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Localization of LM_n - algebras

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Abstract: The aim of this paper is to define the localization LM_n - algebra of an LM_n- algebra L with respect to a topology $\mathcal{F}$ on L; In Section 5 we prove that the maximal LM_n - algebra of fractions (defined in [3]) and the LM_n - algebra of fractions relative to an $\wedge-$ closed system (defined in Section 2) are LM_n - algebras of localization.

Keywords: LM_n *- algebra,topology,* $\mathcal{F}-$ *multiplier, multiplier,* LM_n *- algebra of fractions, maximal* LM_n *- algebra of fractions,* LM_n *- algebra of localization*
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Introduction

The concept of multiplier for distributive lattices was defined by W. H. Cornish in [7]. J. Schmid used multipliers in order to give a non–standard construction of the maximal lattice of quotients for a distributive lattice (see [13]). A direct treatment of the lattices of quotients can be found in [14]. In [10], G. Georgescu exhibited the localization lattice $L_{\mathcal{F}}$ of a distributive lattice L with respect to a topology $\mathcal{F}$ on L mimicking the familiar construction for rings (see [12]) or monoids (see [15]). For the case of Hilbert and Heyting algebras see [4] and [8].

The concepts of LM_n - algebra of fractions relative to an $\wedge-$ closed system, LM_n - algebra of fractions and maximal LM_n - algebra of fractions was defined by the author in Section 2 and [3].

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1 Definitions and preliminaries

Let n be an integer, $n \geq 2$.

Definition 1.1. *([2])An $n-$ valued Lukasiewicz $-$ Moisil algebra (shortly, LM_n - algebra) is an algebra* $\mathcal{L} = (L, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1 \leq i \leq n-1})$ *of type* $(2, 2, 1, 0, 0, \{1\}_{1 \leq i \leq n-1})$ *satisfying the following conditions:*

(1.1) $(L, \wedge, \vee, N, 0, 1)$ *is a De Morgan algebra,*

(1.2) $\varphi_1, ..., \varphi_{n-1} : L \to L$ *are bounded lattice morphisms such that for every* $x, y \in L$*:*

(1.2.1) $\varphi_i(x) \vee N\varphi_i(x) = 1$ *for every* $i = 1, ..., n-1$,

(1.2.2) $\varphi_i(x) \wedge N\varphi_i(x) = 0$ *for every* $i = 1, ..., n-1$,

(1.2.3) $\varphi_i\varphi_j(x) = \varphi_j(x)$ *for every* $i, j = 1, ..., n-1$,

(1.2.4) $\varphi_i(Nx) = N\varphi_j(x)$ *for every* $i, j = 1, ..., n-1$ *with* $i + j = n$,

(1.2.5) $\varphi_1(x) \leq \varphi_2(x) \leq ... \leq \varphi_{n-1}(x)$,

(1.2.6) *If* $\varphi_i(x) = \varphi_i(y)$ *for every* $i = 1, ..., n-1$*, then* $x = y$.

The relation (1.2.6) is called *the determination principle.* As consequences of *the determination principle* we obtain:

(1.2.7) If $x, y \in L$, then $x \leq y$ iff $\varphi_i(x) \leq \varphi_i(y)$ for all $i = 1, ..., n-1$,

(1.2.8.) $\varphi_1(x) \leq x \leq \varphi_{n-1}(x)$ for all $x \in L$.

We denote an LM_n-algebra $\mathcal{L} = (L, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1 \leq i \leq n-1})$ by its universe L.

Remark 1.2. The endomorphisms $\{\varphi_i\}_{1 \leq i \leq n-1}$are called *chrysippian endomorphisms*.

Examples:

1. Let $L_n = \{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\}$.We define $x \vee y = \max\{x, y\}, x \wedge y = \min\{x, y\}, Nx = 1 - x$ $(N(\frac{j}{n-1}) = \frac{n-1-j}{n-1})$ and $\varphi_i : L_n \to L_n, \varphi_i(\frac{j}{n-1}) = 0$ if $i + j < n$ and 1 if $i + j \geq n$, for $i, j = 1, ..., n-1$.

Then $(L_n, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1 \leq i \leq n-1})$ is an LM_n-algebra.

2. If $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra, then $(B, \wedge, \vee, ', 0, 1, \{\varphi_i\}_{1 \leq i \leq n-1})$ is an LM_n-algebra, where $\varphi_i = 1_B$ for every $1 \leq i \leq n-1$.

3. Let $(B, \vee, \wedge, ', 0, 1)$ a Boolean algebra and $D(B) = \{(x_1, ..., x_{n-1}) \in B^{n-1} : x_1 \leq ... \leq x_{n-1}\}$. We define pointwise the infimum and the supremum, $N(x_1, ..., x_{n-1}) = (x'_{n-1}, ..., x'_1)$ and $\varphi_i(x_1, ..., x_{n-1}) = (x_i, ..., x_i)$ for all $i = 1, ..., n-1$.

Then $(D(B), \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1 \leq i \leq n-1})$ is an LM_n-algebra.

In the rest of this paper, by L we denote an LM_n-algebra.

We denote by $C(L)$ the set of all *complemented elements* of L and we call it *the center of* L; it is easy to see that $(C(L), \vee, \wedge, N, 0, 1)$ is a Boolean algebra.

Lemma 1.3. *([2])Let L be an LM_n-algebra.The following are equivalent:*

(i) $e \in C(L)$,

(ii) there are $i \in \{1, ..., n-1\}$ and $x \in L$ such that $e = \varphi_i(x)$,
(iii) there is $i \in \{1, ..., n-1\}$ such that $e = \varphi_i(e)$,
(iv) $e = \varphi_i(e)$ for every $i = 1, ..., n-1$,
(v) $\varphi_i(e) = \varphi_j(e)$ for every $i, j = 1, ..., n-1$.

Remark 1.4. *If $x \in L$, then $\varphi_i(x) \in C(L)$ for every $i = 1, ..., n-1$.*

Lemma 1.5. *([2])Let L be an LM_n-algebra.The following are equivalent:*

(i) $e \in C(L)$,
(ii) $N\ e \in C(L)$,
(iii) $e \wedge Ne = 0$,
(iv) $e \vee Ne = 1$.

Lemma 1.6. *If L is an LM_n-algebra, then for every $x \in L$, $x \wedge \varphi_1(Nx) = 0$ which is equivalent to $x \wedge N\varphi_{n-1}(x) = 0$.*

Proof. For every $x \in L$ we have $x \leq \varphi_{n-1}(x)$, so

$$x \wedge \varphi_1(Nx) = x \wedge N\varphi_{n-1}(x) \leq \varphi_{n-1}(x) \wedge N\varphi_{n-1}(x) = 0 \text{ (by(1.2.2))},$$

hence $x \wedge \varphi_1(Nx) = 0$. □

Theorem 1.7. *([1]) For an LM_n-algebra L (with $0 \neq 1$), the following are equivalent:*

(i) $C(L) = \{0, 1\}$,
(ii) *L is a chain,*
(iii) *L is subdirectly irreducible.*

Corollary 1.8. *([2]) Every chain which is an LM_n-algebra is finite.*

Definition 1.9. *([2])Let L and L' be LM_n-algebras. A function $f : L \to L'$ is a morphism of LM_n-algebras iff it satisfies the following conditions, for every $x, y \in L$:*

(i) $f(x \vee y) = f(x) \vee f(y)$,
(ii) $f(x \wedge y) = f(x) \wedge f(y)$,
(iii) $f(0) = 0, f(1) = 1$,
(iv) $f(\varphi_i(x)) = \varphi_i(f(x))$ for every $i = 1, ..., n-1$.

Remark 1.10. *It follows (from 1.2.4 and 1.2.6) that*

$$f(Nx) = Nf(x)$$

for every $x \in L$.

We denote by $\mathbf{LM}_n$ the *category of LM_n-algebras.*

Definition 1.11. *([2]) Let L an LM_n-algebra. We say that a nonempty subset $I \subseteq L$ in an $n-ideal$ if I is an ideal of the lattice L and if $x \in I$, then $\varphi_{n-1}(x) \in I$.*

Remark 1.12. *From (1.2.5) we deduce that if $I \subseteq L$ is an n-ideal and $x \in I$, then $\varphi_i(x) \in I$ for every $i \in \{1, ..., n-1\}$.*

We denote by $Idn(L)$ the set of all $n-ideals$ of the LM_n- algebra L and by $Id(C(L))$ the set of all ideals of the Boolean algebra $C(L)$.

If $X \subseteq L$ is a nonempty set, we denote by $< X >$ the n-ideal generated by X. We have that:

$$< X >= \{y \in L: \text{ there exist } p \geq 1 \text{ and } x_1, ..., x_p \in X \text{ such that } y \leq \varphi_{n-1}(\bigvee_{i=1}^{p} x_i)\}.$$

In particular, for $a \in L$, $< a >= \{x \in L : x \leq \varphi_{n-1}(a)\}$ and if $a \in C(L)$, then $< a >= \{x \in L : x \leq a\} = (a]$.

Let I be an n-ideal and $x \in L$. We denote by $(I : x) = \{y \in L : x \wedge y \in I\}$.

Lemma 1.13. *The set $(I : x)$ is an n-ideal.*

Proof. Let $y_1, y_2 \in (I : x)$. Then $x \wedge y_1, x \wedge y_2 \in I$, hence $x \wedge (y_1 \vee y_2) = (x \wedge y_1) \vee (x \wedge y_2) \in I$, that is, $y_1 \vee y_2 \in (I : x)$.

