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K-subanalytic rectilinearization and uniformization

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Abstract: We prove rectilinearization and uniformization theorems for K-subanalytic ($\mathbb{R}_{an}^{K}$-definable) sets and functions using the Lion-Rolin formula. Parallel reasoning gives standard results for the subanalytic case.

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1 Introduction

The aim of this paper is to give versions of the uniformization and rectilinearization theorems due to H. Hironaka [5] (see also E. Bierstone and P. Milman ([1]) and Parusiński ([8])), for a larger class of sets which we call K-subanalytic. They can be described by compositions of globally subanalytic functions and power functions $|x|^\lambda$ with $\lambda \in K$, where K is a given subfield of the field of real numbers $\mathbb{R}$. This class of sets was earlier studied by J.-Cl. Tougeron ([11]) and by L. van den Dries and C. Miller (for example [3, 7]). In 1997, J.-M. Lion and J.-P. Rolin in a paper [6] gave an explicit formula for a globally subanalytic function using analytic functions and operations of division and taking roots. This theorem, called the preparation theorem for subanalytic functions is similar to the Weierstrass preparation theorem, and appeared in less explicit form in a paper of A. Parusiński [9]. Lion and Rolin also give a preparation theorem for K-subanalytic functions (called by them x^λ-functions) and logarithmico-exponential functions (LE-functions). All these preparation theorems give more exact descriptions of definable functions in the corresponding o-minimal structures, and do not use model

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theory.

We use the Lion-Rolin formula for K-subanalytic functions, and get uniformization and rectilinearization for K-subanalytic functions and sets. This gives a partial answer to a problem formulated by L. van den Dries and C. Miller in [4].

The main results for the K-subanalytic case are Theorem 4.7 and Theorem 4.8. In the subanalytic case parallel reasoning gives versions of standard uniformization and rectilinearization theorems (see Theorem 5.1 and Theorem 5.2). In particular, we get a version of the characterization of subanalytic continuous functions obtained by A. Parusiński (Theorem 2.7 in [8]; see also Lemma 5.3 in E. Bierstone and P. Milman [1], which follows from uniformization). We do not use equidimensionality as Parusiński does, but admit taking powers in several stages. (An interested reader could also look at Parusiński's exposition [10].)

We give our setting of cylinders and cells in Chapter 2. Chapter 3 gives corollaries of the Lion-Rolin preparation theorem. In Chapter 4, we introduce K-subanalytic modifications, and give lemmas and statements of the main results. Chapter 5 gives analogous theorems for the subanalytic case.

This paper is a rewritten version of my earlier preprint "Rectilinearization and uniformization of k-subanalytic sets and functions".

2 Basic notions

2.1 General notation

Let X be any set. Functions $f, g : X \to \mathbb{R}$ will be called *equivalent* (notation: $f \sim g$) on X, when there exists a constant $M > 0$ such that for $x \in X$ $M^{-1} \cdot |f(x)| \leq |g(x)| \leq M \cdot |f(x)|$. A *unit* is a function equivalent to the constant function 1.

For $Z \subset \mathbb{R}^n$, we consider the *regular part* of Z (denoted reg Z) which is, by definition, the set of these points of Z where Z is a (topological) manifold. The *characteristic function* of Z is defined as 1 on Z, and 0 on the complement of Z. A *compact box* is a cartesian product of closed nonempty intervals (possibly degenerated to a point) in $\mathbb{R}$. By $\mathbb{P}$ we denote the real projective line, which is assumed to contain $\mathbb{R}$.

Denote $D^n = (-1,1)^n$ for $n \in \mathbb{N}$, and if $0 \leq k < n$ identify D^k with $D^k \times \{0\}^{n-k}$.

2.2 Cylinders and cells

Let X be any set and let $\mathcal{F}$ be a finite family of real functions on X. An *$\mathcal{F}$-set* is any subset of X of the form

$$E = \bigcup_{i=1}^{k} \bigcap_{j=1}^{l} E_{ij}, \qquad (k, l \in \mathbb{N})$$

where $E_{ij} = \{x \in X | g_{ij}(x) = 0\}$ or $E_{ij} = \{x \in X | g_{ij}(x) > 0\}$ or $E_{ij} = \{x \in X | g_{ij}(x) < 0\}$ and $g_{ij} \in \mathcal{F}$. Notice that $\mathcal{F}$-sets form a Boolean algebra.

A set $C \subset X \times \mathbb{R}$ is called an *$\mathcal{F}$-cylinder* when it is of the form

(1) $C = \{(x,y) \in B \times \mathbb{R} |\, g_1(x) < y < g_2(x)\}$, where B is an $\mathcal{F}$-set in X and $g_1, g_2 \in \mathcal{F} \cup \{-\infty\}^X \cup \{+\infty\}^X$, with $g_1(x) < g_2(x)$ for every $x \in B$,

or

(2) $C = \{(x,y) \in B \times \mathbb{R} |\, y = g(x)\}$, where B is an $\mathcal{F}$-set in X and $g \in \mathcal{F}$.

Then B is called the *base* of C and the functions g_1, g_2 (or the function g) are called the *generating functions* of the cylinder C.

Now assume that for every $n \in \mathbb{N}$ we have a family $\mathcal{A}_n$ of functions from $\mathbb{R}^n$ into $\mathbb{R}$. Denote $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}$.

An $\mathcal{A}$-*cell* (more precisely: $\mathcal{A}_n$-*cell*) in $\mathbb{R}^{n+1}$ is an $\mathcal{A}_n$-cylinder $\mathcal{C}$ whose base $\mathcal{C}'$ is an $\mathcal{A}_{n-1}$-cell in $\mathbb{R}^n$. ($\{0\}$ is $\mathcal{A}_{-1}$-cell in $\mathbb{R}^0$, where $\mathcal{A}_{-1} = \emptyset$.)

Every $\mathcal{A}$-cell $\mathcal{C} \subset \mathbb{R}^{n+1}$ is associated with a sequence of cells $\mathcal{C} = \mathcal{C}^{n+1}, \mathcal{C}' = \mathcal{C}^n, \ldots, \mathcal{C}^0$ where $\mathcal{C}^j$ is an $\mathcal{A}_{j-1}$-cell in $\mathbb{R}^j$ ($j = n+1, \ldots, 0$) and

(1) $\mathcal{C}^{j+1} = \{(x_1, \ldots, x_{j+1}) \in \mathcal{C}^j \times \mathbb{R} |\, t_{j+1,1}(x_1, \ldots, x_j) < x_{j+1} < t_{j+1,2}(x_1, \ldots, x_j)\}$, where $t_{j+1,1} < t_{j+1,2}$ on $\mathcal{C}^j$,

or

(2) $\mathcal{C}^{j+1} = \{(x_1, \ldots, x_{j+1}) \in \mathcal{C}^j \times \mathbb{R} |\, t_{j+1,1}(x_1, \ldots, x_j) = x_{j+1} = t_{j+1,2}(x_1, \ldots, x_j)\}$, where $t_{j+1,1} = t_{j+1,2}$ on $\mathcal{C}^j$,

with $t_{j+1,i} \in \mathcal{A}_j \cup \{-\infty\}^{\mathbb{R}^j} \cup \{+\infty\}^{\mathbb{R}^j} (i = 1, 2)$, and $t_{j+1,i} \equiv -\infty$ or $t_{j+1,i} \equiv +\infty$ can hold only in the case (1).

