Analysis

Volume 14 · Issue 1 · 1994

Analysis 14, 1 - 18 (1994)

Analysis

HIGH ORDER ERROR CONSTANTS OF CLENSHAW-CURTIS PRODUCT RULES

Knut Petras

Received: May 12, 1992; revised: August 2, 1993

Abstract: The error of the n^{th} Clenshaw-Curtis product rule can be bounded above by $\varrho_n \|f^{(n)}\|_\infty$. Asymptotically best possible estimates depending on the class of weight functions are given for the error constants ϱ_n. The method is based on a result of Braß and Förster.

AMS (1980) Subject Classification: 65 D 30, 41 A 55

1. Introduction

Consider the problem of numerical calculation of the integral

$$I[f] = \int_{-1}^{1} f(x)w(x)\,dx, \tag{1.1}$$

where w is a given integrable weight function. Beside Gauss type rules, i.e., Gauß, Lobatto and Radau rules, another quadrature method of interpolatory type frequently occurs in the present literature – the Clenshaw-Curtis rule. For arbitrary, not necessarily positive weight functions w, the n^{th} Clenshaw-Curtis (product) formula

$$Q_n^{CC}[f] = \sum_{\nu=1}^{n} a_\nu f(x_\nu) \tag{1.2}$$

is defined uniquely by having the nodes $x_\nu = -\cos(\nu-1)\pi/(n-1)$ and by its interpolatory property

$$(1.3) \qquad Q_n^{CC}[p] = I[p] = \int_{-1}^{1} p(x)w(x)\,dx, \qquad \text{for each } p \in \mathcal{P}_{n-1},$$

where $\mathcal{P}_{n-1}$ denotes the space of all polynomials of degree less than or equal to $n-1$. A first reason, why to prefer the Clenshaw-Curtis formulae against Gauss type quadrature formulae is simply that the latter may not exist if w has sign changes. The slightly higher computational effort for the evaluation of the Gaussian formulae cannot play a serious role when using computers.

However, the main advantages of the Clenshaw-Curtis rule may occur in situations, when integrals with respect to different weight functions have to be computed for a fixed given function f. Using Gaussian formulae, the function values of f have to be computed again for each integral, since the nodes may not coincide for different weight functions. This is of particular importance if the effort for the evaluation of function values is essentially higher than that of the summation of the quadrature formula. The latter reason justifies the use of a fixed nodal system.

The main argument for the special choice of Clenshaw-Curtis nodes originates in the Lebesgue inequality,

$$(1.4) \qquad |R_n^{CC}[f]| \le \|R_n^{CC}\| \cdot E_{n-1}[f],$$

where $R_n^{CC} := I - Q_n^{CC}$ is the remainder functional which corresponds to the Clenshaw-Curtis formula. Here, $E_{n-1}[f]$ denotes the error of the best approximation to f from the space $\mathcal{P}_{n-1}$. Indeed, the Lebesgue inequality in this form holds for each quadrature formula of interpolatory type, but the norms of the corresponding remainder functionals may be quite different. Sloan and Smith [5] have shown the important property

$$(1.5) \qquad \lim_{n\to\infty} \|R_n^{CC}\| = 2\|I\|, \qquad \text{if} \quad \int_{-1}^{1} |w(x)|^p\,dx < \infty \quad \text{for some } p > 1.$$

Estimates using the approximation theoretical approach (1.4) are universally applicable. They therefore often lack some precision.

In this paper we will restrict consideration to error estimates using the supremum norm of the n^{th} derivative of the integrand. The method succeeds for each estimate with a norm of a 'high' order derivative of f.

Defining the error constant

$$(1.6) \qquad \varrho_m(R_n^{CC}, w) := \sup\left\{ |R_n^{CC}[f]| \;\middle|\; \|f^{(m)}\|_\infty \le 1 \right\},$$

we obtain by (1.4) and (1.5) that

$$\varrho_n(R_n^{CC}, w) \leq \frac{\text{const}}{2^n n!} \tag{1.7}$$

holds for each weight function w, which satisfies (1.5). For $w(x) \equiv 1$ we can improve upon (1.7) by a factor of the order n^{-3} (cf. Braß and Förster [2] and Fiedler [3]) and further results of this type are given in the sequel.