If $y_1 \in (I : x)$ and $y_2 \leq y_1$, then $x \wedge y_1 \in I$ and $x \wedge y_2 \leq x \wedge y_1$, hence $x \wedge y_2 \in I$, that is, $y_2 \in (I : x)$.

If $y \in (I : x)$ then $x \wedge y \in I$, hence $\varphi_{n-1}(x) \wedge \varphi_{n-1}(y) = \varphi_{n-1}(x \wedge y) \in I$. But $x \wedge \varphi_{n-1}(y) \leq \varphi_{n-1}(x) \wedge \varphi_{n-1}(y)$, so $x \wedge \varphi_{n-1}(y) \in I$, that is, $\varphi_{n-1}(y) \in (I : x)$.

Remark 1.14. *([10]) If I is an $n-ideal$ of L, then $I^b = I \cap C(L)$ is an ideal of the Boolean algebra $C(L)$; also, every ideal J of $C(L)$ induce an n-ideal $\varphi_{n-1}^{-1}(J)$ of L. The mappings $I \mapsto I^b$, $J \mapsto \varphi_{n-1}^{-1}(J)$ establish a bijection between the $n-ideals$ of L and the ideals of $C(L)$.*

Definition 1.15. *([2]) A congruence on an LM_n-algebra L is an equivalence relation on L compatible with the operations $\wedge, \vee, N, \varphi_i$, for every $i = 1, ..., n-1$.*

Proposition 1.16. *([2])For an equivalence relation ρ on an LM_n-algebra L, the following conditions are equivalent:*
(1) ρ is a congruence on L,
(2) ρ is compatible with $\wedge, \vee, \varphi_i$, for every $i = 1, ..., n-1$.

2 LM$_n-$algebra of fractions relative to an $\wedge$-closed system

Definition 2.1. *A nonempty subset $S \subseteq L$ is called $\wedge-$closed system in L if*
(2.1) $1 \in S$,
(2.2) $x, y \in S$ implies $x \wedge y \in S$,

(2.3) *$x \in S$ implies $\varphi_{n-1}(x) \in S$.*

We denote by $S(L)$ the set of all $\wedge-$closed systems of L (clearly $\{1\}$, $L \in S(L)$).

For $S \in S(L)$, on the LM_n-algebra L we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } s \in S \text{ such that } x \wedge \varphi_{n-1}(s) = y \wedge \varphi_{n-1}(s).$$

Lemma 2.2. *θ_S is a congruence on L.*

Proof. The reflexivity (since $1 \in S$) and the symmetry of θ_S are immediately. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$. Thus, there are $s, s' \in S$ such that $x \wedge \varphi_{n-1}(s) = y \wedge \varphi_{n-1}(s)$ and $y \wedge \varphi_{n-1}(s') = z \wedge \varphi_{n-1}(s')$. If denote $s'' = s \wedge s' \in S$, then $x \wedge \varphi_{n-1}(s'') = x \wedge \varphi_{n-1}(s \wedge s') = (x \wedge \varphi_{n-1}(s)) \wedge \varphi_{n-1}(s') = (y \wedge \varphi_{n-1}(s)) \wedge \varphi_{n-1}(s') = y \wedge \varphi_{n-1}(s') \wedge \varphi_{n-1}(s) = z \wedge \varphi_{n-1}(s') \wedge \varphi_{n-1}(s) = z \wedge \varphi_{n-1}(s \wedge s') = z \wedge \varphi_{n-1}(s'')$, hence $(x, z) \in \theta_S$.

To prove the compatibility of θ_S with the operations $\wedge, \vee$, and φ_i for every $i = 1, ..., n-1$, let $x, y, z, t \in L$ such that $(x, y), (z, t) \in \theta_S$. Thus there are $s, s' \in S$ such that $x \wedge \varphi_{n-1}(s) = y \wedge \varphi_{n-1}(s)$ and $z \wedge \varphi_{n-1}(s') = t \wedge \varphi_{n-1}(s')$. If we denote $s'' = s \wedge s' \in S$, then $(x \wedge z) \wedge \varphi_{n-1}(s'') = (y \wedge t) \wedge \varphi_{n-1}(s'')$, hence $(x \wedge z, y \wedge t) \in \theta_S$.

From $x \wedge \varphi_{n-1}(s'') = y \wedge \varphi_{n-1}(s'')$, $z \wedge \varphi_{n-1}(s'') = t \wedge \varphi_{n-1}(s'')$ and the distributivity of L we deduce $(x \vee z) \wedge \varphi_{n-1}(s'') = (y \vee t) \wedge \varphi_{n-1}(s'')$, that is, $(x \vee z, y \vee t) \in \theta_S$.

From $x \wedge \varphi_{n-1}(s) = y \wedge \varphi_{n-1}(s)$ we deduce that $\varphi_i(x \wedge \varphi_{n-1}(s)) = \varphi_i(y \wedge \varphi_{n-1}(s))$, for every $i = 1, ..., n-1$, that is, $\varphi_i(x) \wedge \varphi_{n-1}(s) = \varphi_i(y) \wedge \varphi_{n-1}(s)$, hence $(\varphi_i(x), \varphi_i(y)) \in \theta_S$.□

For $x \in L$ we denote by x/S the equivalence class of x relative to θ_S and by

$$L[S] = L/\theta_S.$$

By $p_S : L \to L[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in L$. Clearly, in $L[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in L$,

$$x/S \wedge y/S = (x \wedge y)/S$$

$$x/S \vee y/S = (x \vee y)/S$$

$$N(x/S) = (Nx)/S$$

$$\bar{\varphi}_i : L[S] \to L[S], \bar{\varphi}_i(x/S) = (\varphi_i(x))/S \text{ for every } i = 1, ..., n-1,$$

(for every $i = 1, ..., n-1, \bar{\varphi}_i$ is correct defined because for $x, y \in L$ such that $x/S = y/S$, there exists $s \in S$ such that $x \wedge \varphi_{n-1}(s) = y \wedge \varphi_{n-1}(s)$, so, $\varphi_i(x) \wedge \varphi_{n-1}(s) = \varphi_i(y) \wedge \varphi_{n-1}(s)$, that is, $\varphi_i(x)/S = \varphi_i(y)/S$).

Remark 2.3. *Since for every $s \in S$, $\varphi_{n-1}(s) \wedge \varphi_{n-1}(s) = 1 \wedge \varphi_{n-1}(s)$, we deduce that $\varphi_{n-1}(s)/S = 1/S = \mathbf{1}$, hence $p_S(\varphi_{n-1}(S)) = \{\mathbf{1}\}$.*

Proposition 2.4. *If $a \in L$, then $a/S \in C(L[S])$ iff there exists $s \in S$ such that $a \wedge \varphi_{n-1}(s) \in C(L)$. So, if $a \in C(L)$, then $a/S \in C(L[S])$.*

Proof. For $a \in L$, we have $a/S \in C(L[S])$ iff $\bar{\varphi}_i(a/S) = a/S$ for all $i = 1, ..., n-1$, that is, $\varphi_i(a)/S = a/S$ for all $i = 1, ..., n-1$. So, $(\varphi_i(a), a) \in \theta_S$, which it means that there exists $s \in S$ such that $\varphi_i(a) \wedge \varphi_{n-1}(s) = a \wedge \varphi_{n-1}(s)$, that is, $\varphi_i(a \wedge \varphi_{n-1}(s)) = a \wedge \varphi_{n-1}(s)$, hence $a \wedge \varphi_{n-1}(s) \in C(L)$.

If $a \in C(L)$, since $1 \in S$ and $a \wedge \varphi_{n-1}(1) = a \in C(L)$, we deduce that $a/S \in C(L[S])$.□

Theorem 2.5. *If L is an LM_n-algebra and $f : L \to L'$ is a morphism of LM_n-algebras such that $f(\varphi_{n-1}(S)) = \{\mathbf{1}\}$, then, there is a unique morphism of LM_n-algebras $f' : L[S] \to L'$, such that the diagram*

$$\begin{array}{ccc} L & \xrightarrow{p_S} & L[S] \\ & \searrow_{f} \quad {}_{f'}\swarrow & \\ & L' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$).

Proof. If $x, y \in L$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence, there is $s \in S$ such that $x \wedge \varphi_{n-1}(s) = y \wedge \varphi_{n-1}(s)$. Since f is morphism of LM_n-algebras, we obtain that $f(x \wedge \varphi_{n-1}(s)) = f(y \wedge \varphi_{n-1}(s))$, that is, $f(x) \wedge f(\varphi_{n-1}(s)) = f(y) \wedge f(\varphi_{n-1}(s))$. But $f(\varphi_{n-1}(s)) = \mathbf{1}$, so $f(x) \wedge \mathbf{1} = f(y) \wedge \mathbf{1}$, that is, $f(x) = f(y)$.

We deduce that the map $f' : L[S] \to L'$ defined for $x \in L$ by $f'(x/S) = f(x)$ is correct defined. Clearly, f' is a morphism of LM_n-algebras. The unicity of f' follows from the fact that p_S is an onto map. □

Remark 2.6. *The previous theorem allows us to call $L[S]$ the LM_n-algebra of fractions relative to the $\wedge-$closed system S.*

Examples:

(1) If $S = \{1\}$ then θ_S is the identity congruence of L, hence $L[S] = L$.