The sequence $\{t_{j,i}\}_{j=1,\ldots,n+1}^{i=1,2}$ is called the *sequence of generating functions* of the cell $\mathcal{C}$.

We say that $\mathcal{C}$ is *full-dimensional* when $\mathcal{C}'$ is full-dimensional and $\mathcal{C}$ is of the form (1). The cell $\{0\}$ is full-dimensional in $\mathbb{R}^0$.

We say that $\mathcal{C}$ is an *analytic* cell, when each of the functions $\{t_{j,i}|\mathcal{C}^{j-1}\}_{j=1,\ldots,n+1}^{i=1,2}$ is a restriction of some analytic function in an open neighbourhood of $\mathcal{C}^{j-1} \subset \mathbb{R}^{n-1}$ (or, equivalently, its sequence of generating functions consists of analytic functions on respective bases as submanifolds).

3 K-subanalytic reduction

3.1 K-subanalytic sets and functions

We call a mapping $g : \mathbb{R}^n \supset U \to \mathbb{R}^m$ *globally subanalytic* if its graph is a subanalytic subset of the ambient $\mathbb{P}^n \times \mathbb{P}^m$ or, equivalently, the graph of g is definable in the structure $\mathbb{R}_{an}$ (see [2]).

For totally defined functions, we denote

$$\mathcal{SAN}_n = \{f : \mathbb{R}^n \to \mathbb{R} |\, f \text{ is globally subanalytic}\},$$

and

$$\mathcal{SAN} = \bigcup_{n \in \mathbb{N}} \mathcal{SAN}_n.$$

For every $\lambda \in \mathbb{R}$, we define a function

$$|\cdot|^{\lambda} : \mathbb{R} \ni x \mapsto \begin{cases} 1, & x = 0, \lambda = 0 \\ 0, & x = 0, \lambda \neq 0 \\ |x|^{\lambda}, & x \neq 0. \end{cases}$$

Notice that for $\lambda \geq 0$ the function $|\cdot|^{\lambda}$ is continuous, and if $\lambda > p \in \mathbb{N}$, then $|\cdot|^{\lambda}$ is of class C^p. If $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$, then $|x|^{\lambda}$ denotes $|x_1|^{\lambda_1} \cdot \ldots \cdot |x_n|^{\lambda_n}$.

Let K be a fixed subfield of the real field $\mathbb{R}$. We put

$$\mathcal{PS}_n^K = \{f : \mathbb{R}^n \to \mathbb{R} |\, f \text{ is a composition of functions from } \mathcal{SAN} \\ \text{and functions of the form } |x|^{\lambda} (\lambda \in K)\}.$$

Here we allow combinations and functions of any arity to appear in our compositions. We will assume that $\mathcal{PS}_n^K \subset \mathcal{PS}_{n+1}^K$ treating functions from $\mathcal{PS}_n^K$ as independent on the last variable. Subsets of $\mathbb{R}^n$ which are $\mathcal{PS}_n^K$-sets will be also called *K-subanalytic*. Mappings will be called *K-subanalytic* if their graphs are K-subanalytic sets.

We adopt the following notation: $K_+ = \{\lambda \in K | \lambda \geq 0\}$ and $K_+^* = \{\lambda \in K | \lambda > 0\}$.

3.2 Preparation on cylinders

Let $E \subset \mathbb{R}_x^n \times \mathbb{R}_y^1$ be a $\mathcal{PS}_{n+1}^K$-set.

A *reducing system R on E* is a system of data $R = (\theta, \{\phi_1, \ldots, \phi_s\}, a, b, \{\lambda_i\}_{i=1}^r, \{\tilde{\lambda_k}\}_{k=1}^q)$, where $s, q, r \in \mathbb{N}^*$, $\theta, \phi_1, \ldots, \phi_s, a, b \in \mathcal{PS}_n^K$, $\lambda_i, \tilde{\lambda_k} \in K_+^*$ and for $(x, y) \in E$ the following conditions hold:

(i) $y - \theta(x) \neq 0 \neq a(x)$, functions $y - \theta(x), a(x), b(x)$ are of constant sign,

(ii) if $\theta(x) \not\equiv 0$, then $y \sim \theta(x)$,

(iii) the mapping $\psi(x, y) = (\phi_1(x), \ldots, \phi_s(x), |\frac{y-\theta(x)}{a(x)}|^{\lambda_1}, \ldots, |\frac{y-\theta(x)}{a(x)}|^{\lambda_r}, |\frac{b(x)}{y-\theta(x)}|^{\tilde{\lambda}_1}, \ldots, |\frac{b(x)}{y-\theta(x)}|^{\tilde{\lambda}_q})$ is bounded.

The mapping $\psi(x, y)$ is called the *reducing morphism* of R.

A finite family of functions $\{f_\nu\}_\nu \subset \mathcal{PS}_{n+1}^K$ will be called *R-reducible on E*, when every function f_ν has on E the form (called *R-reduction*)

($\star$) $f_\nu(x, y) = |y - \theta(x)|^{\gamma_\nu} \cdot A_\nu(x) \cdot V_\nu(\psi(x, y))$,

where $\gamma_\nu \in K$, V_ν is an analytic, nonvanishing function of constant sign on some neighbourhood of the set $\overline{\psi(E)} \subset \mathbb{R}^{s+r+q}$, and $A_\nu \in \mathcal{PS}_n^K$ with $A_\nu(x) \equiv 0$ or $A_\nu(x) \neq 0$ on E. (Thus also $f_\nu \equiv 0$ or $f_\nu \neq 0$ on E.)

The Lion-Rolin preparation theorem can be restated as follows:

Theorem 3.1 (Lion-Rolin [6]). For a finite family of functions $\{f_\nu\}_\nu \subset \mathcal{PS}_{n+1}^K$, there exists a finite family $\{(C_\alpha, R_\alpha)\}_\alpha$, where:

(1) $\{C_\alpha\}_\alpha$ is a decomposition of $\mathbb{R}^{n+1}$ into $\mathcal{PS}_n^K$-cylinders, and R_α is a reducing system on C_α,

(2) the family $\{f_\nu\}_\nu$ is R_α-reducible on C_α.

Remarks.

(1) This statement is a little stronger than the statement in [6]. We have decompositions not coverings, get only one $a(x)$ and only one $b(x)$ by extracting new functions $\phi_i(x)$, and can take subdivisions to obtain constant signs and boundedness of the reducing morphisms. By a careful proof of the theorem, we also have condition (ii).

(2) We can additionally have the cylinders compatible with a given finite family of $\mathcal{PS}^K_{n+1}$-sets by adding the characteristic functions of these sets to the considered family of functions.

(3) By the above theorem, a projection of a K-subanalytic set is K-subanalytic. More generally: K-subanalytic sets are closed under first order definability, and are exactly the class of definable sets in the structure $\mathbb{R}^K_{an}$ considered by L. van den Dries and C. Miller in [3], [4], [7].

(4) We also get

$$\mathcal{PS}^K_n = \{f : \mathbb{R}^n \to \mathbb{R} | \, f \text{ is } K\text{-subanalytic}\},$$

i.e. the class of functions on $\mathbb{R}^n$ definable in the structure $\mathbb{R}^K_{an}$.

(5) $\mathbb{Q}$-subanalytic sets and functions in $\mathbb{R}^n$ are exactly globally subanalytic sets and functions.