Of course, for positive weight functions, the Clenshaw-Curtis rule has to compete with the Gaussian rule, $(R_n^G)_{n\in\mathbb{N}}$, which satisfies

$$\varrho_n(R_n^G, w) \leq \frac{1}{\{[1+o(1)]3\sqrt{3}\}^n n!} \tag{1.8}$$

(cf. Braß and Förster [2]). We may therefore ask , under which assumptions the improvement of the best possible error estimate of the Clenshaw-Curtis rule upon (1.7) is only polynomial for increasing n or under which assumptions it is exponential. Our investigations show that in general, at most a polynomial improvement is possible, while we win a factor of exponential order if $w = w_1 \cdot w_2$, where w_1 is the Gauss-Chebyshev weight function, $w_1(x) = (1-x^2)^{-1/2}$, and w_2 is analytical.

We conjecture that, for each fixed nodal system, there is at most such a small class of weight functions, which yields an exponential improvement upon the standard estimate (1.7), as for the Clenshaw-Curtis rule.

2. The Method

The error constants $\varrho_n(R_n^{CC}, w)$ may be expressed in terms of Peano kernels. For a given linear functional $L : C[-1,1] \to \mathbb{R}$, which vanishes on the space of all polynomials of degree less than m, the Peano kernel $K_m(L,\cdot)$ is defined by

$$K_m(L,x) = L[h_x], \tag{2.1}$$

where

$$h_x(t) = \frac{(t-x)_+^{m-1}}{(m-1)!} = \begin{cases} 0, & \text{if } t \leq x \text{ and} \\ \dfrac{(t-x)^{m-1}}{(m-1)!}, & \text{if } t > x. \end{cases} \tag{2.2}$$

Since L has the representation

$$L[f] = \int_{-1}^{1} f^{(m)}(x) K_m(L,x)\, dx, \tag{2.3}$$

whenever $f^{(m)}$ exists, the error constant is given by

$$\varrho_m(L) = \sup_{\|f^{(m)}\|_\infty \le 1} |L[f]| = \int_{-1}^{1} |K_m(L,x)|\,dx. \tag{2.4}$$

The method, which is used to prove the results in Section 3, is essentially based on a result of Braß and Förster [2]. They have proved a series expansion of $K_m(L,\cdot)$, whose first partial sum $K_{m,1}$ satisfies

$$\varrho_m(L) = \int_{-1}^{1} |K_{m,1}(x)|\,dx + \varepsilon_1, \tag{2.5}$$

where

$$\int_{-1}^{1} |K_{m,1}(x)|\,dx = \frac{1}{m!2^{m-1}}|L[T_m]| \tag{2.6}$$

and

$$|\varepsilon_1| \le \frac{1}{m!2^{m-1}} \left\{ \sum_{\mu=1}^{\infty} \frac{\mu!(2m)!}{(\mu+2m)!} |L[T_{m+\mu}]|^2 \right\}^{1/2} \tag{2.7}$$

(cf. Braß and Förster [2], (3) and (17)). This paper shall show, how the result of Braß and Förster can be used to simplify investigations on high order error constants of the Clenshaw-Curtis rule. We just have to estimate the errors $R_n^{CC}[T_{n+\mu}]$ to obtain reasonable bounds for $\varrho_n(R_n^{CC}, w)$. Fortunately, for the Clenshaw-Curtis formulae, these errors may be expressed rather explicit as follows,

$$\begin{aligned} R_n^{CC}[T_{2l(n-1)+k}] &= \int_{-1}^{1} \left\{ T_{2l(n-1)+k}(x) - T_{|k|}(x) \right\} w(x)\,dx \\ &= -2\int_0^{\pi} \sin l(n-1)t \,\sin\left(l(n-1)+k\right)t \,\sin t \, w(\cos t)\,dt, \qquad |k| \le n-1 \end{aligned} \tag{2.8}$$

