

# On subadditive functions and $\psi$ -additive mappings

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**Abstract:** In [4], assuming among others subadditivity and submultiplicativity of a function  $\psi : [0, \infty) \rightarrow [0, \infty)$ , the authors proved a Hyers-Ulam type stability theorem for “ $\psi$ -additive” mappings of a normed space into a normed space. In this note we show that the assumed conditions of the function  $\psi$  imply that  $\psi = 0$  and, consequently, every “ $\psi$ -additive” mapping must be additive.

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## 1 Introduction

This note is motivated by some recent papers [1], [3] and [4] concerning the Hyers-Ulam type stability theorems for  $\psi$ -additive mapping where the subadditive and submultiplicative functions were used. Recall the definition of  $\psi$ -additive mapping from [3]:

Let  $\psi : [0, \infty) \rightarrow [0, \infty)$  be function,  $E_1$  and  $E_2$  normed spaces. A mapping  $F : E_1 \rightarrow E_2$  is called  $\psi$ -additive if there is  $\theta > 0$  such that

$$|F(x + y) - F(x) - F(y)| \leq \theta (\psi(\|x\|) + \psi(\|y\|))$$

for all  $x, y \in E_1$  (cf. [3]).

In [4] the following Hyers-Ulam type stability result is proved:

**Theorem 1.1.** Suppose that  $\psi : [0, \infty) \rightarrow [0, \infty)$  satisfies the following conditions

(1)  $\lim_{t \rightarrow \infty} \frac{\psi(t)}{t} = 0$

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- (2)  $\psi(st) \leq \psi(s)\psi(t)$  for all  $s, t \geq 0$
- (3)  $\psi(s + t) \leq \psi(s) + \psi(t)$  for all  $s, t \geq 0$
- (4)  $\psi$  is monotone increasing
- (5)  $\psi(t) < t$  for all  $t > 0$ .

Then a mapping  $F : E_1 \rightarrow E_2$  of normed space  $E_1$  into a normed space  $E_2$  is  $\psi$ -additive if, and only if, there exist a constant  $c > 0$  and an additive mapping  $T : E_1 \rightarrow E_2$  such that

$$|F(x) - T(x)| \leq c\psi(\|x\|), \quad \text{for all } x \in E_1.$$

The  $\psi$ -additive functions were also considered in [3] and in a recent paper [1].

In this note we prove some properties of subadditive and submultiplicative functions. By applying them we infer that every function  $\psi : [0, \infty) \rightarrow [0, \infty)$  satisfying the conditions 2 and 5 must be the zero function. Thus, every  $\psi$ -additive mapping with  $\psi$  satisfying only these two conditions is additive.

## 2 Some remarks on subadditive and multiplicative functions

We begin with the following

**Proposition 2.1.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following two conditions:

- (i)  $f$  is subadditive, that is

$$f(s + t) \leq f(s) + f(t), \quad s, t \in \mathbb{R};$$

- (ii) there is a  $c \in \mathbb{R}$  such that for all  $t \in \mathbb{R}$ ,

$$f(t) \leq ct,$$

if, and only if,  $f(t) = ct$  for all  $t \in \mathbb{R}$ .

**Proof.** Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies condition (i) and (ii). Hence, for all  $s, t \in \mathbb{R}$ ,

$$f(t) = f((t - s) + s) \leq f(t - s) + f(s) \leq c(t - s) + f(s),$$

whence

$$f(t) - ct \leq f(s) - cs, \quad s, t \in \mathbb{R}.$$

Changing the roles of  $s$  and  $t$ , we get the converse inequality, and consequently,

$$f(t) - ct = f(s) - cs, \quad s, t \in \mathbb{R}.$$

Taking  $s = 0$  gives

$$f(t) - ct = f(0), \quad t \in \mathbb{R}.$$

Setting  $s = t = 0$  in (i) we get  $f(0) \geq 0$  and from (ii) we have  $f(0) \leq 0$ . Thus  $f(0) = 0$ . This shows that

$$f(t) = ct, \quad t \in \mathbb{R}.$$

Since the converse implication is obvious, the proof is complete.

