

Super-energy and Killing-Yano tensors

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Abstract: In this paper we investigate a class of basic super-energy tensors, namely those constructed from Killing-Yano tensors, and give a generalization of super-energy tensors for cases when we start not with a single tensor, but with a pair of tensors.

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1 Introduction

The Bell-Robinson tensor [1, 2] is a valuable tool in many mathematical formalisms involving the gravitational field, such as hyperbolic formulation [3, 4] of the Einstein field equations [5, 6], the causal propagation of gravity [5, 7, 8], the existence of global solutions of the Cauchy problem [9, 10, 11] and the study of the global stability of spacetimes [11, 12]. It was introduced for the first time to solve the lack of a well-posed definition of a local energy-momentum for the gravitational field, since the Principle of Equivalence tells us that it is always possible to choose a reference system along any timelike curve such that the gravitational field in this particular coordinate system vanishes and so does the gravitational energy density. The analogy between many of its properties and those of energy-momentum tensor of electromagnetic fields has led many authors to look for similar super-energy tensors of fields other than the gravitational one [13, 14, 15, 16].

Killing-Yano tensors, introduced by Yano [17], are geometrical objects closely related with *hidden* symmetries [18] in pseudoclassical models for a spinning particle. They have an important role in generating solutions for generalized Stäckel-Killing equations which in turn lead to new constants of motion [19, 20]. Concrete applications for Euclidean

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Taub-NUT space, pp-wave and Siklos metric can be found in [23, 24, 25, 26, 27]. For the geodesic motions in the Taub-NUT space, the conserved vector analogous to the Runge-Lenz vector of the Kepler type problem is quadratic in 4-velocities, its components are Stäckel-Killing tensors and they can be expressed as symmetrized products of Killing-Yano tensors [21, 22, 23, 28]. A particular class of Killing-Yano tensors, namely those covariantly constant which realize certain square roots of the metric tensor, are involved in generating a Dirac-type operator and the generator of a one-parameter Lie group connecting this operator with the standard Dirac one [29, 30]. An investigation of spaces admitting Killing-Yano tensors based on covariant Petrov classification can be found in [31, 32].

The aim of this paper is to investigate the basic super-energy tensors constructed from Killing-Yano tensors. We also give a generalization of super-energy tensors for cases when the starting point is a pair of tensors.

The paper is organized as follows: In Section 2 is presented a short description of how the super-energy tensors can be constructed starting from an arbitrary tensor. In Section 3 we investigate a particular class of super-energy tensors, namely those obtained from Killing-Yano tensors and write the Killing-Yano equations in terms of electric and magnetic parts. In Section 4 we give an extension of super-energy tensors obtained not from a single tensor, but from a pair of tensors. An analysis of how this generalization is involved in construction of Runge-Lenz vector for Euclidean Taub-NUT space is presented in Section 5. Conclusions are given in Section 6.

2 Preliminaries

In this section we briefly describe how, starting with an arbitrarily tensor field, a super-energy tensor can be constructed. An extensive and detailed presentation can be found in [15].

Let $t_{\mu_1 \dots \mu_m}$ be an arbitrary rank m covariant tensor. The first step is to split its set of m indices in r blocks of completely antisymmetric indices. This way, $t_{\mu_1 \dots \mu_m}$ can be seen as a r -fold (n_1, \dots, n_r) -form which will be schematically denoted by $t_{[n_1] \dots [n_r]}$, where $[n_\Upsilon]$ specify the Υ -th block with n_Υ antisymmetric indices. The next step is to construct a Hodge dual relative to each block and sets of blocks $[n_\Upsilon]$, obtaining the following 2^r duals:

$$t_{[n-n_1] \dots [n_r]}^*, \dots, t_{[n_1] \dots [n-n_r]}^*, t_{[n-n_1][n-n_2] \dots [n_r]}^*, \dots, t_{[n-n_1][n-n_2] \dots [n-n_r]}^* . \tag{1}$$