(cf. Sloan and Smith [5]). The main part of the expansion (2.5)-(2.7) is mostly given by the errors $R_n^{CC}[T_{n-1+k}]$, $k = 1, 2, \ldots, 2n-2$, where eq. (2.8) reads thus

$$R_n^{CC}[T_{n-1+k}] = -2\int_0^{\pi} \sin(n-1)t \,\sin kt \,\sin t \, w(\cos t)\,dt, \qquad k = 1, 2, \ldots, 2n-2. \tag{2.9}$$

It is useful to interpret the quadrature errors for Chebyshev polynomials as multiples of functionals $I^{(\mu,\kappa)}$ given by

$$-2I^{(\mu,\kappa)}[w] = -2\int_0^{\pi} \sin\mu t \,\sin\kappa t \,\sin t \, w(\cos t)\,dt = R_n^{CC}[T_\nu] \tag{2.10}$$

for appropriate combinations of μ, κ and ν.

For the results in Section 3 we generally assume that w is in a certain class such as $w \in \mathcal{K} = V_s := \{f \mid f^{(s)}$ is of bounded variation$\}$ or that, for instance, $w(x)\sqrt{1-x^2}$ is analytic. Since we want to estimate the integral (2.10) in these cases, we may use the typical methods which are applied in quadrature theory.

For example, if w is supposed to be in V_s we define a quadrature formula $Q_s^{(\mu,\kappa)}$ such that the kernel of $L_s^{(\mu,\kappa)} = I^{(\mu,\kappa)} - Q_s^{(\mu,\kappa)}$ includes $\mathcal{P}_s$. The best possible estimate for $L_s^{(\mu,\kappa)}[w]$ under the only assumption $w \in V_s$ is then

$$|L_s^{(\mu,\kappa)}[w]| \leq \mathrm{Var}(w^{(s)}) \cdot \|K_{s+1}(L_s^{(\mu,\kappa)}, \cdot)\|_\infty, \tag{2.11}$$

where $\mathrm{Var}(f)$ denotes the total variation of f. Using these estimates for $R_n^{CC}[T_\nu]$, we obtain bounds for $\varrho_n(R_n^{CC}, w)$, which are asymptotically best possible.

The method may not only yield such sharp bounds for the error constants $\varrho_n(R_n^{CC}, w)$, but would also succeed at least for $\varrho_\mu(R_n^{CC}, w)$, where $\mu \geq n-s$ and s is fixed.

3. The Results

THEOREM 1.

a) *Let $n \geq 4$ and let w be of bounded variation, then,*

$$\varrho_n(R_n^{CC}, w) \leq \frac{1}{n!\,2^{n-2}(n-3)} \left\{ \mathrm{Var}(w) + \frac{2}{\sqrt{n}} (\mathrm{Var}(w) + 2|w(0)|) \right\}. \tag{3.1}$$

b) *Let $n \geq 5$ and let w' be of bounded variation, then,*

$$\begin{aligned} \varrho_n(R_n^{CC}, w) \leq & \frac{1}{n!\,2^{n-2}(n-4)^2} \Big\{ \mathrm{Var}(w') + \\ & \frac{2}{\sqrt{n}} (\mathrm{Var}(w') + 2|w(1)| + 2|w(-1)|) \Big\}. \end{aligned} \tag{3.2}$$

c) *Let $n \geq 6$ and let w'' be of bounded variation, then,*

$$\begin{aligned} \varrho_n(R_n^{CC}, w) \leq & \frac{1}{n!\,2^{n-2}(n-5)^3} \Big\{ \mathrm{Var}(w'') + 2|w(1)| + 2|w(-1)| \\ & + \frac{2}{\sqrt{n}} \left(\mathrm{Var}(w'') + 3|w(1)| + \frac{32}{3}|w(0)| + 3|w(1)| \right) \Big\}. \end{aligned} \tag{3.3}$$

Remark 1. The bounds are asymptotically best possible in the sense that additional factors on the right-hand side must be greater or equal to $1 + O(n^{-1/2})$.

