp_{P_{J_1}}, \check{\psi} \circ \sharp_{P_{J_1}}\}_{\Lambda_{M \times \mathbb{R}}}. \end{aligned}$$

Thus

$$-\{\phi, \psi\}_J \circ \sharp_{J_1} = \{\phi \circ \sharp_{J_1}, \psi \circ \sharp_{J_1}\}_{J_M}$$

if and only if

$$-\{\check{\phi}, \check{\psi}\}_{P_J^c} \circ \sharp_{P_{J_1}} = \{\check{\phi} \circ \sharp_{P_{J_1}}, \check{\psi} \circ \sharp_{P_{J_1}}\}_{\Lambda_{M \times \mathbb{R}}}, \quad (38)$$

which means that J_M and $-\widehat{J}$ are \sharp_{J_1} -related if and only if $\Lambda_{M \times \mathbb{R}}$ and the complete lift of the poissonization $-P_J^c$ are $\sharp_{P_{J_1}}$ -related. Due to Theorem 3.2, we get that $P_{J_1} = P_J$ and the poissonization P_J is a Poisson tensor which, in turn, is equivalent to the fact that J is a Jacobi bracket. Thus we get

$$(J1) \Leftrightarrow (J2) \Leftrightarrow (J3) \Leftrightarrow (J4).$$

Using now linear functions ϕ, ψ , we get in a similar way that (38) is equivalent to

$$-\{\phi, \psi\}_{\widehat{J}^c} \circ \sharp_{J_1} = \{\phi \circ \sharp_{J_1}, \psi \circ \sharp_{J_1}\}_{\Lambda_M}$$

which, due to Theorem 3.2, gives

$$(J1) \Leftrightarrow (J5) \Leftrightarrow (J6) \Leftrightarrow (J7).$$

Finally, completely analogously to (27) we get $(J5) \Leftrightarrow (J8) \Leftrightarrow (J9) \Leftrightarrow (J10)$. ■

Remark. In the above proof we get the lifts \widehat{J} , J^c , and the map \sharp_J in a natural way by using the poissonization and its tangent lift. This is a geometric version of the methods in [Va] for obtaining J^c . Note also that J_M is the canonical Jacobi structure on $T^*M \times \mathbb{R}$ regarded as a contact manifold in a natural way and that the equivalence $(J1) \Leftrightarrow (J8)$ is a version of the characterization in [MMP].

The above theorem characterizing Jacobi structures one can generalize to canonical structures associated with Jacobi algebroids as follows.

Consider now a Jacobi algebroid, i.e. a vector bundle E over M equipped with a Lie algebroid bracket $[\cdot, \cdot]$ and a ‘closed 1-form’ $\Phi \in \Omega^1(E)$. We denote by $\llbracket \cdot, \cdot, \cdot \rrbracket$ the Schouten-Nijenhuis bracket of the Lie algebroid and by $\mathcal{T}(E) \ni X \mapsto X^c \in \mathcal{T}(TE)$ the complete lift from the tensor algebra of E into the tensor algebra of TE . The corresponding Schouten-Jacobi bracket we denote by $\llbracket \cdot, \cdot \rrbracket_\Phi$ and the corresponding complete Jacobi and Poisson lifts by $\mathcal{T}(E) \ni X \mapsto \widehat{X}_\Phi \in \mathcal{T}(TE)$ and $\mathcal{T}(E) \ni X \mapsto X_\Phi^c \in \mathcal{T}(TE)$, respectively.

If the 1-cocycle Φ is exact, $\Phi = ds$, we can obtain the bracket $\llbracket \cdot, \cdot \rrbracket_\Phi$ from $\llbracket \cdot, \cdot \rrbracket$ using the linear automorphism of $\mathcal{A}(E)$ defined by $\mathcal{A}^k(E) \ni X \mapsto e^{-(k-1)s} X$ (cf. [GM1]). This is a version of the Witten’s trick [Wi] to obtain the deformed exterior differential $d^\Phi \mu = d\mu + \Phi \wedge \mu$ via the automorphism of the cotangent bundle given by multiplication by e^s .