**Remark 2.2.** To show that assumption (i) cannot be replaced by the subadditivity of  $f$  on  $(0, \infty)$  or  $[0, \infty)$ , it is enough to observe that  $f : [0, \infty) \rightarrow [0, \infty)$ , given by  $f(t) := \frac{t}{t+1}$ , is subadditive and satisfies the inequality  $f(t) \leq t$  for all  $t \geq 0$ .

**Corollary 2.3.** A function  $\psi : (0, \infty) \rightarrow (0, \infty)$  satisfies the following two conditions:

(i)  $f$  is submultiplicative, that is

$$\psi(st) \leq \psi(s)\psi(t), \quad s, t > 0;$$

(ii) there is a  $c \in \mathbb{R}$  such that for all  $t > 0$ ,

$$\psi(t) \leq t^c,$$

if, and only if,  $\psi(t) = t^c$  for all  $t > 0$ .

**Proof.** Suppose that  $\psi : (0, \infty) \rightarrow (0, \infty)$  satisfies conditions (i) and (ii). Then  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f := \log \circ \psi \circ \exp$  satisfies the conditions (i)–(ii) of Proposition 2.1. Thus  $f(t) = ct$  for all  $t \in \mathbb{R}$ , and consequently,  $\psi(t) = t^c$  for all  $t > 0$ . The converse implication is obvious.

Assuming here  $c := 1$  we obtain the following

**Corollary 2.4.** If a function  $\psi : [0, \infty) \rightarrow [0, \infty)$  is submultiplicative, that is

$$\psi(st) \leq \psi(s)\psi(t), \quad s, t \geq 0,$$

and

$$\psi(t) < t, \quad t > 0,$$

then  $\psi(t) = 0$  for all  $t \geq 0$ .

**Proof.** Suppose that  $\psi(s) = 0$  for some  $s > 0$ . Then from the submultiplicativity of  $\psi$ , we have

$$\psi(t) = \psi\left(\frac{t}{s}\right) \leq \psi(s)\psi\left(\frac{t}{s}\right) = 0$$

for all  $t \geq 0$ , which shows that  $\psi = 0$  in  $[0, \infty)$ .

Suppose that  $\psi(s) > 0$  for all  $s > 0$ . Then with  $c = 1$ , the function  $\psi|_{(0, \infty)}$  satisfies all the assumptions of Corollary 2.3. Consequently, in the other case, we would obtain  $\psi(t) = t$  for all  $t > 0$ , which contradicts the assumption  $\psi(t) < t$  for all  $t > 0$ .

We shall prove the following

**Proposition 2.5.** If  $f : (0, \infty) \rightarrow [0, \infty)$  is subadditive, that is

$$f(s + t) \leq f(s) + f(t), \quad s, t > 0;$$

and

$$\lim_{t \rightarrow 0} f(t) = 0$$

then the limit

$$\lim_{t \rightarrow 0} \frac{f(t)}{t}$$

exists and

$$\lim_{t \rightarrow 0} \frac{f(t)}{t} = \sup \left\{ \frac{f(t)}{t} : t > 0 \right\}.$$

**Proof.** Set

$$\beta := \sup \left\{ \frac{f(t)}{t} : t > 0 \right\},$$

and assume first that  $\beta < +\infty$ . Take  $\varepsilon > 0$  and  $a > 0$  such that

$$\frac{f(a)}{a} > \beta - \frac{\varepsilon}{2}.$$

For every  $t \in (0, a)$  there is a unique positive integer  $n \in \mathbb{N}$ ,  $n = n(t)$ , such that  $\frac{a}{n+1} \leq t < \frac{a}{n}$ , i.e.

$$\frac{an}{n+1} \leq nt < a, \quad n \in \mathbb{N}.$$

Now, by the definition of  $\beta$  and the subadditivity of  $f$ , we have

$$\beta \geq \frac{f(t)}{t} = \frac{nf(t)}{nt} \geq \frac{f(nt)}{nt} \geq \frac{f(a) - f(a - nt)}{nt} \geq \frac{f(a)}{a} - \frac{f(a - nt)}{\frac{an}{n+1}}.$$