Remark 2. For $w(x) \equiv 1$, Theorem 1c) yields the estimate

$$\varrho_n(R_n^{CC}, w) \leq \frac{1}{n!\,2^{n-4}(n-5)^3}\left(1 + \frac{25}{3\sqrt{n}}\right), \tag{3.4}$$

which is asymptotically sharp ut to a factor $1 + O(n^{-1/2})$.

Note that the estimate of Theorem 1 c) may not be replaced by one of the form

$$\varrho_n(R_n^{CC}, w) \leq \frac{\text{const} \cdot \text{Var}(w'') + o(1)}{2^n n! n^3} \tag{3.5}$$

for increasing n, since the case $w(x) \equiv 1$ would already yield a contradiction. The next theorem will give a generalization of this observation. For instance, we can conclude from Theorem 2 that a multiple of $\text{Var}(w^{(s)})$ can only be the main term in a reasonable estimate for $\varrho_n(R_n^{CC}, w)$, if $w^{(\nu)}(1) = w^{(\nu)}(-1) = 0$ for all $\nu \leq (s-3)/2$. Furthermore, we see that for weight functions, whose second derivative is of bounded variation, the error constant $\varrho_n(R_n^{CC}, w)$ will in general be only of the order $O((2^n n! n^3)^{-1})$.

<u>**THEOREM 2.**</u> *Let $w^{(s-1)}$ be absolutely continuous on $[-1,1]$, let $w^{(\nu)}(1) = w^{(\nu)}(-1) = 0$ for all $\nu < r \leq (s-3)/2$, and let $|w^{(r)}(1)| + |w^{(r)}(-1)| > 0$, then,*

$$\frac{r! 2^{r-2}}{(2r+2)!} \cdot \lim_{\substack{n\to\infty \\ n \text{ even}}} n! 2^n n^{3+2r}\, \varrho_n(R_n^{CC}, w) = |w^{(r)}(1) + (-1)^r w^{(r)}(-1)| \tag{3.6}$$

and

$$\frac{r! 2^{r-2}}{(2r+2)!} \cdot \lim_{\substack{n\to\infty \\ n \text{ odd}}} n! 2^n n^{3+2r}\, \varrho_n(R_n^{CC}, w) = |w^{(r)}(1) - (-1)^r w^{(r)}(-1)|. \tag{3.7}$$

For the proof of Theorem 2, we simply have to expand the integral (2.10) similar to the Euler-Maclaurin summation formula.

The Clenshaw-Curtis formula works particularly well for the weight function $w(x) = (1-x^2)^{-1/2}$, i.e., for a weight function which has singularities at both ends of the basic interval. We would therefore like to obtain results as in Theorem 1 without assuming w to be bounded.

THEOREM 3. *Consider the function W given by $W(x) = (1-x^2)w(x)$.*

a) Let $n \geq 4$ and let W be of bounded variation, then,

$$\varrho_n(R_n^{CC}, w) \leq \frac{1}{n!\,2^{n-2}(n-1)} \operatorname{Var}(W) \left\{1 + \frac{3}{\sqrt{n}}\right\}. \tag{3.8}$$

b) Let $n \geq 4$ and let W' be of bounded variation, then,

$$\varrho_n(R_n^{CC}, w) \leq \frac{1}{n!\,2^{n-2}(n-2)^2} \operatorname{Var}(W') \left\{1 + \frac{4}{\sqrt{n}}\right\}. \tag{3.9}$$

c) Let $n \geq 4$ and let W'' be of bounded variation, then,

$$\begin{aligned}\varrho_n(R_n^{CC}, w) \leq \frac{3}{n!\,2^{n-1}(n-3)^3} \Big\{ &\operatorname{Var}(W'') + \frac{8}{3}|w(0)| + \\ &+ \frac{10}{3\sqrt{n}} \left(\operatorname{Var}(W'') + 8|w(0)|\right) \Big\}.\end{aligned} \tag{3.10}$$

Remark 1 holds analogously.