(2) If S is an $\wedge -$ *closed system* such that $0 \in S$ (for example $S = L$ or $S = C(L)$), then for every $x, y \in L, (x, y) \in \theta_S$ (since $x \wedge \varphi_{n-1}(0) = y \wedge \varphi_{n-1}(0)$), hence in this case $L[S] = \{0\}$.

3 Topologies on an LM$_n$-algebra

Definition 3.1. *([10]) A non-empty set $\mathcal{F}$ of n-ideals of L will be called a topology on L if the following properties hold:*

(T_1) *If $I \in \mathcal{F}$, $x \in L$ then $(I : x) \in \mathcal{F}$,*

(T_2) If $I_1, I_2 \in Idn(L)$ and $I_2 \in \mathcal{F}$, if $(I_1 : x) \in \mathcal{F}$ for all $x \in I_2$, then $I_1 \in \mathcal{F}$.

Lemma 3.2. *([10]) If $\mathcal{F}$ is a topology on L, then:*

(i) If $I_1 \in \mathcal{F}$ and I_2 is an n-ideal with $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$,
(ii) If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$,
(iii) $(\mathcal{F} \cup \{\oslash\}, L)$ is a topological space.

Remark 3.3. *([10]) Any intersection of topologies on L is a topology, hence the set $\mathcal{G}(L)$ of the topologies of L is a complete lattice with respect to inclusion.*

Examples:
1. If $I \in Idn(L)$, then the set

$$\mathcal{F}(I) = \{I' \in Idn(L) : I \subseteq I'\}$$

is a topology on L.
2. A non-empty set $I \subseteq L$ will be called *regular* if for every $x, y \in L$ such that $e \wedge x = e \wedge y$ for every $e \in I$, then $x = y$. If we denote

$$R(L) = \{I \subseteq L : I \text{ is a regular subset of } L\},$$

then $Idn(L) \cap R(L)$ is a topology on L .
3. For any $\wedge-$ closed subset S of L(see Definition 2.1) we set

$$\mathcal{F}_S = \{I \in Idn(L) : I \cap S \neq \oslash\}.$$

Then $\mathcal{F}_S$ is a topology on L .

4 $\mathcal{F}$-multipliers and localization LM$_n$-algebra

Let $\mathcal{F}$ be a topology on L. We consider the relation $\theta_{\mathcal{F}}$ of L

$$(x, y) \in \theta_{\mathcal{F}} \text{ iff there exists } I \in \mathcal{F} \text{ such that } e \wedge x = e \wedge y \text{ for every } e \in I.$$

Lemma 4.1. *$\theta_{\mathcal{F}}$ is a congruence on L.*

Proof. The reflexivity and the symmetry of $\theta_{\mathcal{F}}$ are immediate; in order to prove the transitivity of $\theta_{\mathcal{F}}$ let $(x, y), (y, z) \in \theta_{\mathcal{F}}$. Then, there exists $I_1, I_2 \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I_1$ and $f \wedge y = f \wedge z$ for every $f \in I_2$. If we set $I = I_1 \cap I_2 \in \mathcal{F}$, then for every $g \in I$, $g \wedge x = g \wedge z$, hence $(x, z) \in \theta_{\mathcal{F}}$.

For the compatibility of $\theta_{\mathcal{F}}$ with the operations $\wedge, \vee$ let $x, y, z, t \in L$ such that $(x, y), (z, t) \in \theta_{\mathcal{F}}$, that is, there exists $I, J \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I$, and $f \wedge z = f \wedge t$ for every $f \in J$. If we denote $K = I \cap J$, then $K \in \mathcal{F}$ and for every $g \in K$, $g \wedge x = g \wedge y$ and $g \wedge z = g \wedge t$. Then $g \wedge (x \wedge z) = g \wedge (y \wedge t)$ and $g \wedge (x \vee z) = g \wedge (y \vee t)$, that is, $(x \wedge z, y \wedge t) \in \theta_{\mathcal{F}}$ and $(x \vee z, y \vee t) \in \theta_{\mathcal{F}}$.

For the compatibility of $\theta_{\mathcal{F}}$ with φ_i for every $i = 1, ..., n-1$, let $(x, y) \in \theta_{\mathcal{F}}$ and $i \in \{1, ..., n-1\}$ fixed. Then, there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I$. Since for every $j \in \{1, ..., n-1\}$ and $e \in I, \varphi_j(e) \in I$ we deduce that $\varphi_j(e) \wedge x = \varphi_j(e) \wedge y$. Then, $\varphi_i(\varphi_j(e) \wedge x) = \varphi_i(\varphi_j(e) \wedge y) \Leftrightarrow \varphi_j(e) \wedge \varphi_i(x) = \varphi_j(e) \wedge \varphi_i(y) \Leftrightarrow \varphi_j(e \wedge \varphi_i(x)) = \varphi_j(e \wedge \varphi_i(y))$ for every $j \in \{1, ..., n-1\}$. By (1.2.6) we deduce that $e \wedge \varphi_i(x) = e \wedge \varphi_i(y)$ for every $e \in I$. Therefore $(\varphi_i(x), \varphi_i(y)) \in \theta_{\mathcal{F}}$ for every $i = 1, ..., n-1$. □

We shall denote by $x/\theta_{\mathcal{F}}$ the congruence class of an element $x \in L$, by $L/\theta_{\mathcal{F}}$ the quotient LM_n-algebra and by

$$p_{\mathcal{F}} : L \to L/\theta_{\mathcal{F}}$$

the canonical morphism of LM_n-algebras. We denote the chrysippian endomorphisms of $L/\theta_{\mathcal{F}}$ by $\overline{\varphi}_i$ and we have $\overline{\varphi}_i(x/\theta_{\mathcal{F}}) = \varphi_i(x)/\theta_{\mathcal{F}}$.

Proposition 4.2. *For $a \in L, a/\theta_{\mathcal{F}} \in C(L/\theta_{\mathcal{F}})$ iff there exists $I \in \mathcal{F}$ such that $e \wedge \varphi_i(a) = e \wedge a$ for every $e \in I$ and $i \in \{1, ..., n-1\}$. So, if $a \in C(L)$, then $a/\theta_{\mathcal{F}} \in C(L/\theta_{\mathcal{F}})$.*

Proof. For $a \in L$, $a/\theta_{\mathcal{F}} \in C(L/\theta_{\mathcal{F}})$ iff $\overline{\varphi}_i(a/\theta_{\mathcal{F}}) = a/\theta_{\mathcal{F}}$ iff $\varphi_i(a)/\theta_{\mathcal{F}} = a/\theta_{\mathcal{F}}$ for every $i = 1, ..., n-1$. So, $(\varphi_i(a), a) \in \theta_{\mathcal{F}}$, that is, there exists $I \in \mathcal{F}$ such that $e \wedge \varphi_i(a) = e \wedge a$ for every $e \in I$ and $i \in \{1, ..., n-1\}$.

So, if $a \in C(L)$, then for every $I \in \mathcal{F}$ and $e \in I$, $e \wedge \varphi_i(a) = e \wedge a$, hence $a/\theta_{\mathcal{F}} \in C(L/\theta_{\mathcal{F}})$. □

Corollary 4.3. *If $\mathcal{F} = Idn(L) \cap R(L)$, then $a \in C(L)$ iff $a/\theta_{\mathcal{F}} \in C(L/\theta_{\mathcal{F}})$.*

We recall ([3]) some folklore about multipliers.

Let $(A, \leq)$ be a poset. A nonempty subset $I \subseteq A$ is an *order ideal* (also known as a *down-set* or *decreasing set*) in A whenever $x \leq y \in I$ implies $x \in I$; we denote by $I(A)$ the set of all order ideals in A.

If A is an inf-semilattice and $I \in I(A)$, a map $f : I \to A$ is called a *multiplier* (alias a *partial translation*) if $f(a \wedge x) = a \wedge f(x)$ for all $a \in A$ and $x \in I$.

Such maps have been studied extensively by Cornish in [7].

In this paper, we are concerned with multipliers on an LM_n-algebra L; clearly $Idn(L) \subseteq I(L)$.

Definition 4.4. *([3])By a partial multiplier of L we mean a map $f : I \to L$, where $I \in Idn(L)$, which verifies the following condition:*

(2.1) $f(e \wedge x) = e \wedge f(x)$, for every $e \in L$ and $x \in I$.

Sometime we will denote the domain of f by $dom(f)$; if $I = L$ we say that f is *total.*

To simplify the language, we will use *multiplier* instead of *partial multiplier* and *total* to indicate that the domain of a certain multiplier is L.

Examples:

1. The map $\mathbf{0} : L \to L$ defined by $\mathbf{0}(x) = 0$ for every $x \in L$, is a total multiplier of L.

2. The map $\mathbf{1} : L \to L$ defined by $\mathbf{1}(x) = 1$ for every $x \in L$, is also a total multiplier of L.
3. For $a \in L$ and $I \in Idn(L)$, the map $f_a : I \to L$ defined by $f_a(x) = a \wedge x$ for every $x \in I$, is a multiplier of L (called *principal*).
If $dom(f_a) = L$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$ and $\overline{f_1} = \mathbf{1}$.