For a tensor $t^{\{\Omega\}}_{\mu_1 \dots \mu_p} = t^{\{\Omega\}}_{[\mu_1 \dots \mu_p]}$ with a set $\{\Omega\}$ of indices without any symmetry plus a set of $p \leq n$ completely antisymmetric indices $\mu_1 \dots \mu_p$, the dual of t relative to the $\mu_1 \dots \mu_p$ indices is defined as $t^{\{\Omega\}}_{\mu_{p+1} \dots \mu_n}^* \equiv \frac{1}{p!} \eta_{\mu_1 \dots \mu_n} t^{\{\Omega\} \mu_1 \dots \mu_p}$, where \star is placed over the group of indices on which operation of tacking the dual acts. With the aid of duals (1), a canonical *electric-magnetic* decomposition of $t_{\mu_1 \dots \mu_m}$ relative to any timelike unit vector field \vec{u} can be constructed by contracting each of the above duals with r copies of \vec{u} , each one with the first index of each block. When \vec{u} is contracted with a *starred* block $[n - n_\Upsilon]^*$

we have a *magnetic part* in that block, and an *electric part* when \vec{u} is contracted with a *regular block* $[n - n_\Upsilon]$. The super-energy tensor associated to t is defined as follows

$$\begin{aligned}
 T_{\lambda_1\mu_1\dots\lambda_r\mu_r} \{t\} \equiv & \frac{1}{2} \left\{ \left(t_{[n_1]\dots[n_r]} \times t_{[n_1]\dots[n_r]} \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} + \right. \\
 & + \left(t_{[n-n_1][n_2]\dots[n_r]}^* \times t_{[n-n_1][n_2]\dots[n_r]}^* \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} + \dots + \\
 & + \left(t_{[n-n_1][n-n_2][n_3]\dots[n_r]}^* \times t_{[n-n_1][n-n_2][n_3]\dots[n_r]}^* \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} + \dots + \\
 & \left. + \left(t_{[n-n_1]\dots[n-n_r]}^* \times t_{[n-n_1]\dots[n-n_r]}^* \right)_{\lambda_1\mu_1\dots\lambda_r\mu_r} \right\}, \tag{2}
 \end{aligned}$$

where $(t \times t)_{\lambda_1\mu_1\dots\lambda_r\mu_r} \equiv \left(\prod_{\Upsilon=1}^r \frac{1}{(n_\Upsilon-1)!} \right) \tilde{t}_{\lambda_1\rho_2\dots\rho_{n_1}\dots\lambda_r\sigma_2\dots\sigma_{n_r}} \tilde{t}_{\mu_1}^{\rho_2\dots\rho_{n_1}} \dots \tilde{t}_{\mu_r}^{\sigma_2\dots\sigma_{n_r}}$.

Here, $\tilde{t}_{[n_1],\dots,[n_r]}$ denotes the tensor obtained from $t_{[n_1],\dots,[n_r]}$ by permuting its indices such that the first n_1 indices are exactly those from $[n_1]$ block, the next n_2 indices are exactly those from block $[n_2]$ and so on.

3 Super-energy tensor for Killing-Yano tensor

Starting with a Killing-Yano tensor, i.e. an antisymmetric rank two tensor which satisfies

$$Y_{\mu(\nu;\lambda)} \equiv Y_{\mu\nu;\lambda} + Y_{\mu\lambda;\nu} = 0, \tag{3}$$

we obtain the following expression for the associated super-energy tensor (2)

$$T_{\mu\nu} \{Y_{[2]}\} = Y_{\mu a} Y_{\nu}^a - \frac{1}{4} g_{\mu\nu} Y_{ab} Y^{ab} \tag{4}$$

which, taking into account the antisymmetry of Y , turns out to be symmetric $T_{\mu\nu} = T_{\nu\mu}$. Since $Tr(g_{\mu\nu}) = 4$, it follows that $T_{\mu}^{\mu} \{Y_{[2]}\} = 0$. The divergence $T^{\nu}_{\mu;\nu} \{Y_{[2]}\}$ is not necessarily null. Also, the super-energy tensor does not depend on dimension of the underlying space.

The terms $Y_{\mu a} Y_{\nu}^a$ and $Y_{ab} Y^{ab}$ lead to a Stäckel-Killing tensor $K_{\mu\nu} \equiv Y_{\mu a} Y_{\nu}^a$ (a symmetric rank two tensor which satisfies $K_{(\mu\nu;\lambda)} = 0$), respective to its trace $K_a^a \equiv Y_{ab} Y^{ab}$. Thus, a super-energy tensor obtained from a Killing-Yano tensor turns out to be the trace free part of a Stäckel-Killing tensor. We remember that for every rank two tensor $T_{\mu\nu}$ there is a covariant decomposition in an antisymmetric part $T_{[\mu\nu]} \equiv \frac{1}{2} [T_{\mu\nu} - T_{\nu\mu}]$, a trace free symmetric part $T_{(\mu\nu)} - \frac{1}{4} g_{\mu\nu} T_{\rho}^{\rho}$ (where $T_{(\mu\nu)} \equiv \frac{1}{2} [T_{\mu\nu} + T_{\nu\mu}]$) and a part proportional with the metric tensor $\frac{1}{4} g_{\mu\nu} T_{\rho}^{\rho}$.