Even if the 1-cocycle Φ is not exact, there is a nice construction [IM1] which allows one to view Φ as being exact but for an extended Lie algebroid in the bundle $\widehat{E} = E \times \mathbb{R}$ over $M \times \mathbb{R}$. The sections of this bundle may be viewed as parameter-dependent (s -dependent) sections of E . The sections of E form a Lie subalgebra of s -independent sections in the Lie algebroid \widehat{E} which generate the $C^\infty(M \times \mathbb{R})$ -module of sections of \widehat{E} and the whole structure is uniquely determined by putting the anchor $\widehat{\rho}(X)$ of a s -independent section X to be $\widehat{\rho}(X) = \rho(X) + \langle \Phi, X \rangle \partial_s$, where s is the standard coordinate function in \mathbb{R} and ρ is the anchor in E . All this is consistent (thanks to the fact that Φ is a cocycle) and defines a Lie algebroid structure on \widehat{E} with the exterior derivative d satisfying $ds = \Phi$.

Let now $U : \mathcal{T}(E) \rightarrow \mathcal{T}(\widehat{E})$ be the natural embedding of the tensor algebra of E into the tensor subalgebra of s -independent sections of \widehat{E} . It is obvious that on skew-symmetric tensors U is a homomorphism of the corresponding Schouten brackets:

$$[[U(X), U(Y)]]^\wedge = U([X, Y]),$$

where we use the notation $[[\cdot, \cdot]]$ and $[[\cdot, \cdot]]^\wedge$ for the Schouten brackets in E and \widehat{E} , respectively. Let us now gauge $\mathcal{T}(E)$ inside $\mathcal{T}(\widehat{E})$ by putting

$$P^\Phi(X) = e^{-ks}U(X)$$

for any element $X \in \text{Sec}(E^{\otimes(k+1)})$. Note that $X \mapsto P^\Phi(X)$ preserves the grading but not the tensor product. It can be easily proved (cf. [GM1]) that the Schouten-Jacobi bracket $[[\cdot, \cdot]]_\Phi$ can be obtained by this gauging from the Lie algebroid bracket.

Theorem 4.3. ([GM1]) For any $X \in \mathcal{A}(E), Y \in \mathcal{A}(E)$ we have

$$[[P^\Phi(X), P^\Phi(Y)]]^\wedge = P^\Phi([X, Y]_\Phi). \quad (39)$$

We will usually skip the symbol U and write simply $P^\Phi(X) = e^{-ks}X$, regarding $\mathcal{T}(E)$ as embedded in $\mathcal{T}(\widehat{E})$. The complete lift for the Lie algebroid \widehat{E} will be denoted by $X \mapsto X^\wedge$ to distinguish from the lift for E . It is easy to see that

$$(P^\Phi(X))^\wedge = (e^{-ks}X)^\wedge = e^{-ks}(X^c - k\iota_\Phi X^v + \partial_s \wedge (i_\Phi X)^v).$$

Here we understand tensors on E as tensors on $\widehat{E} = E \times \mathbb{R}$ in obvious way. Note that $(\widehat{E^*}) = (\widehat{E})^*$ and the linear Poisson tensor $\Lambda^{\widehat{E^*}}$ reads

$$\Lambda^{\widehat{E^*}} = \Lambda^{E^*} + \Phi^v \wedge \partial_s,$$

where Λ^{E^*} is the Poisson tensor corresponding to the Lie algebroid E and Φ^v is the vertical lift of Φ . Recall that on E^* we have also a canonical Jacobi structure

$$J_\Phi^{E^*} = \Lambda^{E^*} + \Delta_{E^*} \wedge \Phi^v - I \wedge \Phi^v$$

which generates a Jacobi bracket which coincides with the Poisson bracket of Λ^{E^*} on linear functions.

Let us remark that the map P^Φ plays the role of a generalized poissonization. Indeed, for the Jacobi algebroid of first-order differential operators $E = TM \oplus \mathbb{R}$ the extended Lie algebroid $\widehat{E} \times \mathbb{R}$ is canonically isomorphic with $T(M \times \mathbb{R})$, $U((X, f)) = X + f\partial_s$, and for $J \in \text{Sec}(E^{\otimes 2})$ the tensor field $P^\Phi(J)$ coincides with (37).

