Central European Science Journals
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Central European Journal of Mathematics
Central European Science Journals
CEJM 4 (2003) 644–649

On the Hilbert function of curvilinear zero-dimensional subschemes of projective spaces

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Received 26 June 2003; accepted 5 September 2003

Abstract: Here we show the existence of strong restrictions for the Hilbert function of zero-dimensional curvilinear subschemes of $\mathbf{P}^n$ with one point as support and with high regularity index.

Keywords: zero-dimensional scheme, regularity index, minimal free resolution
MSC (1991): 14N05

1 Introduction

For any zero-dimensional scheme $Z \subset \mathbf{P}^n$ let h_Z be its Hilbert function, i.e. define $h_Z : \mathbf{N} \to \mathbf{N}$ by the formula $h_Z(t) := \dim(\mathrm{Im}(\rho_{Z,n,t}))$, where $\rho_{Z,n,t} : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is the restriction map.

The Hilbert functions of zero-dimensional degree d subschemes of $\mathbf{P}^n$ are known since Macaulay: this are the so-called O-sequences. In [4] Geramita, Maroscia and Roberts characterized the Hilbert functions of finite reduced subschemes of $\mathbf{P}^n$: both the function and its first difference function must be O-functions. For some geometrically interesting classes of zero-dimensional subschemes of $\mathbf{P}^n$ there are very strong restrictions for the possible Hilbert functions (see [3], [5] and references therein for the case of fat points).

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Here we study the problem of finding restrictions on the Hilbert function for the case of curvilinear subschemes of $\mathbf{P}^n$ supported by a unique point of $\mathbf{P}^n$. We will find very strong restrictions for the Hilbert function of such a curvilinear scheme Z if Z has large regularity index $\tau(Z)$ (see Theorem 2.5), but that, contrary to the case of fat points, all very high regularity indices may occur (see Proposition 2.8). We will also show that when $\tau(Z) > d/2-1$, then also the Betti numbers $b_{i,j}(Z)$, $1 \leq i \leq n-1$, $j \geq 0$, of the minimal free resolution of Z have some restrictions (see part (d) of Theorem 2.5). If $Z \subset \mathbf{P}^2$ we compute the minimal free resolution of $\mathcal{I}_Z$, too.

2 Restrictions on the Hilbert function

We work over an algebraically closed base field K. Let $R = K[x_0, \ldots, x_n]$ and let $\mathbf{P}^n = \mathrm{Proj}(R)$. If $Z \subset \mathbf{P}^n$ is a closed subscheme, we set I_Z its saturated homogeneous ideal, and $\mathcal{I}_Z$ its ideal sheaf. As usual, we set $h^i(\mathbf{P}^n, \mathcal{I}_Z(j)) = \dim_K H^i(\mathbf{P}^n, \mathcal{I}_Z(j))$.

At first, we state the result for $Z \subset \mathbf{P}^2$.

Proposition 2.1. Fix $d \geq 4$, and x such that $d > x > d/2+1$. Let $P \in \mathbf{P}^2$ and let $Z \subset \mathbf{P}^n$ be a zero-dimensional curvilinear scheme such that $Z_{red} = \{P\}$. If $h^1(\mathbf{P}^2, \mathcal{I}_Z(x-2)) \neq 0$ and $h^1(\mathbf{P}^2, \mathcal{I}_Z(x-1)) = 0$ then

a) there is a line D such that length$(D \cap Z) = x$ and Z is contained in a double line;

b) a minimal free resolution of $\mathcal{I}_Z$ is

$$0 \to \mathcal{O}(-d+x-2) \oplus \mathcal{O}(-x-1) \to \mathcal{O}(-2) \oplus \mathcal{O}(-d+x-1) \oplus \mathcal{O}(-x) \to \mathcal{I}_Z \to 0$$

c) the Hilbert function h_Z of Z is

$$h_Z(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t+1 & \text{if } 0 \leq t \leq d-x \\ t+d-x+1 & \text{if } d-x \leq t \leq x-1 \\ d & \text{if } t \geq x-1. \end{cases}$$

Proof 2.2. Since Z is curvilinear, there is an open subset U containing P in $\mathbf{P}^2$ and a smooth curve $C \subset U$ with $Z \subset C$. Notice that Z is exactly the effective Cartier divisor dP of C. For any integer $y \leq d$ there is a unique curvilinear subscheme Z_y of Z with length$(Z_y) = y$: the effective Cartier divisor yP of C.