The super-energy tensor (4), which we will write as $T_{\mu\nu} \{Y_{[2]}\} = K_{\mu\nu} - \frac{1}{4} g_{\mu\nu} Tr(K)$, is not by itself a Stäckel-Killing tensor since

$$T_{(\mu\nu;\lambda)} \{Y_{[2]}\} = -\frac{1}{4} g_{(\mu\nu} (Tr(K))_{,\lambda)}, \tag{5}$$

which in general does not vanishes.

Since the super-energy tensor for $Y_{\mu\nu}$ can be written [15]

$$T\{Y_{[2]}\} = \frac{1}{2} \left[Y_{\mu a} Y_{\nu}^a + Y_{\mu a}^* Y_{\nu}^{*a} \right] , \tag{6}$$

it follows that if the starting Killing-Yano tensor $Y_{\mu\nu}$ is self-dual ($Y_{\mu\nu}^* = Y_{\mu\nu}$) or antiself-dual ($Y_{\mu\nu}^* = -Y_{\mu\nu}$) we have

$$T_{\mu\nu}\{Y_{[2]}\} = Y_{\mu a} Y_{\nu}^a . \tag{7}$$

A rank r tensor $Y_{\mu_1 \dots \mu_r}$ is said to be a Killing-Yano tensor if it is totally antisymmetric and satisfies

$$Y_{\mu_1 \dots (\mu_r; \lambda)} = 0 . \tag{8}$$

A rank r tensor $K_{\mu_1 \dots \mu_r}$ is said to be a Stäckel-Killing tensor if it is totally symmetric and satisfies

$$K_{(\mu_1 \dots \mu_r; \lambda)} = 0 . \tag{9}$$

The following contraction

$$K_{\mu\nu} \equiv Y_{\mu\rho_2 \dots \rho_r} Y_{\nu}^{\rho_2 \dots \rho_r} \tag{10}$$

is a rank-2 Stäckel-Killing tensor.

The super-energy tensor for the Killing-Yano tensor $Y_{\mu_1 \dots \mu_r}$ is

$$T_{\mu\nu} \{Y_{[r]}\} = \frac{1}{(r-1)!} \left(Y_{\mu\rho_2 \dots \rho_r} Y_{\nu}^{\rho_2 \dots \rho_r} - \frac{1}{2r} g_{\mu\nu} Y_{\rho_1 \rho_2 \dots \rho_r} Y^{\rho_1 \rho_2 \dots \rho_r} \right) . \tag{11}$$

Since $Y_{\mu\rho_2 \dots \rho_r} Y_{\nu}^{\rho_2 \dots \rho_r}$ and $Y_{\rho_1 \rho_2 \dots \rho_r} Y^{\rho_1 \rho_2 \dots \rho_r}$ are a Stäckel-Killing tensor K , with respect to its trace, it follows that (11) is the trace free part of K .

We saw that the super-energy tensors of the various rank Killing-Yano tensors are trace free parts of some Stäckel-Killing tensors. If a trace free Stäckel-Killing tensor of rank two can be written as

$$K_{\mu\nu} = F_{\mu a} F_{\nu}^a - \frac{1}{4} g_{\mu\nu} F_{ab} F^{ab} \tag{12}$$

i.e. it is required to be the super-energy tensor of some 2-form F , the starting 2-form F must satisfies

$$2F_{\lambda}^a F_{a(\mu;\nu)} + 2F_{\mu}^a F_{a(\nu;\lambda)} + 2F_{\nu}^a F_{a(\mu;\lambda)} + g_{\nu\lambda} F_{ab;\mu} F^{ab} + g_{\mu\lambda} F_{ab;\nu} F^{ab} + g_{\mu\nu} F_{ab;\lambda} F^{ab} = 0 . \tag{13}$$

In a more compact form eq. (13) reads:

$$2F_{a\{\mu;\nu\}} F_{\lambda\}^a + F_{ab;\{\mu} g_{\nu\lambda\}} F^{ab} = 0 . \tag{14}$$

where curly braces specify a summation over the circular permutations of enclosed indices.