Remark 4.5. *([3]) If $I \in Idn(L)$, $f : I \to L$ is a multiplier of L, then for all $x, y \in I$,*

$$f(f(x)) = f(x), f(x) \leq x,\ f(x \wedge y) = f(x) \wedge f(y) \text{ and } x \wedge f(y) = y \wedge f(x).$$

For $I \in Idn(L)$, we denote

$$M(I, L) = \{f : I \to L : f \text{ is a multiplier on } L\}$$

and

$$M(L) = \bigcup_{I \in Idn(L)} M(I, L).$$

Remark 4.6. *([3])If we have $f \in M(I, L) \cap M(J, L)$, then $I = J$, that is, the relation $f \in M(I, L)$ determines uniquely I.*

Definition 4.7. *([3])If $I_1, I_2 \in Idn(L)$ and $f_i \in M(I_i, L), i = 1, 2$, we define $f_1 \wedge f_2, f_1 \vee f_2 : I_1 \cap I_2 \to L$ by*

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$

$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$

for every $x \in I_1 \cap I_2$.

Lemma 4.8. *([3]) If $I_1, I_2 \in Idn(L)$ and $f_i \in M(I_i, L), i = 1, 2$, then $f_1 \wedge f_2, f_1 \vee f_2 \in M(I_1 \cap I_2, L)$.*

Definition 4.9. *([3])For $I \in Idn(L)$ and $f \in M(I, L)$ we define $f^* : I \to L$ by*

$$f^*(x) = x \wedge Nf(\varphi_{n-1}(x)),$$

for every $x \in I$.

Remark 4.10. *For $x \in L$ we have $\mathbf{0}^*(x) = x \wedge N0 = x \wedge 1 = x$, that is, $\mathbf{0}^* = \mathbf{1}$, and $\mathbf{1}^*(x) = x \wedge N\varphi_{n-1}(x) = 0$ (by Lemma 1.6), that is, $\mathbf{1}^* = \mathbf{0}$.*

Lemma 4.11. *([3]) If $I \in Idn(L)$ and $f \in M(I, L)$, then $f^* \in M(I, L)$.*

Definition 4.12. *([3])For $I \in Idn(L)$ and $i = 1, ..., n-1$ we define $\tilde{\varphi}_i : M(I, L) \to M(I, L)$ by*

$$\tilde{\varphi}_i(f)(x) = x \wedge \varphi_i(f(\varphi_{n-1}(x))),$$

for every $f \in M(I, L)$ and $x \in I$.

Lemma 4.13. *([3])If $I \in Idn(L)$ and $f \in M(I, L)$, then $\tilde{\varphi}_i(f) \in M(I, L)$ for all $i = 1, ..., n-1$.*

Proposition 4.14. *([3])$(M(L), \wedge, \vee, ^*, \mathbf{0}, \mathbf{1}, \tilde{\varphi}_1, ..., \tilde{\varphi}_{n-1})$ is an LM_n-algebra.*

Lemma 4.15. *([3])The map $v_L : L \to M(L)$ defined by $v_L(a) = \overline{f_a}$ for every $a \in L$ is a monomorphism in $\boldsymbol{LM}_n$.*

Definition 4.16. *Let $\mathcal{F}$ be a topology on L. By an $\mathcal{F}$ -multiplier of L we mean a map $f : I \to L/\theta_{\mathcal{F}}$, where $I \in \mathcal{F}$, which verifies the following condition:*
(3.1) *$f(e \wedge x) = e/\theta_{\mathcal{F}} \wedge f(x)$, for every $e \in L$ and $x \in I$.*

Remark 4.17. *If $f : I \to L/\theta_{\mathcal{F}}$ is an $\mathcal{F}$ -multiplier of L then, for every $x, y \in I$, $f(x) \leq x/\theta_{\mathcal{F}}$, $f(x \wedge y) = f(x) \wedge f(y)$ and $x/\theta_{\mathcal{F}} \wedge f(y) = y/\theta_{\mathcal{F}} \wedge f(x)$.*

If $\mathcal{F} = \{L\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of L and an $\mathcal{F}-$ multiplier is a total multiplier of L in the sense of Definition 4.4.

The maps $\mathbf{0}, \mathbf{1} : L \to L/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in L$ are multipliers in the sense of Definition 4.16.

Also, for $a \in L$ and $I \in \mathcal{F}$, $f_a : I \to L/\theta_{\mathcal{F}}$ defined by $f_a(x) = a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$ for every $x \in I$, is an $\mathcal{F}-$ multiplier. If $dom(f_a) = L$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

We shall denote by $M(I, L/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}$ and by

$$M(L/\theta_{\mathcal{F}}) = \bigcup_{I \in \mathcal{F}} M(I, L/\theta_{\mathcal{F}}).$$

If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$, we have a canonical mapping

$$\varphi_{I_1,I_2} : M(I_2, L/\theta_{\mathcal{F}}) \to M(I_1, L/\theta_{\mathcal{F}})$$

defined by

$$\varphi_{I_1,I_2}(f) = f_{|I_1} \text{ for } f \in M(I_2, L/\theta_{\mathcal{F}}).$$

Let us consider the directed system of sets

$$\langle \{M(I, L/\theta_{\mathcal{F}})\}_{I \in \mathcal{F}}, \{\varphi_{I_1,I_2}\}_{I_1,I_2 \in \mathcal{F}, I_1 \subseteq I_2} \rangle$$

and denote by $L_{\mathcal{F}}$ the inductive limit (in the category of sets):

$$L_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, L/\theta_{\mathcal{F}}).$$

For any $\mathcal{F}-$ multiplier $f : I \to L/\theta_{\mathcal{F}}$ we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $L_{\mathcal{F}}$.

Remark 4.18. *We recall that if $f_i : I_i \to L/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}$-multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $L_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.*

Definition 4.19. *If $I_1, I_2 \in Idn(L)$ and $f_i \in M(I_i, L/\theta_{\mathcal{F}}), i = 1, 2$ we define, $f_1 \wedge f_2, f_1 \vee f_2 : I_1 \cap I_2 \to L/\theta_{\mathcal{F}}$ by*

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$
$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$$

for every $x \in I_1 \cap I_2$.

Let $\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2)$ and $\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2)$.

Definition 4.20. *If $I \in Idn(L)$ and $f \in M(I, L/\theta_{\mathcal{F}})$ we define $f^* : I \to L/\theta_{\mathcal{F}}$ by*

$$f^*(x) = x/\theta_{\mathcal{F}} \wedge N(f(\varphi_{n-1}(x)))$$

for any $x \in I$.

Let $\widehat{(I, f)}^* = \widehat{(I, f^*)}$.

Clearly, the definitions of the operations $\wedge, \vee$ and * on $L_{\mathcal{F}}$ are correct.

Lemma 4.21. *If $I_1, I_2 \in Idn(L)$ and $f_i \in M(I_i, L/\theta_{\mathcal{F}}), i = 1, 2,$ then $f_1 \wedge f_2, f_1 \vee f_2 \in M(I_1 \cap I_2, L/\theta_{\mathcal{F}})$.*

Proof. Routine. □

Remark 4.22. *For $x \in L$ we have $\mathbf{0}^*(x) = x/\theta_{\mathcal{F}} \wedge N(0/\theta_{\mathcal{F}}) = x/\theta_{\mathcal{F}} \wedge 1/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}}$, that is, $\mathbf{0}^* = \mathbf{1}$, and similarly $\mathbf{1}^* = \mathbf{0}$.*

Lemma 4.23. *If $I \in Idn(L)$ and $f \in M(I, L/\theta_{\mathcal{F}})$, then $f^* \in M(I, L/\theta_{\mathcal{F}})$.*

Proof. If $x \in I$ and $e \in L$, then

$$\begin{aligned} f^*(e \wedge x) &= (e \wedge x)/\theta_{\mathcal{F}} \wedge Nf(\varphi_{n-1}(e \wedge x)) = e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge Nf(\varphi_{n-1}(e) \wedge \varphi_{n-1}(x)) \\ &= e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge N(\varphi_{n-1}(e)/\theta_{\mathcal{F}} \wedge f(\varphi_{n-1}(x))) \\ &= e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge (N\varphi_{n-1}(e)/\theta_{\mathcal{F}} \vee Nf(\varphi_{n-1}(x))) \\ &= (e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge N\varphi_{n-1}(e)/\theta_{\mathcal{F}}) \vee (e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge Nf(\varphi_{n-1}(x))) \\ &= 0/\theta_{\mathcal{F}} \vee (e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge Nf(\varphi_{n-1}(x))) = e/\theta_{\mathcal{F}} \wedge f^*(x). \end{aligned}$$

□

Definition 4.24. *For $I \in Idn(L)$ and $i = 1, ..., n-1$ we define $\tilde{\varphi}_i : M(I, L/\theta_{\mathcal{F}}) \to M(I, L/\theta_{\mathcal{F}})$ by*

$$\tilde{\varphi}_i(f)(x) = x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(x))) = x/\theta_{\mathcal{F}} \wedge \varphi_i(f(\varphi_{n-1}(x)))/\theta_{\mathcal{F}},$$

for every $f \in M(I, L/\theta_{\mathcal{F}})$ and $x \in I$.