First we will check the existence of a line $D \subset \mathbf{P}^2$ such that length$(D \cap Z) = x$. By hypothesis, the regularity index $\tau(Z)$ of Z is equal to $x-2$. By [2],Corollaire 2, with $s = 2$ there is a line $D \subset \mathbf{P}^2$ such that length$(D \cap Z) = x$. Let W be the residual scheme of Z with respect to D. We have length$(W) = d - x < x$ and $W \subseteq Z$. Since W is the effective Cartier divisor $(d-x)P$ of C and $d - x \leq x$, W is contained in the Cartier divisor xP of C, i.e. $W \subseteq Z \cap D$. In particular we have $W \subset D$. Thus Z is contained in the double line $2D$.

Furthermore, we have the following exact sequence

$$0 \to \mathcal{I}_W(-1) \to \mathcal{I}_Z \to \mathcal{I}_{Z\cap D|D} \to 0. \tag{1}$$

From the collinearity of W and $Z \cap D$ we deduce that the ideal I_W is minimally generated by L, G where L is a linear form such that $D = V(L)$, and G has degree $d-x$, while the ideal $I_{D\cap Z|D}$ is generated by only one form $\overline{F}$ of degree x. The first map of the sequence is the product by L, while the second map is the restriction onto D. The ideal $I_{Z\cap D} \subset R$ is generated by L, F where F is a lifting of $\overline{F}$ in I_Z. But $I_{Z\cap D} \subset I_W$ and so we get that $F \in I_W$.

Looking at the cohomology sequence associated to the sequence (1) we get that the ideal I_Z is minimally generated by L^2, LG, F, where $F \in I_W$ is a lifting of $\overline{F}$. From $F \in I_W$ we deduce the existence of $a, b \in R$ such that $F = aL + bG$, and so $aL^2 + bLG - LF = 0$. Moreover, L, G is a regular sequence. Then, we get the resolution of $\mathcal{I}_Z$:

$$0 \to \mathcal{O}(-d+x-2) \oplus \mathcal{O}(-x-1) \to \mathcal{O}(-2) \oplus \mathcal{O}(-d+x-1) \oplus \mathcal{O}(-x) \to \mathcal{I}_Z \to 0$$

where $\mathcal{O} = \mathcal{O}_{\mathbf{P}^2}$.

From the free resolution of I_Z we can compute the Hilbert function h_Z of Z and so we have the claim.

Remark 2.3. It is easy to write the last map of the computed resolution of I_Z and it is

$$\begin{pmatrix} G & a \\ -L & b \\ 0 & -L \end{pmatrix}.$$

Remark 2.4. Any smooth curve containing Z has degree at least x as we can see from the generators of the ideal I_Z.

Now, we can state the analogous result if $Z \subset \mathbf{P}^n, n \geq 3$.

Theorem 2.5. Fix positive integers $d \geq 4$, and x such that $d > x > d/2 + 1$. Let $P \in \mathbf{P}^n$, $n \geq 3$, and let $Z \subset \mathbf{P}^n$ be a length d zero-dimensional curvilinear scheme such that $Z_{red} = \{P\}$. If $h^1(\mathbf{P}^n, \mathcal{I}_Z(x-2)) \neq 0$ and $h^1(\mathbf{P}^n, \mathcal{I}_Z(x-1)) = 0$ then:

(a) there is a line $D \subset \mathbf{P}^n$ such that length$(D \cap Z) = x$ and $Z \subset D^{(1)}$, where $D^{(1)}$ denotes the first infinitesimal neighborhood of D in $\mathbf{P}^n$, i.e. the closed subscheme of $\mathbf{P}^n$ with $(\mathcal{I}_D)^2$ as ideal sheaf;