If for a Stäckel-Killing tensor K is required to be the super-energy tensor of a p -form with $p \geq 2$

$$K_{\mu\nu} = F_{\mu\rho_1 \dots \rho_{p-1}} F_{\nu}^{\rho_1 \dots \rho_{p-1}} - \frac{1}{4} g_{\mu\nu} F_{\rho_1 \dots \rho_p} F^{\rho_1 \dots \rho_p} , \tag{15}$$

eqs. (13) becomes:

$$\pm 2F_\nu^{\rho_1 \dots \rho_{p-1}} F_{\rho_1 \dots \rho_{p-1}(\mu; \lambda)} \pm 2F_\mu^{\rho_1 \dots \rho_{p-1}} F_{\rho_1 \dots \rho_{p-1}(\nu; \lambda)} \pm 2F_\lambda^{\rho_1 \dots \rho_{p-1}} F_{\rho_1 \dots \rho_{p-1}(\mu; \nu)} + g_{\mu\nu} F_{\rho_1 \dots \rho_p; \lambda} F^{\rho_1 \dots \rho_p} + g_{\mu\lambda} F_{\rho_1 \dots \rho_p; \nu} F^{\rho_1 \dots \rho_p} + g_{\nu\lambda} F_{\rho_1 \dots \rho_p; \mu} F^{\rho_1 \dots \rho_p} = 0$$

where $+$ ($-$) sign is for even(odd) rank F tensors.

It is easy to see that the super-energy tensors obtained from covariantly constant Killing-Yano tensors ($F_{\rho_1 \dots \rho_{p-1} \rho_p; \mu} = 0$) are trace free Stäckel-Killing tensors, not only trace free parts of some Stäckel-Killing tensors (which in general are not by themselves Stäckel-Killing tensors). For Stäckel-Killing tensors which are the super-energy tensors of some p -forms we have the following properties (which derives from general properties of super-energy tensors). For any causal, future-oriented vectors \vec{u}_1, \vec{u}_2 we have DSEP (dominant super-energy property)

$$T_{\mu\nu}\{Y_{[2]}\}u_1^\mu u_2^\nu \geq 0, \quad (16)$$

when the strict inequality holds provided that $Y_{[2]}$ is not null.

As a particular case for DSEP, super-energy density for Killing-Yano tensor $Y_{[2]}$ can be obtained. Super-energy density of $Y_{[2]}$ relative to temporal vector \vec{u} , denoted by $W_Y(\vec{u})$, is defined by

$$W_Y(\vec{u}) \equiv T_{\mu\nu}\{Y_{[2]}\}u^\mu u^\nu \geq 0 \quad (17)$$

and is non negative. Moreover, if $W_Y(\vec{u}) = 0$ for some \vec{u} then $T_{\mu\nu}\{Y_{[2]}\} = 0$ which implies $Y_{\mu\nu} = 0$. Eq. (17) is important in the following context. It is known that the geodesic motion i.e. $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$ admits a quadratic first integral

$$K_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \text{constant} \quad (18)$$

provided that the symmetric tensor $K_{\mu\nu}$ satisfies the Stäckel-Killing equation $K_{(\mu\nu;\lambda)} = 0$. Here \dot{x}^μ denote the derivatives of coordinate functions in respect with the affine parameter of geodesic.

It follows from (17) that if we consider Stäckel-Killing tensors which are the super-energy tensors of some p -forms then (18) provides a nonnegative first integral. Even if the super-energy density has the mathematical properties of an energy density, it does not always have the physical dimension and signification of an energy density, this fact depends on the starting tensor.