Lemma 4.25. *If $I \in Idn(L)$, $f \in M(I, L/\theta_{\mathcal{F}})$, then $\tilde{\varphi}_i(f) \in M(I, L/\theta_{\mathcal{F}})$ for all $i = 1, ..., n-1$.*

Proof. If $x \in I$ and $e \in L$, then for all $i = 1, ..., n-1$ we have:

$$\begin{aligned}\tilde{\varphi}_i(f)(e \wedge x) &= (e \wedge x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(e \wedge x))) \\ &= e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(e) \wedge \varphi_{n-1}(x))) \\ &= e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\varphi_{n-1}(e)/\theta_{\mathcal{F}} \wedge f(\varphi_{n-1}(x))) \\ &= e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\varphi_{n-1}(e)/\theta_{\mathcal{F}}) \wedge \bar{\varphi}_i(f(\varphi_{n-1}(x))) \\ &= e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge \varphi_i(\varphi_{n-1}(e))/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(x))) \\ &= e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \wedge \varphi_{n-1}(e)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(x))) = e/\theta_{\mathcal{F}} \wedge \tilde{\varphi}_i(f)(x).\end{aligned}$$

□

Let $\varphi_i^{\mathcal{F}} : L_{\mathcal{F}} \to L_{\mathcal{F}}$ defined by $\varphi_i^{\mathcal{F}}(\widehat{I, f}) = (\widehat{I, \tilde{\varphi}_i(f)})$.

Proposition 4.26. $(L_{\mathcal{F}}, \wedge, \vee, ^*, \mathbf{0}, \mathbf{1}, \varphi_1^{\mathcal{F}}, ..., \varphi_{n-1}^{\mathcal{F}})$ *is an LM_n-algebra.*

Proof. We verify the axioms of LM_n- algebras.

In the following we work with $f \in M(I, L/\theta_{\mathcal{F}})$ and $f_i \in M(I_i, L/\theta_{\mathcal{F}})$ where $I, I_i \in Idn(L)$, $i = 1, 2$ and we denote $I' = I_1 \cap I_2 \in Idn(L)$.

(1.1). It is easy to verify that $(L_{\mathcal{F}}, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded distributive lattice.

To prove that it is a *De Morgan* algebra, we have for $x \in I'$:

$$\begin{aligned}(f_1 \vee f_2)^*(x) &= x/\theta_{\mathcal{F}} \wedge N((f_1 \vee f_2)(\varphi_{n-1}(x))) \\ &= x/\theta_{\mathcal{F}} \wedge N(f_1(\varphi_{n-1}(x)) \vee f_2(\varphi_{n-1}(x))) \\ &= x/\theta_{\mathcal{F}} \wedge Nf_1(\varphi_{n-1}(x)) \wedge Nf_2(\varphi_{n-1}(x)) \\ &= (x/\theta_{\mathcal{F}} \wedge Nf_1(\varphi_{n-1}(x))) \wedge (x/\theta_{\mathcal{F}} \wedge Nf_2(\varphi_{n-1}(x))) \\ &= f_1^*(x) \wedge f_2^*(x) = (f_1^* \wedge f_2^*)(x),\end{aligned}$$

that is, $(f_1 \vee f_2)^* = f_1^* \wedge f_2^*$.

Also, for every $x \in I$:

$$\begin{aligned}(f^*)^*(x) &= x/\theta_{\mathcal{F}} \wedge Nf^*(\varphi_{n-1}(x)) \\ &= x/\theta_{\mathcal{F}} \wedge N(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge Nf(\varphi_{n-1}(\varphi_{n-1}(x)))) \\ &= x/\theta_{\mathcal{F}} \wedge N(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge Nf(\varphi_{n-1}(x))) \\ &= x/\theta_{\mathcal{F}} \wedge (N\varphi_{n-1}(x)/\theta_{\mathcal{F}} \vee f(\varphi_{n-1}(x))) \\ &= (x/\theta_{\mathcal{F}} \wedge N\varphi_{n-1}(x)/\theta_{\mathcal{F}}) \vee (x/\theta_{\mathcal{F}} \wedge f(\varphi_{n-1}(x))) \\ &= 0/\theta_{\mathcal{F}} \vee (x/\theta_{\mathcal{F}} \wedge f(\varphi_{n-1}(x))) = \varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge f(x) \\ &= f(x),\end{aligned}$$

that is, $(f^*)^* = f$.

Then,

$$(f_1 \wedge f_2)^* = (f_1^{**} \wedge f_2^{**})^* = ((f_1^* \vee f_2^*)^*)^* = (f_1^* \vee f_2^*)^{**} = f_1^* \vee f_2^*.$$

(1.2). $\varphi_i^{\mathcal{F}} : L_{\mathcal{F}} \to L_{\mathcal{F}}$, for all $i = 1, ..., n-1$, are bounded lattice morphisms that satisfy $(1.2.1) - (1.2.6)$.

For $x \in I'$ we have:

$$\begin{aligned}\tilde{\varphi}_i(f_1 \vee f_2)(x) &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f_1(\varphi_{n-1}(x)) \vee f_2(\varphi_{n-1}(x)))\\ &= x/\theta_{\mathcal{F}} \wedge (\bar{\varphi}_i(f_1(\varphi_{n-1}(x))) \vee \bar{\varphi}_i(f_2(\varphi_{n-1}(x))))\\ &= (x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f_1(\varphi_{n-1}(x)))) \vee (x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f_2(\varphi_{n-1}(x))))\\ &= \tilde{\varphi}_i(f_1(x)) \vee \tilde{\varphi}_i(f_2(x))\\ &= (\tilde{\varphi}_i(f_1) \vee \tilde{\varphi}_i(f_2))(x),\end{aligned}$$

hence $\tilde{\varphi}_i(f_1 \vee f_2) = \tilde{\varphi}_i(f_1) \vee \tilde{\varphi}_i(f_2)$ for all $i = 1, ..., n-1$, that is, $\varphi_i^{\mathcal{F}}(\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)}) = \varphi_i^{\mathcal{F}}(\widehat{(I_1, f_1)}) \vee \varphi_i^{\mathcal{F}}(\widehat{(I_2, f_2)})$ and similarly $\varphi_i^{\mathcal{F}}(\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)}) = \varphi_i^{\mathcal{F}}(\widehat{(I_1, f_1)}) \wedge \varphi_i^{\mathcal{F}}(\widehat{(I_2, f_2)})$,for all $i = 1, ..., n-1$.

Also, for all $x \in L$ and $i = 1, ..., n-1, \tilde{\varphi}_i(\mathbf{0})(x) = x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\mathbf{0}(\varphi_{n-1}(x))) = x/\theta_{\mathcal{F}} \wedge 0/\theta_{\mathcal{F}} = 0/\theta_{\mathcal{F}} = \mathbf{0}(x)$, that is, $\tilde{\varphi}_i(\mathbf{0}) = \mathbf{0}$ and similarly $\tilde{\varphi}_i(\mathbf{1}) = \mathbf{1}$. So, we have $\varphi_i^{\mathcal{F}}(\widehat{L, \mathbf{0}}) = (\widehat{L, \mathbf{0}})$ and $\varphi_i^{\mathcal{F}}(\widehat{L, \mathbf{1}}) = (\widehat{L, \mathbf{1}})$.

(1.2.1). For $x \in I$, then:

$$\begin{aligned}(\tilde{\varphi}_i(f) \vee (\tilde{\varphi}_i(f))^*)(x) &= \tilde{\varphi}_i(f)(x) \vee (\tilde{\varphi}_i(f))^*(x)\\ &= (x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(x)))) \vee (x/\theta_{\mathcal{F}} \wedge N\tilde{\varphi}_i(f)(\varphi_{n-1}(x)))\\ &= x/\theta_{\mathcal{F}} \wedge (\bar{\varphi}_i(f(\varphi_{n-1}(x))) \vee N(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(\varphi_{n-1}(x))))))\\ &= x/\theta_{\mathcal{F}} \wedge (\bar{\varphi}_i(f(\varphi_{n-1}(x))) \vee N\varphi_{n-1}(x)/\theta_{\mathcal{F}} \vee N\bar{\varphi}_i(f(\varphi_{n-1}(x))))\\ &= x/\theta_{\mathcal{F}} \wedge 1/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}},\end{aligned}$$

hence $\tilde{\varphi}_i(f) \wedge (\tilde{\varphi}_i(f))^* = \mathbf{1}$ for all $i = 1, ..., n-1$, that is, $\varphi_i^{\mathcal{F}}(\widehat{(I, f)}) \wedge (\varphi_i^{\mathcal{F}}(\widehat{(I, f)}))^* = (\widehat{L, 1})$.

To prove (1.2.2) we use the *De Morgan* relations and (1.2.1) :

$$\tilde{\varphi}_i(f) \wedge (\tilde{\varphi}_i(f))^* = (\tilde{\varphi}_i(f))^{**} \wedge (\tilde{\varphi}_i(f))^* = [(\tilde{\varphi}_i(f))^* \vee \tilde{\varphi}_i(f)]^* = \mathbf{1}^* = \mathbf{0},$$

that is, $\varphi_i^{\mathcal{F}}(\widehat{(I, f)}) \vee (\varphi_i^{\mathcal{F}}(\widehat{(I, f)}))^* = (\widehat{L, \mathbf{0}})$.