Since  $n = n(t) \rightarrow \infty$  iff  $t \rightarrow 0$ , and

$$0 < a - nt \leq \frac{a}{n+1},$$

we infer that

$$\lim_{t \rightarrow 0} (a - nt) = 0.$$

The assumption that  $\lim_{t \rightarrow 0} f(t) = 0$  implies there is a  $\delta > 0$  such that

$$0 < t < \delta \Rightarrow \frac{f(a - nt)}{\frac{an}{n+1}} < \frac{\varepsilon}{2}.$$

Now for all  $t \in (0, \delta)$  we have

$$\beta \geq \frac{f(t)}{t} \geq \frac{f(a)}{a} - \frac{f(a - nt)}{\frac{an}{n+1}} \geq \left( \beta - \frac{\varepsilon}{2} \right) - \frac{\varepsilon}{2} = \beta - \varepsilon,$$

which proves that  $\lim_{t \rightarrow 0} \frac{f(t)}{t}$  exists and

$$\lim_{t \rightarrow 0} \frac{f(t)}{t} = \beta.$$

In the case when  $\beta = \infty$  we can argue in a similar way replacing  $\beta$  by an arbitrary number  $M > 0$ . This completes the proof.

**Remark 2.6.** Under the measurability assumptions the above result is proved in [2].

**Corollary 2.7.** Suppose that  $f : (0, \infty) \rightarrow [0, \infty)$  is subadditive, that is

$$f(s + t) \leq f(s) + f(t), \quad s, t > 0;$$

and

$$\lim_{t \rightarrow 0} f(t) = 0.$$

Then

$$\liminf_{t \rightarrow 0} \frac{f(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{f(t)}{t} \tag{1}$$

if and only if, there is a  $c \in \mathbb{R}$  such that  $f(t) = ct$  for all  $t > 0$ .

**Proof.** The assumptions imply that  $f$  is bounded on every finite subinterval of  $[0, \infty)$ . By the subadditivity of  $f$  (cf. Hille, Phillips [2, p.244, Theorem7.6.1] there exists a finite limit

$$\alpha := \lim_{t \rightarrow \infty} \frac{f(t)}{t}$$

and

$$\alpha = \inf_{t > 0} \left\{ \frac{f(t)}{t} : t > 0 \right\}.$$

Putting

$$\beta := \sup \left\{ \frac{f(t)}{t} : t > 0 \right\},$$

in view of Proposition 2.5, we have

$$\beta = \lim_{t \rightarrow 0} \frac{f(t)}{t}.$$

Of course we have  $\beta \geq \alpha$ . Condition (1) implies that  $\alpha \geq \beta$  and, consequently,  $\alpha = \beta$ . Setting  $c := \alpha$  we obtain  $f(t) = ct$  for all  $t > 0$ .

### 3 Remarks on $\psi$ -additive maps

From Corollary 2.4 we obtain the following

**Remark 3.1.** Suppose that  $\psi : [0, \infty) \rightarrow [0, \infty)$  satisfies the conditions 2 and 5 of Theorem 1.1. Then every mapping of normed space  $E_1$  into a normed space  $E_2$  is  $\psi$ -additive if, and only if, it is additive.

Applying Proposition 2.5 we get the following

**Remark 3.2.** Let  $\psi : [0, \infty) \rightarrow [0, \infty)$  satisfy the conditions 2 and 3 of Theorem 1.1. Suppose that  $\lim_{t \rightarrow 0} \psi(t) = 0$  and 0 is a cluster point of the set

$$\{t > 0 : \psi(t) < ct\}$$

for some  $c < 1$ . Applying Proposition 2.5 we infer that  $\psi(t) < t$  for all  $t > 0$ , that is, condition 5 is satisfied.

These remarks show that condition 5 of Theorem 1.1 should be either removed or replaced by a weaker one.

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