Let now $J \in \mathcal{A}^2(E)$. The tensor J is a canonical structure for the Jacobi algebroid (E, Φ) , i.e. $[[J, J]]_\Phi = 0$, if and only if $P^\Phi(J)$ is a canonical structure for the Lie algebroid \widehat{E} , i.e. $[[P^\Phi(J), P^\Phi(J)]]^\wedge = 0$. Moreover,

$$\sharp_{P^\Phi(J)}(u_x, s) = (e^{-s}\sharp_J(u_x), s).$$

Like above, for any function $\phi \in C^\infty(E)$ we denote by $\check{\phi}$ the function on $\widehat{E} = E \times \mathbb{R}$ given by $\check{\phi}(v_x, s) = e^s \phi(v_x)$ and for any function $\varphi \in C^\infty(E^*)$ we denote by $\tilde{\varphi}$ the function on $\widehat{E}^* = E^* \times \mathbb{R}$ given by $\tilde{\varphi}(u_x, s) = e^s \varphi(e^{-s} u_x)$. Recall that (cf. Section 2)

$$\widehat{J}_\Phi = J^c - \iota_\Phi J^v + I \wedge (i_\Phi J)^v$$

and

$$\widehat{J}_\Phi^c = J^c - \iota_\Phi J^v + \Delta_E \wedge (i_\Phi J)^v.$$

The corresponding brackets on functions on E coincide on linear functions and define a bracket $[\cdot, \cdot]_J$ on sections of E^* in the standard way:

$$\iota_{[\mu, \nu]_J} = \{\iota_\mu, \iota_\nu\}_{\widehat{J}_\Phi} = \{\iota_\mu, \iota_\nu\}_{\widehat{J}_\Phi^c}$$

Completely analogously to Lemma 4.2 we get the following.

Lemma 4.4. (a) The maps $\phi \mapsto \check{\phi}$ and $\varphi \mapsto \tilde{\varphi}$ are injective.

(b) For any $J \in \mathcal{A}^2(E)$

$$\check{\phi} \circ \sharp_{P^\Phi(J)} = (\phi \circ \sharp_J)^\sim.$$

(c) For any $\phi, \psi \in C^\infty(E)$

$$\{\check{\phi}, \check{\psi}\}_{(P^\Phi(J))^c} = (\{\phi, \psi\}_{\widehat{J}_\Phi})^\sim.$$

(d) For any $\phi, \psi \in C^\infty(E^*)$

$$\{\tilde{\phi}, \tilde{\psi}\}_{\Lambda^{\widehat{E}^*}} = (\{\phi, \psi\}_{J_\Phi^{E^*}})^\sim.$$

(e) For linear $\phi, \psi \in C^\infty(E)$

$$\{\check{\phi}, \check{\psi}\}_{(P^\Phi(J))^c} = (\{\phi, \psi\}_{\widehat{J}_c})^\sim.$$

(f) For linear $\phi, \psi \in C^\infty(E^*)$

$$\{\tilde{\phi}, \tilde{\psi}\}_{\Lambda^{\widehat{E}^*}} = (\{\phi, \psi\}_{\Lambda^{E^*}})^\sim.$$

Now, repeating the arguments from the classical case, one easily derives the following.

Theorem 4.5. For any bisection $J \in \mathcal{A}^2(E)$ of the vector bundle E of a Jacobi algebroid (E, Φ) the following are equivalent:

- (1) J is a canonical structure, i.e. $\llbracket J, J \rrbracket_\Phi = 0$;
- (2) The canonical Jacobi bracket $J_\Phi^{E^*}$ and $-\widehat{J}_\Phi$ are \sharp_J -related;
- (3) The bivector fields Λ^{E^*} and $-\widehat{J}_\Phi^c$ are \sharp_J -related;
- (4) For any ‘1-forms’ $\mu, \nu \in \Omega^1(E)$,

$$\sharp_J([\mu, \nu]_J) = [\sharp_J(\mu), \sharp_J(\nu)],$$

where the bracket on the right-hand-side is the Lie algebroid bracket on E .

Note that a canonical structure for a Jacobi algebroid gives rise to a *triangular Jacobi bialgebroid* [GM] (or a *triangular generalized Lie bialgebroid* in the terminology of [IM1]).

Acknowledgments

Supported by KBN, grant No 2 P03A 041 18.

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