From Braß' and Förster's result we can readily deduce that, if we have a positive weight function, the n^{th} error constants of the corresponding Gaussian rule, $Q_n = Q_n^G$, satisfy

$$\limsup_{n\to\infty} \sqrt[n]{n!\,\varrho_n(R_n, w)} \leq \frac{1}{3\sqrt{3}}, \tag{3.11}$$

where, in general, equality holds. Furthermore, if we have any (not necessarily positive) weight function, for which a bounded sequence of quadrature formulae of algebraic degree greater or equal to $2n-k$, k fixed, exists, we can obtain the same constant on the right-hand side of (3.11).

For the classes of weight functions considered in Theorem 2, $1/2$ as a constant on the right-hand side of (3.11) is not improvable for the Clenshaw-Curtis rule. In the case that $w(x) = (1-x^2)^{-1/2}$, the Clenshaw-Curtis rule is the corresponding Lobatto rule and therefore also satisfies the limit relation (3.11) as well as the Gaussian rule. We may thus ask, for which weight functions we can guarantee, that the Clenshaw-Curtis rule has such small error constants as described in (3.11), i.e., can compete with formulae of high algebraic degree. The answer will be that $w(x) = W(x)(1-x^2)^{-1/2}$ with W being analytic in the interior C_r of an ellipse, which is given by its foci -1 and 1 and the sum $r \geq 3\sqrt{3}/2$ of its semi-axis.

THEOREM 4. *Let the function W be defined on the closure of $\mathcal{C}_r$ such that its restriction to $[-1,1]$ is given by*

$$W(z) = \sqrt{1-z^2} \cdot w(z). \tag{3.12}$$

Assuming W to be analytic in $\mathcal{C}_r$ and bounded on $\partial\mathcal{C}_r$, we have

$$\limsup_{n\to\infty} \sqrt[n]{n!\, \varrho_n(R_n^{CC}, w)} \le \max\left\{\frac{1}{2r}, \frac{1}{3\sqrt{3}}\right\}. \tag{3.13}$$

Remark 3. If $2r \le 3\sqrt{3}$ and we assume analyticity of w only in $\mathcal{C}_r \setminus \{z_0\}$, where z_0 lies on the boundary of the ellipse $\mathcal{C}_{r^*}$, $r^* < r$, we obtain $1/2r^*$ as an unimprovable constant on the right-hand side of (3.13). $1/2r^*$ is indeed the constant if $W(z) = 1/(z-z_0)$.

The only situation, in which the limit on the left-hand side of (3.13) may be estimated by a constant less than $1/2$ is similar to that of Theorem 4.

THEOREM 5. *The relation*

$$\limsup_{n\to\infty} \sqrt[n]{n!\, \varrho_n(R_n^{CC}, w)} < \frac{1}{2} \tag{3.14}$$

holds if and only if there is an analytic 2π-periodic function v, which satisfies

$$v(t) = w(\cos t)|\sin t|, \tag{3.15}$$

whenever t is real.

4. Proof of the Results

Proof of Theorem 1: The moments $m_\nu^{(\mu,\kappa)}$ of the functionals $I^{(\mu,\kappa)}$ defined in (2.10) are given by

$$m_\nu^{(\mu,\kappa)} = I^{(\mu,\kappa)}[p_\nu] = \int_0^\pi \sin\mu t\, \sin\kappa t\, \sin t\, \cos^\nu t\, dt, \qquad p_\nu(x) = x^\nu. \tag{4.1}$$