3.3 Cells with reducers

A *reducer* $\mathcal{R}$ *on* $\mathcal{C}$, where $\mathcal{C}$ is a $\mathcal{PS}^K$-cell in $\mathbb{R}^{n+1}$, is a sequence $\mathcal{R} = (R^1, \ldots, R^{n+1})$, where R^j is a reducing system on $\mathcal{C}^j \subset \mathbb{R}^j$.

For a reducer $\mathcal{R}$ on $\mathcal{C}$ with

$$R^j = (\theta_j, \{\phi_{j,1}, \ldots, \phi_{j,s_j}\}, a_j, b_j, \{\lambda_{j,i}\}_{i=1}^{r_j}, \{\tilde{\lambda}_{j,k}\}_{k=1}^{q_j}), j = 1, \ldots, n+1,$$

we say that a finite family of K-subanalytic functions $\{f_\nu\}_\nu$ defined on $\mathcal{C}$ is $\mathcal{R}$-*reducible on* $\mathcal{C}$ when the following two conditions hold:

(1) $\{f_\nu\}_\nu$ is R^{n+1}-reducible on $\mathcal{C}$:

$$\begin{aligned} f_\nu(x,y) = {} & |y - \theta_{n+1}(x)|^{\gamma_\nu} \cdot A_\nu(x) \cdot \\ & \cdot V_\nu(\phi_{n+1,1}(x), \ldots, \phi_{n+1,s_{n+1}}(x), |\frac{y - \theta_{n+1}(x)}{a_{n+1}(x)}|^{\lambda_{n+1,1}}, \ldots, \\ & |\frac{y - \theta_{n+1}(x)}{a_{n+1}(x)}|^{\lambda_{n+1,r}}, |\frac{b_{n+1}(x)}{y - \theta_{n+1}(x)}|^{\tilde{\lambda}_{n+1,1}}, \ldots, |\frac{b_{n+1}(x)}{y - \theta_{n+1}(x)}|^{\tilde{\lambda}_{n+1,q}}), \end{aligned}$$

(2) the family $\{\theta_{n+1}, \phi_{n+1,1}, \ldots, \phi_{n+1,s_{n+1}}, a_{n+1}, b_{n+1}\} \cup \{A_\nu\}_\nu$ is $(R^1, \ldots, R^n)$-reducible on $\mathcal{C}^n$. (Constant functions on $\mathbb{R}^0$ are reducible.)

A *cell with a reducer in* $\mathbb{R}^{n+1}$ is a pair $(\mathcal{C}, \mathcal{R})$, where $\mathcal{C}$ is a $\mathcal{PS}^K_{n+1}$-cell, and $\mathcal{R}$ is a reducer on $\mathcal{C}$ such that for $j = 1, \ldots, n$ the family $\{t_{j+1,1}, t_{j+1,2}\}$ is $(R^1, \ldots, R^j)$-reducible on $\mathcal{C}^j$. We say that a finite family $\{f_\nu\}_\nu$ (defined at least on $\mathcal{C}$) is *reducible on* $(\mathcal{C}, \mathcal{R})$, when it is $\mathcal{R}$-reducible on $\mathcal{C}$.

The following proposition gives a kind of "partial desingularization".

Proposition 3.2. For a finite family $\{f_\nu\}_\nu$ of K-subanalytic functions reducible on a cell with a reducer $(\mathcal{C}, \mathcal{R})$ in $\mathbb{R}^{n+1}$, there exist: an open neighbourhood U of the cell $\mathcal{C}$ and a K-subanalytic and analytic isomorphism $\Omega : U \ni z \mapsto \tilde{z} \in \tilde{U}$ of open sets in $\mathbb{R}^{n+1}$ such that $\tilde{U} \subset (0, +\infty)^{n+1}$ and for every ν

$$\left(f_\nu \circ \Omega^{-1} |\, \Omega(K)\right)(\tilde{z}_1, \ldots, \tilde{z}_{n+1}) = |\tilde{z}|^{\gamma_\nu} \cdot h_\nu(\tilde{z}),$$

where $\gamma_\nu \in K^{n+1}$, h_ν are zeroes or analytic and K-subanalytic units on $\tilde{U}$. Thus $\{f_\nu |\, \mathcal{C}\}$ are restrictions of analytic functions on U.

Proof. This goes by a straightforward induction on n due to the possibility of making an analytic and K-subanalytic shift $\tilde{z}_{n+1} = |z_{n+1} - \theta(\tilde{z}_1, \ldots, \tilde{z}_n)|$, where $\theta(\tilde{z}_1, \ldots, \tilde{z}_n)$ is already analytic. □

Corollary 3.3. If $(\mathcal{C}, \mathcal{R})$ is a cell with a reducer, then $\mathcal{C}$ is an analytic cell.

Applying Theorem 3.1 recursively, we get the following

Corollary 3.4. Let $E \subset \mathbb{R}^{n+1}$ be a K-subanalytic set, and $\{f_\nu\}_\nu$ a finite family of K-subanalytic functions defined on E. Then there exists a finite family of cells with reducers $\{(\mathcal{C}_\alpha, \mathcal{R}_\alpha)\}_\alpha$ such that $\{\mathcal{C}_\alpha\}_\alpha$ form a decomposition of E and the family $\{f_\nu\}_\nu$ is reducible on every $(\mathcal{C}_\alpha, \mathcal{R}_\alpha)$.

If the family $\{f_\nu\}_\nu$ is already R-reducible on E with some reducing system R, then we can additionally get $R = R_\alpha^{n+1}$ for every α.

Applying this to the characteristic functions of K-subanalytic sets, we get

Corollary 3.5 (Miller [7], van den Dries and Miller [4]). The class of K-subanalytic sets admit analytic cell decomposition and (finite) analytic stratification.

4 Rectilinearization and uniformization

4.1 Modifications

A function $V : U \to \mathbb{R}$, where U is a K-subanalytic open set in $\mathbb{R}^n$, will be called *K-analytic* if it is of the form $V(x) = \tilde{V}(\Theta(x))$, where $\Theta(x) = (x_1, \ldots, x_n, |x_{\sigma(1)}|^{\lambda_1}, \ldots, |x_{\sigma(m)}|^{\lambda_m})$, for some $m \in \mathbb{N}$, some sequence $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$, $\lambda_i \in K_+^*$ $(i = 1, \ldots, m)$, and $\tilde{V}$ an analytic and globally subanalytic function in some neighbourhood of $\Theta(U) \subset \mathbb{R}^{n+m}$.

Remark. K-analytic functions are continuous, analytic outside of $\{x \in \mathbb{R}^n |\, x_1 \cdot \ldots \cdot x_n = 0\}$, and if all $\lambda_i > p \in \mathbb{N}^*$ then they are of class C^p.

Example. The function $z(x,y) = ||x|^{\pi} + y|^{\pi}$ is a composition of $\mathbb{R}$-analytic functions which is not $\mathbb{R}$-analytic.

A mapping $\Omega : \mathbb{R}^n \supset A \to B \subset \mathbb{R}^m (m,n \in \mathbb{N})$ will be called *K-analytic* if its coordinates are K-analytic functions.

A *K-analytic isomorphism* is a bijective mapping which is K-analytic together with its inverse. K-analytic isomorphisms are K-subanalytic homeomorphisms of open sets.

A *K-power mapping* is any composition of mappings of the form

$$P_i : \mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto (x_1, \dots, |x_i|^{\gamma}, \dots, x_n) \in \mathbb{R}^n,$$

where $\gamma \in K_+^*$.

A K-analytic function is called a *K-normal crossing* if it is a product of a monomial with exponents from K_+ and a K-analytic unit. A mapping is called a *K-normal crossing* if all its coordinates are K-normal crossings.