Relative to a orthonormal coordinate basis $\vec{\epsilon}_a$ such that $\vec{\epsilon}_0 = \vec{u}$, we have

$$W_Y(\vec{\epsilon}_0) = T_{00}\{Y_{[2]}\} = \frac{1}{2} \sum_{\mu, \nu=0}^3 |Y_{\mu\nu}|^2 \quad (19)$$

If the *positive metric* $h_{\mu\nu}$ relative to \vec{u} is introduced as

$$h_{\mu\nu}(\vec{u}) \equiv g_{\mu\nu} + 2u_\mu u_\nu \quad (20)$$

then the super-energy density for $Y_{[2]}$ relative to \vec{u} can be written

$$W_Y(\vec{u}) = \frac{1}{4} Y_{\mu\nu} Y_{\rho\sigma} h^{\mu\rho} h^{\nu\sigma}. \quad (21)$$

Since the divergence of $T\{Y_{[2]}\}$ is

$$T_{\lambda ;\mu}^{\mu} \{Y_{[2]}\} = Y_{\lambda a ;\mu} Y^{\mu a} + Y_{\lambda a} Y^{\mu a}{}_{;\mu} - \frac{1}{2} \delta_{\lambda}^{\mu} Y_{ab ;\mu} Y^{ab} \tag{22}$$

it follows that if Y is covariantly constant ($Y_{ab ;\mu} = 0$) then

$$T_{\lambda ;\mu}^{\mu} \{Y_{[2]}\} = 0 . \tag{23}$$

Eq. (23) cannot lead by itself to an integral conservation law since there is no general Gauss law for rank two or higher tensor fields. Anyway, if the space admits a Killing vector field i.e. a vector field ξ_{μ} for which $\xi_{(\mu ;\nu)} = 0$ and if (23) holds, we have the following local conservation law

$$(T^{\mu\nu} \{Y_{[2]}\} \xi_{\nu})_{;\mu} = 0 \tag{24}$$

which via Gauss theorem for vector fields can lead to an integral conservation law. Moreover, if $Tr(T_{\mu\nu} \{Y_{[2]}\}) = 0$, then a conformal Killing vector field is sufficient in order to have the above conservation law.

3.1 Killing-Yano equations

As Y is a simple 2-form, i.e. can be written as a wedge product of two 1-forms, we have a single Hodge dual:

$$Y_{\mu\nu}^* = \frac{1}{2} \eta_{\alpha\beta\mu\nu} Y^{\alpha\beta} \tag{25}$$

which is a Killing-Yano tensor by itself ($Y_{\mu\nu ;\lambda}^* + Y_{\mu\lambda ;\nu}^* = 0$) if $Y_{\mu\nu}$ is covariantly constant. Therefore we have two tensors, the initial Y and its dual Y^* for constructing electric and magnetic parts of the Killing-Yano tensor Y .

The electric part of Y relative to a temporal vector \vec{u} is defined as

$$\left(\frac{Y}{\vec{u}} E\right)_{\nu} \equiv Y_{\mu\nu} u^{\mu} . \tag{26}$$

If \vec{u} is covariantly constant, the electric part $\frac{Y}{\vec{u}} \vec{E}$ is a Killing vector field ($\left(\frac{Y}{\vec{u}} E\right)_{(\mu ;\nu)} = 0$). However, this condition is only a sufficient one. The necessary condition for the electric part of Y to be a Killing vector field reads

$$Y_{\alpha\mu} u_{;\nu}^{\alpha} + Y_{\alpha\nu} u_{;\mu}^{\alpha} = 0 . \tag{27}$$

The magnetic part of Y relative to the temporal vector \vec{u} is

$$\left(\frac{Y}{\vec{u}} H\right)_{\nu} = Y_{\mu\nu}^* u^{\mu} = \frac{1}{2} \eta_{\alpha\beta\mu\nu} Y^{\alpha\beta} u^{\mu} . \tag{28}$$

Since the dual $Y_{\mu\nu}^*$ of a covariantly constant Killing-Yano tensor $Y_{\mu\nu}$ is by itself a Killing-Yano tensor and the magnetic part of $Y_{\mu\nu}$ is $\left(\frac{Y}{\vec{u}} H\right)_{\nu} = Y_{\mu\nu}^* u^{\mu}$ it follows that for a covariantly constant Killing-Yano tensor, the magnetic part is a Killing vector field with the same conditions the electric part, we just having to replace $Y_{\mu\nu}$ by $Y_{\mu\nu}^*$

$$Y_{\alpha\mu}^* u_{;\nu}^{\alpha} + Y_{\alpha\nu}^* u_{;\mu}^{\alpha} = 0 . \tag{29}$$