(b) every quadric hypersurface containing D in its singular locus contains Z and in particular $h^0(\mathbf{P}^n, \mathcal{I}_Z(2)) \geq n(n-1)/2$;

(c) $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) \leq x - 1 - t$ for $d - x - 1 \leq t \leq x - 2$;

(d) $b_{i,x+i-2}(Z) \neq 0$ and $b_{i,j}(Z) = 0$ for $1 \leq i \leq n$ and $j \geq x + i - 1$.

The non-vanishing of the Betti number $b_{i,x+i-2}(Z)$ follows from part (a) because it is forced by the existence of a line D such that $\text{length}(D \cap Z) = x$; for instance the existence of the collinear subscheme $Z \cap D$ of Z shows that the homogeneous ideal of Z requires at least a minimal generator of degree $\geq x$.

In the proof we shall use the following well-known and easy lemma, usually called Horace lemma, that we state without proof.

Lemma 2.6. Let W be a projective variety, D an effective Cartier divisor of W, E a vector bundle on W and Z a closed subscheme of W. Let Y be the residual scheme of Z with respect to D, i.e. the closed subscheme of W with $Hom(\mathcal{I}_{D,W}, \mathcal{I}_{Z,W})$ as ideal sheaf. Then $h^i(W, E \otimes \mathcal{I}_{Z,W}) \leq h^i(D, (E|D) \otimes \mathcal{I}_{Z\cap D,D}) + h^i(W, E(-D) \otimes \mathcal{I}_{Y,W})$ for $i = 0, 1$.

Proof 2.7. As in the proof of Proposition 2.1 let U be an open subset of $\mathbf{P}^n$ containing P and let $C \subset U$ be a smooth curve with $Z \subset C$. Z is the effective Cartier divisor dP on C. Let $D \subseteq \mathbf{P}^n$ be the Zariski tangent space of C at P. Since C is smooth at P, D is a line, and it is the only line of $\mathbf{P}^n$ with $w := \text{length}(Z \cap D) \geq 2$. Since $h^1(\mathbf{P}^n, \mathcal{I}_Z(x-1)) = 0$, and $x - 1 \leq d - 2$ we have $Z \nsubseteq D$, i.e. $w < d$. Moreover, $w \leq x$. Let E be the osculating plane of C at P, i.e. the only plane E' of $\mathbf{P}^n$ such that $D \subset E'$ and the intersection multiplicity of C with E' at P is at least $1 + \mu_P(C \cap D)$, where $\mu_P(C \cap D)$ is the multiplicity of the intersection between C and D in P. Since $w < d$, the plane E is the unique plane E' of $\mathbf{P}^n$ such that $\text{length}(E' \cap Z) > w$. There is a unique subscheme Y of Z with $\text{length}(Y) = w + 1$: the effective Cartier divisor $(w + 1)P$ of C. The plane E is the linear span of Y.

Claim We have $w = x$.

Proof (of the Claim) Fix a general codimension 3 linear suspace A of $\mathbf{P}^n$ and let $f : \mathbf{P}^n \backslash A \to \mathbf{P}^n$ be the linear projection from A. Since A is general, $A \cap E = \emptyset$ and in particular $D \cap A = \emptyset$. Hence by Nakayama's lemma f induces an isomorphism of Z with $f(Z)$. Hence, taking cones with A as vertex, we easily see that $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) \leq h^1(\mathbf{P}^2, \mathcal{I}_{f(Z)}(t))$. Hence we may apply Proposition 2.1 to $f(Z)$ taking instead of x the maximal integer x' such that $h^1(\mathbf{P}^2, \mathcal{I}_{f(Z)}(x'-2)) \neq 0$. The line $f(D)$ is the tangent line to $f(C)$ at $f(P)$. The condition $A \cap E = \emptyset$ implies that $f(Y)$ spans $\mathbf{P}^2$. Thus $f(Y) \cap f(D) = f(Y \cap D) = f(Z \cap D)$, i.e. $\text{lenght}(f(Y) \cap f(D)) = w + 1$. Hence $x' = w$. Since $x' \geq x$ and $w \leq x$, we obtain $w = x$, proving the Claim.