We will use the notion of a *blowing-up* in the most standard sense: any mappping of the form

$$B_{i,j} : \mathbb{R}^n \ni (x_1, \dots, x_i, \dots, x_j, \dots, x_n) \mapsto (x_1, \dots, x_i, \dots, \overset{(j)}{x_j \cdot x_i}, \dots, x_n) \in \mathbb{R}^n,$$

where $n \geq 2$, $1 \leq i,j \leq n$ and $i \neq j$.

Remarks.

(1) A composition of K-normal crossings is a K-normal crossing. Substitution of a K-normal crossing to a K-analytic mapping is a K-analytic mapping.

(2) If a quotient of (functions) K-normal crossings restricted to the regular part of a compact box in $\mathbb{R}_+^n$ is bounded, then the resulting function is a restriction of a K-normal crossing.

A *K-subanalytic modification* is a mapping $\Phi : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is open and K-subanalytic, and Φ is a composition of blowings-up, K-power mappings and K-analytic isomorphisms.

Lemma 4.1. For every compact box $K \subset \mathbb{R}_+^n$ of dimension $k \leq n$ there is a finite family of mappings $m_\mu : D^n \to \mathbb{R}^n$ and $L \subset D^k$ such that:

(1) $\bigcup_\mu m_\mu(D^k) = \bigcup_\mu m_\mu(L) = K$,

(2) each m_μ is a composition of taking squares $(x_1, \dots, x_n) \mapsto (x_1^2, \dots, x_n^2)$ and a linear isomorphism.

Proof. As we can make cartesian products of m_μ-s, it suffices to consider the case $n = 1$, when the proof is easy. □

4.2 Lemmas

In this chapter we choose and fix $p \in \mathbb{N}$. We consider pairs $(E, \{f_\nu\}_\nu)$ where E is a compact, K-subanalytic subset of $\mathbb{R}^{n+1}$, and $\{f_\nu\}_\nu$ is a finite family of K-subanalytic functions bounded on E. Such a pair will be called *p-smooth* if the following holds:

there exists a finite family of pairs (π_α, I_α), where $\pi_\alpha : N_\alpha \to \mathbb{R}^{n+1}$ (N_α open in $\mathbb{R}^{n+1}$) is a K-subanalytic modification of class C^p, I_α is a compact box in $\mathbb{R}^{n+1}_+ \cap N_\alpha$, such that $\bigcup_\alpha \pi_\alpha(I_\alpha) = E$ and for all α, ν the function $f_\nu \circ \pi_\alpha|$ reg I_α is a restriction of a function $\tilde{f}_{\nu,\alpha} : N_\alpha \to \mathbb{R}$, which is a K-normal crossing of class C^p, or the zero function.

Remarks.

a) Boxes I_α do not have to be full-dimensional.

b) If each $(E_i, \{f_\nu\}_\nu)$ is a p-smooth $(i = 1, \ldots, k.)$ and $E = \bigcup_{i=1}^k E_i$, then $(E, \{f_\nu\}_\nu)$ is p-smooth.

c) By Remark (2) from chapter 4.1, we only need to show that nonzero functions $\tilde{f}_{\nu,\alpha}$ are quotients of K-normal crossings.

d) If f_{ν_0} is continuous on E, then, for every α, $\tilde{f}_{\nu_0,\alpha}|I_\alpha = f_{\nu_0} \circ \pi_\alpha|I_\alpha$.

Lemma 4.2. If $n = 0$, then every pair $(E, \{f_\nu\}_\nu)$ is p-smooth.

Proof. We may assume that $E = [t_1, t_2]$ $(t_1, t_2 \in \mathbb{R})$ and $\{f_\nu\}_\nu$ is reducible on (t_1, t_2):

$$f_\nu(y) = |y - \theta|^{\gamma_\nu} \cdot A_\nu \cdot V_\nu(|y - \theta|),$$

where $\gamma_\nu \in K$; $\theta, A_\nu \in \mathbb{R}$; V_ν is a K-analytic unit on an open interval containing $\phi(E) \subset \mathbb{R}$, where $\phi : \mathbb{R} \ni y \mapsto |y - \theta| \in \mathbb{R}$.

Let

$$\pi : \mathbb{R} \ni y_1 \mapsto \epsilon \cdot |y_1|^l + \theta \in \mathbb{R},$$

where $\epsilon = \operatorname{sgn}\left(\frac{1}{2}(t_1 + t_2) - \theta\right)$ and l is some sufficiently large positive even integer.

Put $I = \tilde{\phi}(E)$, where $\tilde{\phi}(y) = |\phi(y)|^{\frac{1}{l}}$. Take an open interval $W \supset I$ such that every $V_\nu(|\cdot|^l)$ is defined on W.

Each $f_\nu \circ \pi$ is equal on the interior of I to a function

$$y_1 \mapsto |y_1|^{l\cdot\gamma_\nu} \cdot A_\nu \cdot V_\nu(|y_1|^l),$$

which is zero or a K-normal crossing of class C^p on W (for sufficiently large l), and π is a polynomial mapping. We take the family $\{((\pi|W), I)\}$. □

Lemma 4.3. If E is fat (i.e. $E = \overline{\text{int } E}$), then $(E, \{f_\nu\}_\nu)$ is p-smooth with full-dimensional boxes I_α.

Proof. We use induction on n. The case $n = 0$ was proved in Lemma 4.2. Let $n > 0$ and let us assume that the Lemma holds for $n - 1$.

By Corollary 3.4 and Remark b) above, we can assume that $E = \overline{\mathcal{C}}$ where $(\mathcal{C}, \mathcal{R})$ is a full-dimensional cell with a reducer on which the family $\{f_\nu\}_\nu$ is reducible. In particular, on $\mathcal{C}$, we have

$$f_\nu(x, y) = |y - \theta(x)|^{\gamma_\nu} A_\nu(x) V_\nu(\psi(x, y)), \text{ where}$$
$$\psi(x, y) = \left(\phi_1(x), \ldots, \phi_s(x), \left|\frac{y - \theta(x)}{a(x)}\right|^{\lambda_1}, \ldots, \left|\frac{y - \theta(x)}{a(x)}\right|^{\lambda_r}, \left|\frac{b(x)}{y - \theta(x)}\right|^{\tilde{\lambda}_1}, \ldots, \left|\frac{b(x)}{y - \theta(x)}\right|^{\tilde{\lambda}_q}\right).$$

Extracting some bounded functions as new functions $\phi_i(x)$, we can assume that

$$a(x) = \max\{|t_1(x) - \theta(x)|, |t_2(x) - \theta(x)|\},$$

$$b(x) = \min\{|t_1(x) - \theta(x)|, |t_2(x) - \theta(x)|\},$$

where t_1 and t_2 are functions generating the cylinder $\mathcal{C}$. (We can assume that the number of functions $\phi_i(x)$ remains unchanged.)

Using again Corollary 3.4, we can assume that on $\mathcal{C}$ for each ν either A_ν or its reciprocal is bounded, and denote this bounded function by A'_ν.

Put $b'(x) = a(x) - b(x)$. Denote by $\{g_\mu\}_\mu$ the family

$$\{a(x), b(x), b'(x), \theta(x), \phi_1(x), \ldots, \phi_s(x)\} \cup \{A'_\nu(x)\}_\nu$$

of bounded K-subanalytic functions on the basis $\mathcal{C}'$ of the cell $\mathcal{C}$.