Hence, by the equation $\cos t\, \sin\mu t = (\sin(\mu+1)t + \sin(\mu-1)t)/2$, we obtain the recurrence relation:

$$2m_{s+1}^{(\mu,\kappa)} = m_s^{(\mu-1,\kappa)} + m_s^{(\mu+1,\kappa)}, \tag{4.2}$$

and

$$(4.3)\qquad m_0^{(\mu,\kappa)} = \begin{cases} -\dfrac{2\{1+(-1)^{\mu-\kappa}\}\mu\kappa}{\{(\mu-\kappa)^2-1\}\{(\mu+\kappa)^2-1\}}, & \text{if } |\mu-\kappa| \neq 1 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Defining the quadrature formulae

$$(4.4)\qquad Q_1^{(\mu,\kappa)}[f] = m_0^{(\mu,\kappa)} f(0),$$

$$(4.5)\qquad Q_2^{(\mu,\kappa)}[f] = \frac{m_0^{(\mu,\kappa)} + m_1^{(\mu,\kappa)}}{2} f(1) + \frac{m_0^{(\mu,\kappa)} - m_1^{(\mu,\kappa)}}{2} f(-1)$$

and

$$(4.6)\quad Q_3^{(\mu,\kappa)}[f] = \frac{m_2^{(\mu,\kappa)} + m_1^{(\mu,\kappa)}}{2} f(1) - (m_2^{(\mu,\kappa)} - m_0^{(\mu,\kappa)}) f(0) + \frac{m_2^{(\mu,\kappa)} - m_1^{(\mu,\kappa)}}{2} f(-1),$$

the functionals $L_s^{(\mu,\kappa)} = I^{(\mu,\kappa)} - Q_{s+1}^{(\mu,\kappa)}$, where $s = 0, 1, 2$, map all polynomials of degree less than or equal to s onto zero.

The theorem will almost be proved, if we are able to estimate the corresponding Peano kernels appropriately. They are given by

$$(4.7)\qquad \begin{aligned} K_{s+1}(L_s^{(\mu,\kappa)}, \cos t) = &\int_0^t \frac{(\cos x - \cos t)^s}{s!} \sin \mu x \sin \kappa x \sin x \, dx \\ &- Q_{s+1}^{(\mu,\kappa)} \left[\frac{(\cdot - \cos t)^s}{s!} \right]. \end{aligned}$$

For $s = 0$ and $k \leq n - 3$, we have

$$(4.8)\qquad \begin{aligned} K_1(L_0^{(n-1,k)}, \cos t) = &\frac{1}{4}\left(\frac{1}{n+k-2} - \frac{1}{n+k} - \frac{1}{n-k-2} + \frac{1}{n-k} \right) \\ &- \frac{1}{4}\left(\frac{\cos(n+k-2)t}{n+k-2} - \frac{\cos(n+k)t}{n+k} - \frac{\cos(n-k-2)t}{n-k-2} + \frac{\cos(n-k)t}{n-k} \right), \end{aligned}$$

such that

$$(4.9)\qquad \begin{aligned} |K_1(L_0^{(n-1,k)}, \cos t)| \leq &\frac{1}{4}\left| \frac{1}{n+k-2} - \frac{1}{n+k} - \frac{1}{n-k-2} + \frac{1}{n-k} \right| \\ &+ \frac{1}{4}\left(\frac{1}{n+k-2} + \frac{1}{n+k} + \frac{1}{n-k-2} + \frac{1}{n-k} \right) \\ = &\frac{n-1}{(n+k)(n-k-2)}. \end{aligned}$$

Considering the representation,

$$(4.10)\qquad K_1(L_0^{(n-1,1)},\cos t) = \frac{-2}{(n-3)(n-1)(n+1)} - \frac{1}{n-1}\left(\sin^2 t\cos(n-1)t + \frac{\cos(n+1)t}{2(n+1)} - \frac{\cos(n-3)t}{2(n-3)}\right),$$

we readily see that the estimate (4.9) may be improved at most by a summand of order n^{-2}, if $k=1$.