(1.2.3). For $x \in I$ and $i, j \in \{1, ..., n-1\}$ then:

$$\begin{aligned}\tilde{\varphi}_i\tilde{\varphi}_j(f)(x) &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\tilde{\varphi}_j(f)(\varphi_{n-1}(x)))\\ &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_j(f(\varphi_{n-1}(\varphi_{n-1}(x)))))\\ &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\varphi_{n-1}(x)/\theta_{\mathcal{F}}) \wedge \bar{\varphi}_i(\bar{\varphi}_j(f(\varphi_{n-1}(x))))\\ &= x/\theta_{\mathcal{F}} \wedge \varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_j(f(\varphi_{n-1}(x)))\\ &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_j(f(\varphi_{n-1}(x))) = \tilde{\varphi}_j(f)(x),\end{aligned}$$

that is, $\tilde{\varphi}_i\tilde{\varphi}_j(f) = \tilde{\varphi}_j(f)$, hence $\varphi_i^{\mathcal{F}}(\varphi_j^{\mathcal{F}}(\widehat{(I, f)})) = \varphi_j^{\mathcal{F}}(\widehat{(I, f)})$.

(1.2.4). For $i = 1, ..., n-1$ we have to prove that:

$$\tilde{\varphi}_i(f^*) = (\tilde{\varphi}_{n-i}(f))^*.$$

Indeed, for $x \in I$:

$$\begin{aligned}\tilde{\varphi}_i(f^*)(x) &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f^*(\varphi_{n-1}(x)))\\ &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge Nf(\varphi_{n-1}(\varphi_{n-1}(x))))\\ &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\varphi_{n-1}(x)/\theta_{\mathcal{F}}) \wedge \bar{\varphi}_i(Nf(\varphi_{n-1}(x)))\\ &= x/\theta_{\mathcal{F}} \wedge \varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(Nf(\varphi_{n-1}(x)))\\ &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(Nf(\varphi_{n-1}(x))),\end{aligned}$$

and

$$\begin{aligned}(\tilde{\varphi}_{n-i}(f))^*(x) &= x/\theta_{\mathcal{F}} \wedge N\tilde{\varphi}_{n-i}(f)(\varphi_{n-1}(x))\\ &= x/\theta_{\mathcal{F}} \wedge N(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_{n-i}(f(\varphi_{n-1}(\varphi_{n-1}(x)))))\\ &= x/\theta_{\mathcal{F}} \wedge N(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_{n-i}(f(\varphi_{n-1}(x))))\\ &= x/\theta_{\mathcal{F}} \wedge (N\varphi_{n-1}(x)/\theta_{\mathcal{F}} \vee N\bar{\varphi}_{n-i}(f(\varphi_{n-1}(x))))\\ &= (x/\theta_{\mathcal{F}} \wedge N\varphi_{n-1}(x)/\theta_{\mathcal{F}}) \vee (x/\theta_{\mathcal{F}} \wedge N\bar{\varphi}_{n-i}(f(\varphi_{n-1}(x))))\\ &= 0/\theta_{\mathcal{F}} \vee (x/\theta_{\mathcal{F}} \wedge N\bar{\varphi}_{n-i}(f(\varphi_{n-1}(x))))\\ &= x/\theta_{\mathcal{F}} \wedge N\bar{\varphi}_{n-i}(f(\varphi_{n-1}(x)))\\ &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(Nf(\varphi_{n-1}(x))),\end{aligned}$$

so, $\tilde{\varphi}_i(f^*) = (\tilde{\varphi}_{n-i}(f))^*$, that is, $\varphi_i^{\mathcal{F}}(\widehat{(I,f)}^{\,*}) = \varphi_{n-i}^{\mathcal{F}}(\widehat{(I,f)})$.

(1.2.5). For $x \in I$ we obtain successively:

$$\begin{aligned}\bar{\varphi}_1(f(\varphi_{n-1}(x))) &\le ... \le \bar{\varphi}_{n-1}(f(\varphi_{n-1}(x)))\\ x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_1(f(\varphi_{n-1}(x))) &\le ... \le x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_{n-1}(f(\varphi_{n-1}(x)))\\ \tilde{\varphi}_1(f)(x) &\le ... \le \tilde{\varphi}_{n-1}(f)(x)\\ \tilde{\varphi}_1(f) &\le ... \le \tilde{\varphi}_{n-1}(f),\end{aligned}$$

that is, $\varphi_1^{\mathcal{F}}(\widehat{(I,f)}) \le ... \le \varphi_{n-1}^{\mathcal{F}}(\widehat{(I,f)})$.

(1.2.6). If $\varphi_i^{\mathcal{F}}(\widehat{(I_1,f_1)}) = \varphi_i^{\mathcal{F}}(\widehat{(I_2,f_2)})$, that is, $(\widehat{I_1,\tilde{\varphi}_i(f_1)}) = (\widehat{I_2,\tilde{\varphi}_i(f_2)})$, for all $i = 1,...,n-1$, then we get in turn, for all $i = 1,...,n-1$, according to Remark 4.18 there exists $J_i \subseteq I_1 \cap I_2$ such that $\tilde{\varphi}_i(f_{1|J_i}) = \tilde{\varphi}_i(f_{2|J_i})$. For $x \in J_1 \cap ... \cap J_{n-1}$ we have:

$$\begin{aligned}\tilde{\varphi}_i(f_1)(x) &= \tilde{\varphi}_i(f_2)(x),\\ x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f_1(\varphi_{n-1}(x))) &= x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f_2(\varphi_{n-1}(x))),\\ \varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f_1(\varphi_{n-1}(\varphi_{n-1}(x)))) &= \varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(f_2(\varphi_{n-1}(\varphi_{n-1}(x)))),\\ \bar{\varphi}_i(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge f_1(\varphi_{n-1}(x))) &= \bar{\varphi}_i(\varphi_{n-1}(x)/\theta_{\mathcal{F}} \wedge f_2(\varphi_{n-1}(x))),\\ \bar{\varphi}_i(f_1(\varphi_{n-1}(x))) &= \bar{\varphi}_i(f_2(\varphi_{n-1}(x))),\\ f_1(\varphi_{n-1}(x)) &= f_2(\varphi_{n-1}(x)).\end{aligned}$$

So,

$$\begin{aligned} f_1(x) &= f_1(x \wedge \varphi_{n-1}(x)) = x/\theta_{\mathcal{F}} \wedge f_1(\varphi_{n-1}(x)) = x/\theta_{\mathcal{F}} \wedge f_2(\varphi_{n-1}(x)) \\ &= f_2(x \wedge \varphi_{n-1}(x)) = f_2(x), \text{hence } f_1 = f_2 \text{ on } J_1 \cap ... \cap J_{n-1} \subseteq I_1 \cap I_2, \end{aligned}$$

that is, $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$. □

Definition 4.27. *The LM_n-algebra $L_{\mathcal{F}}$ will be called the localization $LM_n - algebra$ of L with respect to the topology $\mathcal{F}$.*

Lemma 4.28. *Let the map $v_{\mathcal{F}} : L \to L_{\mathcal{F}}$ defined by $v_{\mathcal{F}}(a) = \widehat{(L, \overline{f_a})}$ for every $a \in L$. Then:*
(i) $v_{\mathcal{F}}$ is a morphism of LM_n-algebras,
(ii) For every $a \in C(L)$, $\widehat{(L, \overline{f_a})} \in C(L_{\mathcal{F}})$,
(iii) $v_{\mathcal{F}}(L) \in R(L_{\mathcal{F}})$.

Proof. (i). Clearly, $v_{\mathcal{F}}(0) = \widehat{(L, \overline{f_0})} = \widehat{(L, \mathbf{0})}$, $v_{\mathcal{F}}(1) = \widehat{(L, \overline{f_1})} = \widehat{(L, \mathbf{1})}$. For $a, b \in L$ we have:

$$v_{\mathcal{F}}(a) \vee v_{\mathcal{F}}(b) = \widehat{(L, \overline{f_a})} \vee \widehat{(L, \overline{f_b})} = \widehat{(L, \overline{f_a} \vee \overline{f_b})} = \widehat{(L, \overline{f}_{a \vee b})} = v_{\mathcal{F}}(a \vee b),$$

and

$$v_{\mathcal{F}}(a) \wedge v_{\mathcal{F}}(b) = \widehat{(L, \overline{f_a})} \wedge \widehat{(L, \overline{f_b})} = \widehat{(L, \overline{f_a} \wedge \overline{f_b})} = \widehat{(L, \overline{f}_{a \wedge b})} = v_{\mathcal{F}}(a \wedge b).$$

Also, for every $x \in L :$

$$v_{\mathcal{F}}(\varphi_i(a)) = \widehat{(L, \overline{f}_{\varphi_i(a)})}$$

and

$$\varphi_i^{\mathcal{F}}(v_{\mathcal{F}}(a)) = \widehat{(L, \widetilde{\varphi}_i(\overline{f_a}))}.$$

But $\widetilde{\varphi}_i(\overline{f_a})(x) = x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(\overline{f_a}(\varphi_{n-1}(x))) = x/\theta_{\mathcal{F}} \wedge \bar{\varphi}_i(a/\theta_{\mathcal{F}} \wedge \varphi_{n-1}(x)/\theta_{\mathcal{F}}) = x/\theta_{\mathcal{F}} \wedge \varphi_i(a)/\theta_{\mathcal{F}} \wedge \varphi_{n-1}(x)/\theta_{\mathcal{F}} = \varphi_i(a)/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} = \overline{f}_{\varphi_i(a)}(x)$, hence $v_{\mathcal{F}}(\varphi_i(a)) = \varphi_i^{\mathcal{F}}(v_L(a))$ for all $i = 1, ..., n-1$. Therefore $v_{\mathcal{F}}$ is a morphism in LM_n.