Both the electric and magnetic part of Y are 4-vectors. Apparently they have together 8 independent components. Anyway, since each of them is orthogonal on the chosen temporal vector field \vec{u} , the number of independent components is reduced to 6, as is necessary to univocally determine an antisymmetric 4×4 tensor, as is the Killing-Yano tensor. Taking into account (32) and that $\eta_{\mu\nu\lambda\rho} = \sqrt{-g}\Delta_{\mu\nu\lambda\rho}$ implies $\eta_{\mu\nu\lambda\rho;\alpha} = 0$, where $\Delta_{\mu\nu\lambda\rho}$ is the Levi-Civita pseudotensor, the Killing-Yano equation (3) can be written in terms of Y 's electric and magnetic parts as

$$E_{\mu}u_{(\mu;\alpha)} - u_{\mu}E_{(\nu;\alpha)} + E_{\mu;(\nu}u_{\alpha)} - u_{\mu;(\nu}E_{\alpha)} + \frac{1}{2}\eta_{\lambda\rho\mu(\nu}u_{;\alpha)}^{\lambda}H^{\rho} + \frac{1}{2}\eta_{\lambda\rho\mu(\nu}H_{;\alpha)}^{\rho}u^{\lambda} = 0. \quad (30)$$

These equations have to be considered together with the orthogonality conditions between electric, respective magnetic parts of Y and the vector field \vec{u} , namely

$$\frac{Y}{\vec{u}}E_{\lambda}u^{\lambda} = 0, \quad \frac{Y}{\vec{u}}H_{\lambda}u^{\lambda} = 0. \quad (31)$$

There is no need to impose the antisymmetry condition $Y_{\mu\nu} = -Y_{\nu\mu}$ in terms of $\frac{Y}{\vec{u}}E$ and $\frac{Y}{\vec{u}}H$, since the expression of Y from (32) is by construction an antisymmetric quantity.

If the electric $\frac{Y}{\vec{u}}E$ and magnetic $\frac{Y}{\vec{u}}H$ parts of a Killing-Yano tensor are determined, the original tensor can be reconstructed [15] through

$$Y_{\mu\nu} = -2u_{[\mu}E_{\nu]} + \frac{1}{2}\eta_{\lambda\rho\mu\nu}u^{\lambda}H^{\rho}. \quad (32)$$

If \vec{u} would be chosen such that $u_{;\mu}^{\lambda} = 0$, eqs. (30) would have a more simplified form. Such a choice is not always possible since when the metric is required to be a solution of Einstein equations for empty space, a covariantly constant vector field is a null type one, or the metric is flat, or in the definitions for electric and magnetic parts a temporal vector field is required. Also, the equations would become simpler if \vec{u} is chosen to be a Killing vector field. If \vec{u} is covariantly constant, the Killing-Yano equations (30) becomes

$$E_{\mu;(\alpha}u_{\nu)} - u_{\mu}E_{(\nu;\alpha)} + \frac{\sqrt{-g}}{2}\Delta_{\lambda\rho\mu(\nu}H_{;\alpha)}^{\rho}u^{\lambda} = 0 \quad (33)$$

which have to be considered together with (31).

If the Killing-Yano tensor Y is selfdual (anti-selfdual) its electric $\frac{Y}{\vec{u}}E$ and magnetic $\frac{Y}{\vec{u}}H$ parts becomes equal (equal up to a sign) leading to the following Killing-Yano equations

$$E_{\mu}u_{(\mu;\alpha)} - u_{\mu}E_{(\nu;\alpha)} + E_{\mu;(\nu}u_{\alpha)} - u_{\mu;(\nu}E_{\alpha)} \pm \frac{1}{2}\eta_{\lambda\rho\mu(\nu}u_{;\alpha)}^{\lambda}E^{\rho} \pm \frac{1}{2}\eta_{\lambda\rho\mu(\nu}E_{;\alpha)}^{\rho}u^{\lambda} = 0, \quad (34)$$

where $+(-)$ sign is for the selfdual (anti-selfdual) case.

4 Super-energy tensor for a pair of simple 2-forms

Be A and B two simple 2-forms ($A_{(ab)} = A_{ab}$ respective $B_{(ab)} = B_{ab}$). We note

$$X \equiv A + B, \quad Y \equiv A - B. \quad (35)$$

It is manifest that X and Y are also simple 2-forms, so that the corresponding super-energy tensors can be immediately written:

$$\begin{aligned}
 T_{\mu\nu}\{X_{[2]}\} = & X_{\mu a}X_{\nu}^a - \frac{1}{4}g_{\mu\nu}X_{ab}X^{ab} = \\
 & A_{\mu a}A_{\nu}^a + A_{\mu a}B_{\nu}^a + B_{\mu a}A_{\nu}^a + B_{\mu a}B_{\nu}^a - \\
 & \frac{1}{4}g_{\mu\nu}[A_{ab}A^{ab} + A_{ab}B^{ab} + B_{ab}A^{ab} + B_{ab}B^{ab}] = \\
 & T_{\mu\nu}\{A_{[2]}\} + T_{\mu\nu}\{B_{[2]}\} + \\
 & A_{\mu a}B_{\nu}^a + B_{\mu a}A_{\nu}^a - \frac{1}{4}g_{\mu\nu}A_{ab}B^{ab} - \frac{1}{4}g_{\mu\nu}B_{ab}A^{ab} .
 \end{aligned} \tag{36}$$

Analogously,

$$\begin{aligned}
 T_{\mu\nu}\{Y_{[2]}\} = & T_{\mu\nu}\{A_{[2]}\} + T_{\mu\nu}\{B_{[2]}\} - \\
 & A_{\mu a}B_{\nu}^a - B_{\mu a}A_{\nu}^a + \\
 & \frac{1}{4}g_{\mu\nu}A_{ab}B^{ab} + \frac{1}{4}g_{\mu\nu}B_{ab}A^{ab} .
 \end{aligned} \tag{37}$$

If we take as definition for the super-energy tensor of a pair of two simple 2-forms A and B

$$T_{\mu\nu}\{A_{[2]}, B_{[2]}\} \equiv \frac{1}{4} [T_{\mu\nu}\{X_{[2]}\} - T_{\mu\nu}\{Y_{[2]}\}] , \tag{38}$$

we obtain

$$T_{\mu\nu}\{A_{[2]}, B_{[2]}\} = \frac{1}{2} [A_{\mu a}B_{\nu}^a + B_{\mu a}A_{\nu}^a] - \frac{1}{4}g_{\mu\nu}A_{ab}B^{ab} . \tag{39}$$

If the two 2-forms A and B are taken to be equal, then (39) resemble the standard way of constructing the super-energy tensor of a simple 2-form. Since every term in r.h.s. of (39) contains elements from each tensor in pair, it is obvious that (39) is a divergence free Stäckel-Killing tensor if both tensors in pair are covariantly constant. The above construction can be applied to a pair of two arbitrarily chosen p -forms, the only difference being on summation indices, namely: whenever we have a contraction of type $A_{\mu a}B_{\nu}^a$ we will replace it with $A_{\mu\rho_2\dots\rho_p}B_{\nu}^{\rho_2\dots\rho_p}$ and instead of $A_{ab}B^{ab}$ we will write $A_{\rho_1\dots\rho_p}B^{\rho_1\dots\rho_p}$. The same technique can be used to construct a super-energy tensor for a pair of two r -fold forms which have the same structure of indices blocks, i.e. both have the same number of blocks and the same number of indices in corresponding blocks. Anyway, the formulas are rather complicated and we do not present them here.

Since from (6) the following alternative form of the super-energy tensor (39) can be derived

$$T_{\mu\nu}\{A_{[2]}, B_{[2]}\} = A_{\mu a}B_{\nu}^a + B_{\mu a}A_{\nu}^a + A_{\mu a}^*B_{\nu}^{*a} + B_{\mu a}^*A_{\nu}^{*a} , \tag{40}$$

it follows that if the starting 2-forms are simultaneously selfdual or anti-selfdual the super-energy tensor becomes

$$T_{\mu\nu}\{A_{[2]}, B_{[2]}\} = 2 [A_{\mu a}B_{\nu}^a + B_{\mu a}A_{\nu}^a] , \tag{41}$$

while in the case where one is selfdual and the other anti-selfdual the super-energy tensor vanishes.

5 Euclidean Taub-NUT space analysis

In a special choice of coordinates the Euclidean Taub-NUT line element takes the form

$$ds^2 = V(r) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2) + 16m^2 V^{-1}(r) (d\chi + \cos \theta d\varphi)^2 \quad (42)$$

with $V(r) = 1 + \frac{4m}{r}$.