We use the induction assumption for $(\overline{\mathcal{C}'}, \{g_\mu\}_\mu)$. There exists a finite family of pairs (Φ_β, J_β), where $\Phi_\beta : M_\beta \to \mathbb{R}^n$ (M_β open in $\mathbb{R}^n$) is a K-subanalytic modification of class C^p, a J_β is a full-dimensional box in $\mathbb{R}^n_+ \cap M_\beta$, such that $\bigcup_\beta \Phi_\beta(J_\beta) = \overline{\mathcal{C}'}$ and for all β, μ the function $g_\mu \circ \Phi_\beta |$ reg J_β is a restriction of a function $\tilde{g}_{\mu,\beta}$, which is zero or a K-normal crossing of class C^p on M_β.

The mappings $\Psi_\beta = \Phi_\beta \times id_\mathbb{R}$ are K-subanalytic modifications of class C^p. Let $L_\beta = \{(u, y) \in (\text{ int } J_\beta) \times \mathbb{R} |\ (t_1 \circ \Phi_\beta)(u) < y < (t_2 \circ \Phi_\beta)(u)\}$. Then $\bigcup_\beta \Psi_\beta(\overline{L_\beta}) = \overline{\mathcal{C}}$, so it suffices to prove the Lemma for each pair $(\overline{L_\beta}, \{f_\nu \circ \Psi_\beta\}_\nu)$. Fix β (we will drop this index).

Let J be the compact box $[\zeta_1, \eta_1] \times \ldots \times [\zeta_n, \eta_n]$ and $\mathcal{J} = \{j | \zeta_j = 0\}$, $K^{\mathcal{J}}_+ = \{\gamma \in K^n_+ |\ \gamma_j \neq 0 \Rightarrow j \in \mathcal{J}\}$. Set

$$\tilde{b}(u) = l_1(u)|u|^{c_1},$$

$$\tilde{a}(u) = l_2(u)|u|^{c_2},$$

$$\tilde{b}'(u) = l_3(u)|u|^{c_3},$$

where $c_1, c_2, c_3 \in K^{\mathcal{J}}_+$, functions l_2, l_3 are K-analytic positive units on M, and so is l_1 if is not zero (in this case $c_1 = c_2$).

Set

$$C^1 = \{(u, y_1) \in (\text{ int } J) \times \mathbb{R}_+ |\ \tilde{b}(u) < y_1 < \tilde{a}(u)\}$$

$$\text{and } \pi : M \times \mathbb{R} \ni (u, y_1) \mapsto (u, \epsilon \cdot y_1 + \tilde{\theta}(u)) \in M \times \mathbb{R},$$

where $\epsilon =$ sgn $(\frac{1}{2}(t_1(x) + t_2(x)) - \theta(x))$ for $x \in \mathcal{C}'$ (does not depend on x).

Then $\pi(\overline{C^1}) = \overline{L}$ and

$$\begin{aligned} \hat{f}_\nu(u, y_1) &= (f_\nu \circ \Psi \circ \pi |\ C^1)(u, y_1) \\ &= |y_1|^{\gamma_\nu} \tilde{A}_\nu(u) V_\nu(\tilde{\phi_1}(u), \ldots, \tilde{\phi_s}(u), |\frac{y_1}{\tilde{a}(x)}|^{\lambda_1}, \ldots, |\frac{y_1}{\tilde{a}(x)}|^{\lambda_r}, |\frac{\tilde{b}(x)}{y_1}|^{\tilde{\lambda}_1}, \ldots, |\frac{\tilde{b}(x)}{y_1}|^{\tilde{\lambda}_q}), \end{aligned}$$

where

$$\tilde{A}_\nu = \begin{cases} \tilde{A'}_\nu & \text{when } A'_\nu = A_\nu \\ \frac{1}{\tilde{A}'_\nu} & \text{when } A'_\nu = \frac{1}{A_\nu} \end{cases}.$$

By the induction assumption, $\tilde{A}_\nu(u) = |u|^{\delta_\nu} h_\nu(u)$ with $\delta_\nu \in K^{\mathcal{J}}$ and $h_\nu = 0$ or h_ν are K-analytic units on M. Extracting the units from $\frac{1}{\tilde{a}}, \tilde{b}$ as new functions $\tilde{\phi}_i$ and joining h_ν with V_ν, we get

$$\hat{f}_\nu(u, y_1) = |y_1|^{\gamma_\nu} |u|^{\delta_\nu} \hat{V}_\nu(u, \frac{y_1}{|u|^{c_2}}, \frac{|u|^{c_1}}{y_1}),$$

where $\hat{V}_\nu$ are zeroes or K-analytic units. By boundedness of the function $\frac{\tilde{b}(u)}{\tilde{a}(u)}$ on J, we have $c_2 \leq c_1$. Put

$$C^2 = \{(u, y_2) \in (\text{ int } J) \times \mathbb{R} |\, l_1(u)\, |u|^{c_1 - c_2} < y_2 < l_2(u)\}$$

and

$$R_1 : \mathbb{R}^{n+1} \ni (u, y_2) \mapsto (u, y_2 \cdot |u|^{c_2}) \in \mathbb{R}^{n+1}.$$

Then $R_1(\overline{C^2}) = \overline{C^1}$ and

$$(\hat{f}_\nu \circ R_1 |\, C^2)(u, y_2) = |y_2|^{\gamma_\nu} |u|^{\omega_\nu} \hat{V}_\nu(u, y_2, \frac{|u|^{c_1 - c_2}}{y_2}),$$

where $\omega_\nu = \delta_\nu + \gamma_\nu \cdot c_2$.

Case 1: $\tilde{b}(u) \neq 0$, $c_1 = c_2$.

Then y_2 is a unit on C^2 and $l_2 - l_1 = l_3 |u|^{c_3 - c_2}$ $(c_3 \geq c_2)$.

Let us introduce the mappings

$$T : M \times \mathbb{R} \ni (u, y_3) \mapsto (u, y_3 \cdot l_3(u) + l_1(u)) \in M \times \mathbb{R}$$

$$R_2 : \mathbb{R}^{n+1} \ni (u, y_4) \mapsto (u, y_4 \cdot |u|^{c_3 - c_2}) \in \mathbb{R}^{n+1}$$

and cells

$$\begin{aligned} C^3 &= \{(u, y_3) \in (\text{ int } J) \times \mathbb{R} |\, 0 < y_3 < |u|^{c_3 - c_2}\} \\ C^4 &= \{(u, y_4) \in (\text{ int } J) \times \mathbb{R} |\, 0 < y_4 < 1\}. \end{aligned}$$

Then $T(\overline{C^3}) = \overline{C^2}$, $R_2(\overline{C^4}) = \overline{C^3}$ and

$$(\hat{f}_\nu \circ R_1 \circ T \circ R_2 |\, C^4)(u, y_4) = |u|^{\omega_\nu} |y_2(u, y_4)|^{\gamma_\nu} \hat{V}_\nu(u, y_2(u, y_4), \frac{1}{y_2}(u, y_4)).$$

Each of the functions $(\hat{V}_\nu(u, y_2, \frac{1}{y_2}))(u, y_4)$ is zero or a K-analytic unit on a common open K-subanalytic neighbourhood $\hat{W}$ of a set $\overline{C^4}$. Now we get the result by substituting a K-power mapping with sufficiently large exponents.

Case 2: $\tilde{b}(u) = 0$.