(ii). Is a direct consequence of (i), since any LM_n-algebra morphism preserves the boolean elements.

(iii). To prove that $v_{\mathcal{F}}(L)$ is a regular subset of $L_{\mathcal{F}}$, let $\widehat{(I_i, f_i)} \in L_{\mathcal{F}}$, $I_i \in \mathcal{F}$, $i = 1, 2$, such that $\widehat{(L, \overline{f_a})} \wedge \widehat{(I_1, f_1)} = \widehat{(L, \overline{f_a})} \wedge \widehat{(I_2, f_2)}$ for every $a \in L$.

Then, for any $a \in L$ there exists $J_a \subseteq I_1 \cap I_2$ such that for any $x \in J_a$ we have $(f_1 \wedge \overline{f_a})(x) = (f_2 \wedge \overline{f_a})(x)$, hence $f_1(x) \wedge a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} = f_2(x) \wedge a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$, that is, $f_1(x) \wedge a/\theta_{\mathcal{F}} = f_2(x) \wedge a/\theta_{\mathcal{F}}$.

In particular, for $a = 1$ we obtain that $f_1(x) = f_2(x)$ for every $x \in J_a \subseteq I_1 \cap I_2$, hence $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$, that is, $v_{\mathcal{F}}(L) \in R(L_{\mathcal{F}})$. □

5 Applications

In the following we describe the localization LM_n-algebra $L_{\mathcal{F}}$ in some special instances.
1. If $I \in Idn(L)$ and $\mathcal{F}$ is the topology

$$\mathcal{F}(I) = \{I' \in Idn(L) : I \subseteq I'\}$$

(see Example 1 in Section 3), then $L_{\mathcal{F}}$ is isomorphic with $M(I, L/\theta_{\mathcal{F}})$ and

$$v_{\mathcal{F}} : L \to L_{\mathcal{F}}$$

is defined by $v_{\mathcal{F}}(a) = \overline{f_{a|I}}$ for every $a \in L$.
2. If $\mathcal{F} = Idn(L) \cap R(L)$ is the topology of regular sets of $Idn(L)$ (see Example 2 in Section 3), then $\theta_{\mathcal{F}}$ is the identity congruence of L and

$$L_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, L),$$

where $M(I, L)$ is the set of multipliers of L having the domain I (see [3]).

We recall the construction of maximal LM_n-algebra of fractions of L from [3](where is denoted by $Q_n(L)$).

We denote

$$M_r(L) = \{f \in M(L) : dom(f) \in Idn(L) \cap R(L)\}.$$

Lemma 5.1. *([3]) If $I_1, I_2 \in Idn(L) \cap R(L)$, then $I_1 \cap I_2 \in Idn(L) \cap R(L)$.*

Remark 5.2. *([3])By Lemma 5.1, we deduce that $M_r(L)$ is an LM_n-subalgebra of $M(L)$.*

Definition 5.3. *Define the relation ρ_L on the LM_n- algebra $M_r(L)$ by the prescription:*

$$(f_1, f_2) \in \rho_L \text{ iff } f_1 \text{ and } f_2 \text{ agree on the intersection of their domains.}$$

Lemma 5.4. *([3]) ρ_L is a congruence on $M_r(L)$.*

Definition 5.5. *For $f \in M_r(L)$ with $I = dom(f) \in Idn(L) \cap R(L)$, we denote by $[f, I]$ the congruence class of f modulo ρ_L and $L_{\mathcal{M}} = M_r(L)/\rho_L$.*

Remark 5.6. *For every $I \in Idn(L) \cap R(L)$ and $a \in L$ we have $[\overline{f_a}, L] = [f_a, I]$ (because for every $x \in I \cap L = I$ we have $\overline{f_a}(x) = f_a(x) = a \wedge x$).*

Remark 5.7. *([3])As was proved by Cignoli [5](see also [2], Theorem 2.4), the class of LM_n-algebras is equational, therefore $L_{\mathcal{M}}$ is an LM_n-algebra, where, for $[f, I], [g, J] \in L_{\mathcal{M}}$, $[f, I] \wedge [g, J] = [f \wedge g, I \cap J]$, $[f, I] \vee [g, J] = [f \vee g, I \cap J]$, and for $i = 1, ..., n-1$, $\widetilde{\widetilde{\varphi}}_i : L_{\mathcal{M}} \to L_{\mathcal{M}}$, $\widetilde{\widetilde{\varphi}}_i([f, I]) = [\widetilde{\varphi}_i(f), I]$.*

Lemma 5.8. *([3])Let the map $\overline{v_L} : L \to L_{\mathcal{M}}$ be defined by $\overline{v_L}(a) = [\overline{f_a}, L]$ for every $a \in L$. Then:*

(i) $\overline{v_L}$ is a monomorphism in $\boldsymbol{LM}_n$,
(ii) For every $a \in C(L)$, $[\overline{f_a}, L] \in C(L_{\mathcal{M}})$,
(iii) $\overline{v_L}(L) \in R(L_{\mathcal{M}})$.

Remark 5.9. *([3]) Since by Lemma 4.15 and Lemma 5.8, for every $a, b \in L, [\overline{f_a}, L] = [\overline{f_b}, L]$ iff $\overline{f_a} = \overline{f_b}$ iff $a = b$, the elements of L can be identified with the elements of the sets $\{[\overline{f_a}, L] : a \in L\}$ and $\{\overline{f_a} : a \in L\}$. So, $v_L(L) \approx \overline{v_L}(L) \approx L$ (as LM_n-algebras).*

Definition 5.10. *([3]) An LM_n-algebra L' is called an LM_n-algebra of fractions of L if:*
(3.1) *L is an LM_n-subalgebra of L',*
(3.2) *For every $a', b', c' \in L', a' \neq b'$, there is $e \in L$ such that $e \wedge a' \neq e \wedge b'$ and $\varphi_{n-1}(e) \wedge c' \in L$ (where by φ_{n-1} we denote the chrysippian endomorphism of L which is the restriction to L of the chrisippian endomorphism φ'_{n-1} of L').*

Remark 5.11. *As a notational convenience, we write $L \preccurlyeq L'$ to indicate that L' is an LM_n-algebra of fractions for L.*

Remark 5.12. *([3]) If $L \preccurlyeq L', e \in L$ and $a', b' \in L'$ are such that $e \wedge a' \neq e \wedge b'$, then $\varphi_{n-1}(e) \wedge a' \neq \varphi_{n-1}(e) \wedge b'$.*

Remark 5.13. *([3]) If $L \preccurlyeq L', e \in L, c' \in L'$ and $\varphi_{n-1}(e) \wedge c' \in L$, then $e \wedge c' = [e \wedge \varphi_{n-1}(e)] \wedge c' = e \wedge [\varphi_{n-1}(e) \wedge c'] \in L$.*

Theorem 5.14. *([3]) For every LM_n-algebra L, the LM_n-algebra $L_{\mathcal{M}}$ in Definition 5.5 has the following properties:*

(i) $\overline{v_L}(L) \preccurlyeq L_{\mathcal{M}}$,
(ii) For every LM_n-algebra L' such that $L \preccurlyeq L'$, there exists a monomorphism of LM_n-algebras $u : L' \to L_{\mathcal{M}}$ which induces the canonical monomorphism $\overline{v_L}$ of L into $L_{\mathcal{M}}$.

Theorem 5.14 provides the motivation for the following:

Definition 5.15. *([3]) For any LM_n-algebra L, $L_{\mathcal{M}}$ is called a maximal LM_n-algebra of fractions of L. To range whith the tradition ([4], [10], [13], [14]) we denote $L_{\mathcal{M}}$ by $Q_n(L)$.*

So, in the case of the topology $\mathcal{F} = Idn(L) \cap R(L)$ we obtain:

Proposition 5.16. *For $\mathcal{F} = Idn(L) \cap R(L)$, $L_{\mathcal{F}}$ is exactly the maximal LM_n-algebra $Q_n(L)$ of fractions of L introduced in [3](where it is denoted by $L_{\mathcal{M}}$).*

3. If $S \subseteq L$ is an $\wedge-$closed system of L, we consider the following congruence on L :

$(x, y) \in \theta_S$ iff there exists $s \in S$ such that $x \wedge \varphi_{n-1}(s) = y \wedge \varphi_{n-1}(s)$ (see Section 2).

The quotient LM_n-algebra $L[S] = L/\theta_S$ is called in Section 2 the *LM_n-algebra of fractions of L relative to the $\wedge-$closed system*

Proposition 5.17. *If $\mathcal{F}_S$ is the topology associated with an $\wedge-$closed system $S \subseteq L$ (see Example 3 in Section 3), then the LM_n-algebra $L_{\mathcal{F}_S}$ is isomorphic with $L[S]$.*

Proof. Let $x, y \in L$. If $(x, y) \in \theta_{\mathcal{F}_S}$ then there exists $I \in \mathcal{F}_S$ (hence $I \cap S \neq \oslash$) such that $x \wedge e = y \wedge e$ for any $e \in I$. Since $I \cap S \neq \oslash$ there exists $e_0 \in I \cap S$ such that $x \wedge e_0 = y \wedge e_0$. But $e_0 \in I$ implies that $\varphi_{n-1}(e_0) \in I$, so in particular $x \wedge \varphi_{n-1}(e_0) = y \wedge \varphi_{n-1}(e_0)$, that is, $(x, y) \in \theta_S$. So, $\theta_{\mathcal{F}_S} \subseteq \theta_S$.