There are four Killing vectors for metric (42)

$$D^{(A)} = R^{(A)\mu} \partial_\mu, \quad A = 0, 1, 2, 3, \quad (43)$$

where

$$\begin{aligned} D^{(0)} &= \partial_\chi \\ D^{(1)} &= -\sin \varphi \partial_\theta - \cos \varphi \cot \theta \partial_\varphi + \frac{\cos \varphi}{\sin \theta} \partial_\chi \\ D^{(2)} &= \cos \varphi \partial_\theta - \sin \varphi \cot \theta \partial_\varphi + \frac{\sin \varphi}{\sin \theta} \partial_\chi \\ D^{(3)} &= \partial_\varphi. \end{aligned} \quad (44)$$

The first vector $D^{(0)}$ generates the $U(1)$ of χ translations and commutes with the other Killing vectors, while the remaining three vectors, corresponding to the invariance of the metric (42) under spatial rotations, obey an $SU(2)$ algebra. These symmetries would correspond to conservation of the so called *relative electric charge* and the angular momentum:

$$q = 16m^2 V(r) (\dot{\chi} + \cos \theta \dot{\varphi}) \quad (45)$$

$$\vec{j} = \vec{r} \times \vec{p} + q \frac{\vec{r}}{r}, \quad (46)$$

where $\vec{p} = V^{-1}(r) \dot{\vec{r}}$ is the *mechanical momentum* which is only part of the momentum canonically conjugate to \vec{r} .

Taub-NUT space admits the following Killing-Yano tensors

$$f^{(i)} = 8m (d\chi + \cos \theta d\varphi) \wedge dx_i - \epsilon_{ijk} \left(1 + \frac{4m}{r}\right) dx_j \wedge dx_k \quad i, j, k = 1, 2, 3 \quad (47)$$

$$f^{(y)} = 8m (d\chi + \cos \theta d\varphi) \wedge dr + 4r (r + 2m) \left(1 + \frac{r}{4m}\right) \sin \theta d\theta \wedge d\varphi, \quad (48)$$

where the first three are covariantly constant, while the last one has only one non-vanishing component of the field strength

$$f_{r\theta;\varphi}^{(y)} = 2 \left(1 + \frac{r}{4m}\right) r \sin \theta. \quad (49)$$

For the Taub-NUT metric there is a conserved vector, analogous to the Runge-Lenz vector of the Kepler type problem

$$\vec{K} = \frac{1}{2} \vec{K}_{\mu\nu} \Pi^\mu \Pi^\nu = \vec{p} \times \vec{j} + \left(\frac{q^2}{4m} - 4mE\right) \frac{\vec{r}}{r}, \quad (50)$$

where the conserved energy is

$$E = \frac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu = \frac{1}{2}V^{-1}(r) \left[\dot{r}^2 + \left(\frac{q}{4m} \right)^2 \right]. \quad (51)$$

Taking $A = f^{(y)}$ and $B = f^{(i)}$ in (39) and observing that the last term, being proportional with metric tensor, leads to trivial constants of motion and can be dropped out, we obtain the following three nontrivial Stäckel-Killing tensors

$$\tilde{T}_{\mu\nu}^{(i)}\{f^{(y)}, f^{(i)}\} = \frac{1}{2} \left[f^{(y)}_{\mu a} f^{(i)a}_\nu + f^{(i)}_{\mu a} f^{(y)a}_\nu \right] \quad i = 1, 2, 3. \quad (52)$$

An explicit evaluation [23] shows that the components of the Runge-Lenz vector (50) can be written in terms of the geometrical objects (52), (43)

$$K_{\mu\nu}^{(i)} = 2m\tilde{T}_{\mu\nu}^{(i)}\{f^{(y)}, f^{(i)}\} + \frac{1}{8m} \left(R_\mu^{(0)} R_\nu^{(i)} + R_\nu^{(0)} R_\mu^{(i)} \right). \quad (53)$$

6 Conclusions

In this paper we have investigated a class of basic super-energy tensors, namely those constructed from Killing-Yano tensors. We found that they are trace free parts of some rank two Stäckel-Killing tensors and determined the conditions for these super-energy tensors to be trace free Stäckel-Killing tensors not only trace free part of some Stäckel-Killing tensors. We have also investigated the special cases when the Killing-Yano tensors are covariantly constant, self-dual and antiself-dual. An alternative form for Killing-Yano equations in terms of electric and magnetic parts of the Killing-Yano tensor was written. We have also introduced a generalized definition of super-energy tensors for cases when the starting object is not a single tensor, but a pair of tensors, which we used to obtain an analogous of the conserved Runge-Lenz vector from Kepler problem in an Euclidean Taub-NUT space.

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