Then $\hat{V}_\nu$ does not depend on the last variable. Applying the mapping

$$T' : M \times \mathbb{R} \ni (u, y_3) \mapsto (u, y_3 \cdot l_2(u)) \in M \times \mathbb{R}$$

we get

$$T'(\overline{C^3}) = \overline{C^2} \text{ and } (\hat{f}_\nu \circ R_1 \circ T')(u, y_3) = |y_3|^{\gamma_\nu} |u|^{\omega_\nu} |l_2(u)|^{\gamma_\nu} \hat{\hat{V}}_\nu(u, y_3)$$

for $(u, y_3) \in C^3 = (\text{ int } J) \times (0,1)$, where $\hat{\hat{V}}_\nu$ are zeroes or K-analityc units on a common open K-subanalytic neighbourhood $\hat{W}$ of a set $\overline{C^3}$. Here also we get the result by substituting a K-power mapping with sufficiently large exponents.

Case 3: $c_1 \neq c_2$ and $\tilde{b}(u) \neq 0$.

We put

$$C^3 = \{(u, y_3) \in \text{ int } J \times \mathbb{R} : \, |u|^{c_1 - c_2} < y_3 < \frac{l_2}{l_1}(u)\} \text{ and}$$

$$T'' : M \times \mathbb{R} \ni (u, y_3) \mapsto (u, y_3 \cdot l_1(u)) \in M \times \mathbb{R}.$$

Then

$$T''(\overline{C^3}) = \overline{C^2} \text{ and } (\hat{f}_\nu \circ R_1 \circ T''| \, C^3)(u, y_3) = |y_3|^{\gamma_\nu} |u|^{\omega_\nu} \hat{\hat{V}}_\nu(u, y_3, \frac{|u|^{c_1 - c_2}}{y_3}).$$

Making, if necessary, a decomposition of the cell C^3, we can assume that $|\eta|^{c_1 - c_2} < \frac{l_2}{l_1}(u)$ on C^3 or else $\mathcal{J} = \emptyset$, what reduces the situation to Case 1.

Let $0 = \kappa_0 < \kappa_1 < \ldots < \kappa_k = c_1 - c_2$, where $\kappa_j \in K_+^{\mathcal{J}}$ and $\kappa_j - \kappa_{j-1} = \rho_j \cdot e_{i(j)}$, where $\rho_j \in K_+^*$ and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the i-th place. We set

$$C_0^3 = \{(u, y_3) \in \text{ int } J \times \mathbb{R} : \, |\eta|^{\kappa_k} < y_3 < \frac{l_2}{l_1}(u)\}$$

$$C_j^3 = \{(u, y_3) \in \text{ int } J \times \mathbb{R} : \, |\eta|^{\kappa_k - \kappa_j} |u|^{\kappa_j} < y_3 < |\eta|^{\kappa_k - \kappa_{j-1}} |u|^{\kappa_{j-1}}, j = 1, \ldots, k.$$

It suffices to prove the Lemma for each pair $(\overline{C_j^3}, \{\hat{f}_\nu \circ R_1 \circ T''\}_\nu)$, $j = 0, \ldots, k$.

For $j = 0$, Case 1 holds.

For $j = 1, \ldots, k$, we put

$$C_j^4 = \{(u, y_4) \in \text{ int } J \times \mathbb{R} : \, |\eta|^{\kappa_k - \kappa_j} |u_{i(j)}|^{\rho_j} < y_4 < |\eta|^{\kappa_k - \kappa_j} |\eta_{i(j)}|^{\rho_j}\}$$

$$R_2 : \mathbb{R}^{n+1} \ni (u, y_4) \mapsto (u, y_4 \cdot |u|^{\kappa_{j-1}}) \in \mathbb{R}^{n+1}.$$

Then $R^2(\overline{C_j^4}) = \overline{C_j^3}$ and

$$(\hat{f}_\nu \circ R_1 \circ T'' \circ R_2 | \, C_j^4)(u, y_4) = |y_4|^{\gamma_\nu} |u|^{\omega'_\nu} H_\nu(u, y_4, \frac{|u_{i(j)}|^{\rho_j}}{y_4}),$$

where $\omega'_\nu = \omega_\nu + \gamma_\nu \cdot |u|^{\kappa_{j-1}}$, H_ν is zero or a K-analytic unit in a neighbourhood W of the set $\overline{\hat{\psi}(C_j^4)}$, where $\hat{\psi}(u, y_4) = (u, y_4, \frac{|u_{i(j)}|^{\rho_j}}{y_4})$. Moreover let

$$C_j^5 = \{(u, y_5) \in \text{ int } J \times \mathbb{R} : \, |u_{i(j)}|^{\rho_j} < y_5 < |\eta_{i(j)}|^{\rho_j}\}$$

$$T''' : \mathbb{R}^{n+1} \ni (u, y_5) \mapsto (u, y_5 \cdot |\eta|^{\kappa_k - \kappa_j}) \in \mathbb{R}^{n+1}.$$

Then $T'''(\overline{C_j^5}) = \overline{C_j^4}$. We can assume that $i(j) = n$. Set $J' = [\zeta_1, \eta_1] \times \ldots \times [\zeta_{n-1}, \eta_{n-1}]$ and

$$\hat{C}_j^5 = \{(\hat{u}, y_5) \in \text{ int } J' \times \mathbb{R}^2 |\, 0 < \hat{u}_n < 1, 0 < y_5 < \eta_n\}$$

$$\hat{R} : \mathbb{R}^{n+1} \ni (\hat{u}, y_5) \mapsto (\hat{u}_1, \ldots, \hat{u}_{n-1}, |\hat{u}_n \cdot y_5|^{\frac{1}{\rho_j}}, y_5) \in \mathbb{R}^{n+1}.$$

Then $\hat{R}(\overline{\hat{C}_j^5}) = \overline{C_j^5}$ and

$$(\hat{f}_\nu \circ R_1 \circ T'' \circ R_2 \circ T''' \circ \hat{R}|\, \hat{C}_j^5)(\hat{u}, y_5) = |y_5|^{l_\nu'} |\hat{u}|^{\omega_\nu'} H_\nu^\star(\hat{u}_1, \ldots, \hat{u}_{n-1}, |\hat{u}_n \cdot y_5|^{\frac{1}{\rho_j}}, y_5, \hat{u}_n),$$

where $l_\nu' = \gamma_\nu + (\omega_\nu')_n$. Thus the above functions are of the form $|\hat{z}|^{\sigma_\nu} \hat{H}_\nu(\hat{z})$, where $\hat{z} = (\hat{u}, y_5)$, $\sigma_\nu \in K^{n+1}$, and every $\hat{H}_\nu$ is zero or a K-analytic unit in a common open K-subanalytic neighbourhood $\hat{W}$ of the set $\overline{\hat{C}_j^5}$. By substituting a K-power mapping with sufficiently large exponents, the result follows. □

Lemma 4.4. For a bounded cell with a reducer $(\mathcal{C}, \mathcal{R})$ in $\mathbb{R}^{n+1}$ with the functions t_1, t_2 generating $\mathcal{C}$ as a cylinder, if the pair $(\overline{\mathcal{C}'}, \{t_1, t_2\})$ is p-smooth, then for any finite family $\{f_\nu\}_\nu$ of K-subanalytic functions bounded on $\overline{\mathcal{C}}$, the pair $(\overline{\mathcal{C}}, \{f_\nu\}_\nu)$ is p-smooth.