If $(x, y) \in \theta_S$, then there exists $e_0 \in S$ such that $x \wedge \varphi_{n-1}(e_0) = y \wedge \varphi_{n-1}(e_0)$. If we set $I_0 =< e_0 >= \{x \in L : x \leq \varphi_{n-1}(e_0)\}$, then $I_0 \in Idn(L)$. Since $e_0 \leq \varphi_{n-1}(e_0)$, we have that $e_0 \in I_0$, so $e_0 \in I_0 \cap S$, hence $I_0 \cap S \neq \oslash$, that is, $I_0 \in \mathcal{F}_S$. For every $e \in I_0$, $e \leq \varphi_{n-1}(e_0)$, then $e = e \wedge \varphi_{n-1}(e_0)$, so $x \wedge e = x \wedge (e \wedge \varphi_{n-1}(e_0)) = (x \wedge \varphi_{n-1}(e_0)) \wedge e = (y \wedge \varphi_{n-1}(e_0)) \wedge e = y \wedge (e \wedge \varphi_{n-1}(e_0)) = y \wedge e$, hence $(x, y) \in \theta_{\mathcal{F}_S}$, that is, $\theta_S \subseteq \theta_{\mathcal{F}_S}$. Therefore $\theta_{\mathcal{F}_S} = \theta_S$.

Then $L/\theta_{\mathcal{F}_S} = L/\theta_S = L[S]$, hence an $\mathcal{F}_S-$multiplier can be considered in this case (see Definition 4.16) as a mapping $f : I \to L[S]$ $(I \in \mathcal{F}_S)$ having the property $f(e \wedge x) = e/S \wedge f(x)$ for every $x \in I$ and $e \in L$ (x/S denotes the congruence class of x relative to θ_S).

If $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in L_{\mathcal{F}_S} = \varinjlim_{I \in \mathcal{F}} M(I, L[S])$ and $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ then there exists $I \in \mathcal{F}_S$ such that $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$. Since $I, I_1, I_2 \in \mathcal{F}_S$, there exists $s \in I \cap S, s_1 \in I_1 \cap S$ and $s_2 \in I_2 \cap S$. We shall prove that $f_1(\varphi_{n-1}(s_1)) = f_2(\varphi_{n-1}(s_2))$. If we denote $s' = \varphi_{n-1}(s \wedge s_1 \wedge s_2) = \varphi_{n-1}(s) \wedge \varphi_{n-1}(s_1) \wedge \varphi_{n-1}(s_2)$, then $s' \in I \cap S$ and $s' \leq \varphi_{n-1}(s_1), \varphi_{n-1}(s_2)$. Since $\varphi_{n-1}(s_1) \wedge s' = \varphi_{n-1}(s_2) \wedge s' \in I$ then $f_1(\varphi_{n-1}(s_1) \wedge s') = f_2(\varphi_{n-1}(s_2) \wedge s')$, hence $f_1(\varphi_{n-1}(s_1)) \wedge s'/S = f_2(\varphi_{n-1}(s_2)) \wedge s'/S$, so $f_1(\varphi_{n-1}(s_1)) \wedge \mathbf{1} = f_2(\varphi_{n-1}(s_2)) \wedge \mathbf{1}$ (since $s \in S$ implies $\varphi_{n-1}(s)/S = \mathbf{1}$ by Remark 2.3), that is, $f_1(\varphi_{n-1}(s_1)) = f_2(\varphi_{n-1}(s_2))$. In a similar way, we can show that $f_1(\varphi_{n-1}(s_1)) = f_2(\varphi_{n-1}(s_2))$ for any $s_1, s_2 \in I \cap S$.

In accordance with these considerations we can define the mapping:

$$\alpha : L_{\mathcal{F}_S} = \varinjlim_{I \in \mathcal{F}} M(I, L[S]) \to L[S]$$

by putting

$$\alpha(\widehat{(I, f)}) = f(\varphi_{n-1}(s_0)) \in L[S], \text{ where } s_0 \in I \cap S.$$

We have $\alpha(\mathbf{0}) = \alpha(\widehat{(L, \mathbf{0})}) = \mathbf{0}(\varphi_{n-1}(s)) = 0/S = \mathbf{0}$ and $\alpha(\mathbf{1}) = \alpha(\widehat{(L, \mathbf{1})}) = \mathbf{1}(\varphi_{n-1}(s)) = \varphi_{n-1}(s)/S = \mathbf{1}$ by Remark 2.3 for every $s \in S$.

Also, for every $\widehat{(I_i, f_i)} \in L_{\mathcal{F}_S}, i = 1, 2$ we have:

$$\begin{aligned}\alpha(\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)}) &= \alpha(\widehat{(I_1 \cap I_2, f_1 \wedge f_2)}) = (f_1 \wedge f_2)(\varphi_{n-1}(s)) \\ &= f_1(\varphi_{n-1}(s)) \wedge f_2(\varphi_{n-1}(s)) = \alpha(\widehat{(I_1, f_1)}) \wedge \alpha(\widehat{(I_2, f_2)}),\end{aligned}$$

and

$$\begin{aligned}\alpha(\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)}) &= \alpha(\widehat{(I_1 \cap I_2, f_1 \vee f_2)}) = (f_1 \vee f_2)(\varphi_{n-1}(s)) \\ &= f_1(\varphi_{n-1}(s)) \vee f_2(\varphi_{n-1}(s)) = \alpha(\widehat{(I_1, f_1)}) \vee \alpha(\widehat{(I_2, f_2)})\end{aligned}$$

with $s \in I_1 \cap I_2 \cap S$.

If $\widehat{(I, f)} \in L_{\mathcal{F}_S}$ and $s \in I \cap S$, for every $i = 1, ..., n-1$ we have

$$\begin{aligned}\alpha(\varphi_i^{\mathcal{F}}(\widehat{(I, f)})) &= \alpha(\widehat{(I, \widetilde{\varphi}_i(f))}) = \widetilde{\varphi}_i(f)(\varphi_{n-1}(s)) = \varphi_{n-1}(s)/S \wedge \overline{\varphi}_i(f(\varphi_{n-1}(\varphi_{n-1}(s)))) \\ &= \mathbf{1} \wedge \bar{\varphi}_i(f(\varphi_{n-1}(s))) = \bar{\varphi}_i(f(\varphi_{n-1}(s))) = \bar{\varphi}_i(\alpha(\widehat{(I, f)})).\end{aligned}$$

Therefore, this mapping is a morphism of LM_n-algebras.

We shall prove that α is injective and surjective. To prove the injectivity of α, let $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in L_{\mathcal{F}_S}$ such that $\alpha(\widehat{(I_1, f_1)}) = \alpha(\widehat{(I_2, f_2)})$. Then, for any $s_1 \in I_1 \cap S$, $s_2 \in I_2 \cap S$ we have $f_1(\varphi_{n-1}(s_1)) = f_2(\varphi_{n-1}(s_2))$. If $f_1(\varphi_{n-1}(s_1)) = x/S$ and $f_2(\varphi_{n-1}(s_2)) = y/S$ with $x, y \in L$, since $x/S = y/S$, there exists $s \in S$ such that $x \wedge \varphi_{n-1}(s) = y \wedge \varphi_{n-1}(s)$.

If we consider $s' = s \wedge s_1 \wedge s_2 \in I_1 \cap I_2 \cap S$, we have that $\varphi_{n-1}(s') \leq \varphi_{n-1}(s_1), \varphi_{n-1}(s_2)$. It follows that

$$\begin{aligned}f_1(\varphi_{n-1}(s')) &= f_1(\varphi_{n-1}(s') \wedge \varphi_{n-1}(s_1)) = \varphi_{n-1}(s')/S \wedge f_1(\varphi_{n-1}(s_1)) \\ &= \varphi_{n-1}(s')/S \wedge f_2((\varphi_{n-1}(s_2)) = f_2(\varphi_{n-1}(s') \wedge \varphi_{n-1}(s_2)) = f_2(\varphi_{n-1}(s')).\end{aligned}$$

If we denote $I =< s' >= \{x \in L : x \leq \varphi_{n-1}(s')\}$, then $s' \in I$, so $I \in \mathcal{F}_S$, $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$ (since if $x \in I$, then $x \leq \varphi_{n-1}(s')$, hence $x = x \wedge \varphi_{n-1}(s')$, so $f_1(x) = f_1(x \wedge \varphi_{n-1}(s')) = x/S \wedge f_1(\varphi_{n-1}(s')) = x/S \wedge f_2(\varphi_{n-1}(s')) = f_2(x \wedge \varphi_{n-1}(s')) = f_2(x)$), hence $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$, that is, α is injective.

To prove the surjectivity of α, let $a/S \in L[S]$ and $\bar{f}_a : L \to L[S]$ defined by

$$\bar{f}_a(x) = a/S \wedge x/S = (a \wedge x)/S \quad \text{for every } x \in L.$$

It is easy to see that $\bar{f}_a$ is an $\mathcal{F}_S-$multiplier and $\alpha(\widehat{(L, f_a)}) = \bar{f}_a(\varphi_{n-1}(s)) = (a \wedge \varphi_{n-1}(s))/S = a/S \wedge \varphi_{n-1}(s)/S = a/S \wedge \mathbf{1} = a/S$, where $s \in S$. So α is surjective.

Therefore, α is an isomorphism of LM_n-algebras. □

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