During the proof of the Claim we also checked the inequalities in part (c). Let $H \subset \mathbf{P}^n$ be the hyperplane spanned by D and a general $(n-3)$-dimensional linear space and W the residual scheme of Z with respect to H. In the proof of the Claim we also obtained $\text{length}(Z \cap H) = x$, i.e. $\text{length}(W) = d - x$. Since Z is curvilinear, $W \subset Z$ and $\text{length}(W) = d - x \leq x = \text{length}(Z \cap D)$, we saw at the beginning of the proof that $W \subset D$. Hence W is contained in any hyperplane H' with $D \subset H'$. Thus Z is contained in any reducible quadric hypersurface containing D in its singular locus (use for instance Horace Lemma). Taking linear combinations of such quadric hypersurfaces we see that every quadric hypersurface singular along D contains Z. Since $D^{(1)}$ is the intersection of all such quadrics, we have $Z \subset D^{(1)}$, concluding the proof of parts (a), (b) and (c). To prove part (d) it is sufficient to prove $h^1(\mathbf{P}^n, \Omega^i_{\mathbf{P}^n}(i+x-1) \otimes \mathcal{I}_Z)) \neq 0$ and $h^1(\mathbf{P}^n, \Omega^i_{\mathbf{P}^n}(j) \otimes \mathcal{I}_Z)) = 0$ for $1 \leq i \leq n-1$ and $j \geq i+x$. We have $\Omega_{\mathbf{P}^n}(1)|D \cong \mathcal{O}_D^{\oplus(n-1)} \oplus \mathcal{O}_D(-1)$ and hence for every integer i with $1 \leq i \leq n-1$ we have $\Omega^i_{\mathbf{P}^n}(i)|D \cong \mathcal{O}^a \oplus \mathcal{O}_D(-1)^b$, where $a = \binom{n-1}{i}$ and $b = \binom{n-1}{i-1}$. Hence the vanishing result in part (d) follows by applying twice Horace Lemma with respect to a general hyperplane H containing D, while the non-vanishing part follows from the non-vanishing of the corresponding Betti numbers of the subscheme $Z \cap D$ of Z.

At last, we want to give an existence result, proving that we can construct such curvilinear schemes.

Proposition 2.8. Fix integers $n \geq 2$, $x > 0$ and $d \geq x+n-1$, $P \in \mathbf{P}^n$ and a line $D \subset \mathbf{P}^n$. Then there exist a curvilinear zero-dimensional scheme $Z \subset \mathbf{P}^n$ spanning $\mathbf{P}^n$ such that length$(Z) = d$ and length$(Z \cap D) = x$. If $d \geq 4$ and $x > d/2+1$, then $\tau(Z) = x-2$.

Proof 2.9. Set $X := \mathrm{Spec}(K[t]/(t^d))$. Use the homogeneous polynomials associated to the $n+1$ polynomials $1, t, t^x, t^{x+1} \ldots, t^{x+n-2}$ to obtain an embedding $j : X \to \mathbf{P^n}$ such that $Z := j(X)$ is a solution of the first part of Proposition 2.8. The last part follows from the first one and Theorem 2.5.

Remark 2.10. Assume char$(K) = 0$. Fix positive integers n, d and $P \in \mathbf{P}^n$. There is a zero-dimensional curvilinear scheme $Z \subset \mathbf{P}^n$ such that $Z_{red} = \{P\}$, length$(Z) = d$ and with maximal rank ([1]). The regularity index $\tau(Z)$ of such scheme Z is the first positive integer x such that $\binom{n+x}{n} \geq d$.

Acknowledgments

The authors were partially supported by MURST and GNSAGA of INdAM (Italy).

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