Proof. By the assumption, there exists a finite family of pairs (Φ_β, J_β), where $\Phi_\beta : M_\beta \to \mathbb{R}^n$ is a K-subanalytic modification of class C^p, J_β is a box in $\mathbb{R}^n_+ \cap M_\beta$, such that $\bigcup_\beta \Phi_\beta(J_\beta) = \overline{\mathcal{C}'}$ and for each β the functions $t_i \circ \Phi_\beta|$ reg J_β $(i = 1, 2)$ are restrictions of functions $\tilde{t}_{i,\beta}$ that are zeroes or K-normal crossings of class C^p on M_β.

Let $\Psi_\beta = \Phi_\beta \times id_{\mathbb{R}}$. The box J_β is isomorphic to a fat box $\tilde{J}_\beta$ in $\mathbb{R}^k$ $(0 \leq k \leq n)$ through the mapping $L_\beta : \mathbb{R}^k \supset \tilde{J}_\beta \to J_\beta \subset \mathbb{R}^n$. Let $L_{\beta,1} = L_\beta \times id_{\mathbb{R}}$. We can assume that $J_\beta = \{b\} \times \tilde{J}_\beta$, with $b \in \mathbb{R}^{n-k}_+$.

Case 1. C is of the form $\{(x, y) \in B \times \mathbb{R}|\, t_1(x) = y = t_2(x)\}$.

We define

$$C_\beta = \{(u, y) \in (\text{reg } J_\beta) \times \mathbb{R}|\, (t_1 \circ \Phi_\beta)(u) = y\}$$
$$\tilde{C}_\beta = \text{ int } \tilde{J}_\beta \subset \mathbb{R}^k.$$

Then $\bigcup_\beta \Psi_\beta \circ L_{\beta,1} \circ (id_{\mathbb{R}^k}, \tilde{t}_{1,\beta})(\tilde{J}_\beta) = \overline{C}$.

We apply Lemma 4.3 to each pair $(\tilde{J}_\beta, \{f_\nu \circ \Psi_\beta \circ L_{\beta,1} \circ (id_{\mathbb{R}^k}, \tilde{t})\}_\nu)$, where $\tilde{t}(v) = \tilde{t}_{1,\beta}(b, v)$.

There exists a finite family of pairs $(\hat{\pi}_{\hat{\alpha}}, \hat{I}_{\hat{\alpha}})$, where $\hat{\pi}_{\hat{\alpha}} : \hat{N}_{\hat{\alpha}} \to \mathbb{R}^k$ is a K-subanalytic modification of class C^p, $\hat{I}_{\hat{\alpha}}$ is a fat box in $\mathbb{R}^k_+ \cap \hat{N}_{\hat{\alpha}}$, such that $\bigcup_{\hat{\alpha}} \hat{\pi}_{\hat{\alpha}}(\hat{I}_{\hat{\alpha}}) = \tilde{J}_\beta$ and for every $\hat{\alpha}$ and ν the function $f_\nu \circ \Psi_\beta \circ L_{\beta,1} \circ (id_{\mathbb{R}^k}, \tilde{t}) \circ \hat{\pi}_{\hat{\alpha}}|$ int $\hat{I}_{\hat{\alpha}}$ is a restriction of the function $\tilde{f}_{\nu,\hat{\alpha}}$, which is zero or a K-normal crossing of class C^p on $\hat{N}_{\hat{\alpha}}$.

Let $N_\alpha = \mathbb{R}^{n-k} \times \hat{N}_{\hat{\alpha}} \times \mathbb{R}$, $I_\alpha = \{b\} \times \hat{I}_{\hat{\alpha}} \times \{0\}$. The mapping $\pi_\alpha = \Psi_\beta \circ \Gamma \circ (id_{\mathbb{R}^{n-k}} \times \hat{\pi}_{\hat{\alpha}} \times id_{\mathbb{R}}) : N_\alpha \to \mathbb{R}^{n+1}$, where $\Gamma(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, x_{n+1} + \tilde{t}(x_{n-k+1}, \ldots, x_n))$, is a K-subanalytic modification of class C^p.

Then $\bigcup_\alpha \pi_\alpha(I_\alpha) = \overline{C}$ and each of functions

$$f_\nu \circ \pi_\alpha|\text{ reg } I_\alpha = f_\nu \circ \Psi_\beta \circ L_{\beta,1} \circ (id_{\mathbb{R}^k}, \tilde{t}) \circ \hat{\pi}_{\hat{\alpha}} \circ \rho'|\text{ reg } I_\alpha,$$

where $\rho' : \mathbb{R}^{n+1} \ni (w, v, y) \mapsto v \in \mathbb{R}^k$, is zero or a restriction of a K-normal crossing on N_α. This proves that $(\overline{C}, \{f_\nu\}_\nu)$ is p-smooth.

Case 2. C is of the form $\{(x, y) \in B \times \mathbb{R} |\, t_1(x) < y < t_2(x)\}$.

The proof of Case 2 is similar to the proof of Case 1.

□

Corollary 4.5. Each pair $(E, \{f_\nu\}_\nu)$ is p-smooth. Moreover, all π_α's can be taken as combinations of K-normal crossings.

Proof. For the first part of statement, use Corollary 3.4, Remark b) and (recursively) Lemma 4.4. For the second part, it suffices to use the fact that $(I_\alpha, \{f_\nu \circ \pi_\alpha\} \cup \{\pi_{\alpha,1}, \ldots, \pi_{\alpha,n+1}\})$ are p-smooth pairs for all α.

4.3 Rectilinearization and uniformization theorems

Theorem 4.6 (on rectilinearization in $\mathbb{R}^n$). For any $p \in \mathbb{N}$, a finite family of K-subanalytic bounded sets $\{E_\nu\}_\nu$ in $\mathbb{R}^n$ and a family of K-subanalytic continuous functions $\{f_\nu : E_\nu \to \mathbb{R}\}_\nu$ there exists a finite family of mappings $\Phi_\alpha : D^{l(\alpha)} \to \mathbb{R}^n$ such that:

a) every Φ_α is a composition of an inclusion $D^{l(\alpha)} \to D^n$ and a K-subanalytic modification of class C^p, which is a K-normal crossing,

b) there are compact boxes $L_\alpha \subset D^{l(\alpha)}$ such that $\bigcup_\alpha \Phi_\alpha(L_\alpha)$ is a neighbourhood of $\bigcup_\nu \overline{E_\nu}$ and each of $\overline{E_\nu}$ is a union of some $\Phi_\alpha(L_\alpha)$,

c) each of the sets $\Phi_\alpha^{-1}(E_\nu)$ is a Boolean combination of "hyperplanes" $H_i^{l(\alpha)} = \{x \in D^{l(\alpha)} : x_i = 0\}$,

d) if $\Phi_\alpha^{-1}(E_\nu)$ is dense in $D^{l(\alpha)}$, then every $f_\nu \circ \Phi_\alpha : \Phi_\alpha^{-1}(E_\nu) \to \mathbb{R}$ is a restriction of a function $f_\nu^\star : D^{l(\alpha)} \to \mathbb{P}$, which is the zero function, a K-normal crossing of class C^p or the reciprocal of a K-normal crossing of class C^p.

Proof. By restricting to smaller sets E_ν and taking reciprocals if needed, we can assume that each f_ν is bounded. Let F be a compact and K-subanalytic neighbourhood of the set $\bigcup_\nu \overline{E_\nu}$.

Each of the sets E_ν is a Boolean combination of K-subanalytic compact sets E_ν^j, $j = 1, \ldots, r_\nu$. We consider a family of continuous K-subanalytic functions $d_\nu^j = \operatorname{dist}(\cdot, E_\nu^j)$ on F, which we add to the family $\{f_\nu\}_\nu$.

By Corollary 3.5, there exists a stratification of F into K-subanalytic analytic cells compatible with the family of sets $\{E_\nu\}$. For a stratum S of this stratification, the pair $(\overline{S}, \{f_\nu\})$ is p-smooth: there exists a finite family of pairs (ϕ_β, I_β), where $\phi_\beta : M_\beta \to \mathbb{R}^n$ is a K-subanalytic modification of class C^p and a combination of K-normal crossings, I_β is a compact box in $M_\beta \cap \mathbb{R}^n_+$ such that $\bigcup_\beta \phi_\beta(I_\beta) = \overline{S}$, and for any ν, β the function $f_\nu \circ \phi_\beta |\text{reg } I_\beta$ is a restriction of some function $\tilde{f}_{\nu,\beta} : M_\beta \to \mathbb{R}$ which is zero or a K-normal crossing of class C^p. We can additionally have $\bigcup_\beta \phi_\beta(\text{reg } I_\beta) \subset S$.

If $S \cap E_\nu = \emptyset$, then, $\phi_\beta^{-1}(E_\nu) \cap \operatorname{reg} I_\beta = \emptyset$ for each β.

If $S \subset E_\nu$, then, by the continuity of functions f_ν on E_ν,

$$f_\nu \circ \phi_\beta = \tilde{f}_{\nu,\beta} \text{ on } \phi_\beta^{-1}(E_\nu) \cap I_\beta.$$

Moreover, $d_\nu^j \circ \phi_\beta = \tilde{d}_{\nu,\beta}^j$ on I_β. Thus each of the sets $(\phi_\beta|I_\beta)^{-1}(E_\nu^j) = (d_\nu^j \circ \phi_\beta|I_\beta)^{-1}(0)$ is of the form $I_\beta \cap H$, where H is a union of hyperplanes $\{x \in \mathbb{R}^n : x_i = 0\}$ or the whole space $\mathbb{R}^n$.

We apply Lemma 4.1 on each box I_β, and the theorem follows. □

The above version of the rectilinearization theorem can be moved to (Hausdorff, second countable) analytic manifolds with the analytic-geometric category of *locally K-subanalytic sets* corresponding to the o-minimal structure of K-subanalytic sets (see [4]):

Theorem 4.7 (rectilinearization theorem). Let M be an analytic manifold of dimension n, $\{E_\nu\}_\nu$ a locally finite family of locally K-subanalytic subsets of M, and $\{f_\nu : E_\nu \to \mathbb{R}\}_\nu$ a family of continuous functions whose graphs are locally K-subanalytic in $M \times \mathbb{P}$. Then there exists a locally finite family of mappings $\Phi_\alpha : D^{l(\alpha)} \to M$ such that:

a) every Φ_α is a composition of an inclusion $D^{l(\alpha)} \to D^n$, a K-subanalytic modification of class C^p, and an inverse chart of M,

b) there are compact boxes $L_\alpha \subset D^{l(\alpha)}$ such that $\bigcup_\alpha \Phi_\alpha(L_\alpha) = M$ and each of $\overline{E_\nu}$ is a union of some $\Phi_\alpha(L_\alpha)$,

c) each of sets $\Phi_\alpha^{-1}(E_\nu)$ is a Boolean combination of "hyperplanes" $H_i^{l(\alpha)} = \{x \in D^{l(\alpha)} : x_i = 0\}$,

d) if $\Phi_\alpha^{-1}(E_\nu)$ is dense in $D^{l(\alpha)}$, then $f_\nu \circ \Phi_\alpha : \Phi_\alpha^{-1}(E_\nu) \to \mathbb{R}$ is a restriction of the function $f_\nu^\star : D^{l(\alpha)} \to \mathbb{P}$ which is the zero function, a K-normal crossing of class C^p or the reciprocal of a K-normal crossing of class C^p.

Proof. Take a locally finite stratification of M compatible with all E_ν with all strata locally K-subanalytic, relatively compact, and contained with their closures in domains of charts (see D.11 and D.12 in [4]). Use Theorem 4.6 in respective coordinate systems. □

For analytic manifolds M, N, we say that a mapping $F : N \to M$ is *locally a K-normal crossing* if for each $x \in N$ there are charts ϕ around x, and ψ around $F(x)$ such that $\psi \circ F \circ \phi^{-1}$ is a K-normal crossing.

Theorem 4.8 (uniformization theorem). Let $p \in \mathbb{N}$, and E be a closed and locally K-subanalytic set of dimension k on an analytic manifold M. Then there exists an analytic manifold N of dimension k and a proper mapping $f : N \to M$ which is locally a K-normal crossing of class C^p and such that $f(N) = E$.

Proof. The set E is a locally finite union of compact, locally K-subanalytic sets E_n contained in the domains of charts ψ_n. Apply Corollary 4.5 to $\psi_n(E_n)$ and the empty set of functions, and use projections of the torus $\mathbb{P}^k$ onto the boxes I_α. □

5 Subanalytic case

In the subanalytic case, we can make parallel reasonings and obtain versions of classical rectilinearization and uniformization. It is due to the fact that rational exponents appearing in the Parusiński-Lion-Rolin formula always have a common divisor. (Here, instead of exponents from K_+, we use exponents from $\mathbb{N}$, and define normal crossings, subanalytic modifications, and related notions analogously.)

Theorem 5.1 (rectilinearization theorem). *(cf. Desingularization II in [5], thm. 0.2 in [1], thm. 2.6 and thm. 2.7 in [8])* Let M be an analytic manifold of dimension n, $\{E_\nu\}_\nu$ a locally finite family of subanalytic subsets of M, and let $\{f_\nu : E_\nu \to \mathbb{R}\}_\nu$ be a family of continuous functions whose graphs are subanalytic in $M \times \mathbb{P}$. Then there exists a locally finite family of mappings $\Phi_\alpha : D^{l(\alpha)} \to M$ such that:

a) every Φ_α is a composition of an inclusion $D^{l(\alpha)} \to D^n$, a subanalytic modification, and an inverse chart of M,

b) there exist compact boxes $L_\alpha \subset D^{l(\alpha)}$ such that $\bigcup_\alpha \Phi_\alpha(L_\alpha) = M$ and each of $\overline{E_\nu}$ is a union of some $\Phi_\alpha(L_\alpha)$,

c) each of sets $\Phi_\alpha^{-1}(E_\nu)$ is a Boolean combination of "hyperplanes" $H_i^{l(\alpha)} = \{x \in D^{l(\alpha)} : x_i = 0\}$,

d) if $\Phi_\alpha^{-1}(E_\nu)$ is dense in $D^{l(\alpha)}$, then $f_\nu \circ \Phi_\alpha : \Phi_\alpha^{-1}(E_\nu) \to \mathbb{R}$ is a restriction of a function $f_\nu^\star : \mathbb{R}^{l(\alpha)} \to \mathbb{P}$, which is the zero function, a normal crossing or the reciprocal of a normal crossing. □

Theorem 5.2 (uniformization theorem). *(cf. Desingularization I in [5], thm. 0.1 in [1])* Let E be a closed and subanalytic set of dimension k in an analytic manifold M. Then there exist: an analytic manifold N of dimension k and an analytic proper mapping $f : N \to M$ (locally a normal crossing) such that $f(N) = E$. □

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