Analogously, by elementary calculations, we also obtain

$$(4.11)\qquad \begin{aligned} &|K_2(L_1^{(n-1,k)},\cos t)| \\ &\le \frac{1}{4}\left(\frac{1}{n-k-3} - \frac{1}{n-k-2} + \frac{1}{n-k} - \frac{1}{n-k+1} + \frac{2}{n+k-2} - \frac{2}{n+k}\right) \\ &\le \begin{cases} \dfrac{1}{n(n-4)}, & \text{if } k=1 \text{ and} \\ \dfrac{22}{9(n+2)(n-3)}, & \text{if } 2 \le k \le (n-3)/2, \end{cases} \end{aligned}$$

and

$$(4.12)\qquad \begin{aligned} &|K_3(L_2^{(n-1,k)},\cos t)| \\ &\le \frac{1}{16}\left(\frac{1}{n-k-4} - \frac{1}{n-k-3} - \frac{1}{n-k} + \frac{1}{n-k+1} + \right. \\ &\qquad \left. \frac{1}{n+k-3} - \frac{1}{n+k-2} - \frac{1}{n+k+1} + \frac{1}{n+k+2}\right) \\ &\le \frac{k}{n(n-1)(n-5)}, \qquad \text{if } k \le (n-4)/2. \end{aligned}$$

Since in (4.7), μ is always greater or equal to $n-1$, we may estimate the Peano kernels for all occurring combinations of μ and κ by

$$(4.13)\qquad \|K_1(L_0^{(\mu,\kappa)},\cdot)\|_\infty \le 2/3,$$

$$(4.14)\qquad \|K_2(L_1^{(\mu,\kappa)},\cdot)\|_\infty \le 1/3$$

and

$$\|K_3(L_2^{(\mu,\kappa)},\cdot)\|_\infty \le 1/16. \tag{4.15}$$

This estimation requires some distinctions of cases, quite a lot of elementary calculations and numerical investigations for small μ and κ, but this is all standard work and may partially be automated on a computer.

By (4.4) and (4.9), we first obtain

$$|R_n[T_n]| \le \frac{2(n-1)}{(n-3)(n+1)}\left(\operatorname{Var}(w) + \frac{4}{(n-1)^2}|w(0)|\right) \tag{4.16}$$

In order to estimate ε_1, we consider the bound in (4.9) for $k \le (n-3)/2$, which yields

$$|R_n[T_{n+\mu}]| \le \frac{8(n-1)}{(n-3)(3n-1)}\left(\operatorname{Var}(w) + \frac{8(n-1)}{(n+1)(3n-5)}|w(0)|\right), \tag{4.17}$$

such that

$$\begin{aligned} &|n!\,2^{n-2}\varepsilon_1|^2 \\ &\le \left\{\frac{4(n-1)}{(n-3)(3n-1)}\left(\operatorname{Var}(w) + \frac{8(n-1)}{(n+1)(3n-5)}|w(0)|\right)\right\}^2 \sum_{\mu=1}^{\infty} \frac{\mu!(2n)!}{(\mu+2n)!} \\ &\quad + \frac{4}{9}\left(\operatorname{Var}(w) + \frac{54}{35}|w(0)|\right)^2 \sum_{\mu=[(n-1)/2]}^{\infty} \frac{\mu!(2n)!}{(\mu+2n)!}. \end{aligned} \tag{4.18}$$

An explicit expression for the series is given by

$$\sum_{\mu=s}^{\infty} \frac{\mu!}{(\mu+2m)!} = \frac{s!}{(2m+s)!}\,\frac{2m+s}{2m-1} \tag{4.19}$$

(cf. Braß and Förster [2], eq. (17)), such that a further estimation yields the bound (3.1) of Theorem 1.

The bounds in the cases that w' respectively w'' are of bounded variation may be proved analogously. □

Proof of Theorem 2: We first state the following 'Euler-Maclaurin-type' result for the integral (2.10).

LEMMA. *Let*

$$C_{k,m}(x) := C_k(x) := \frac{m}{2^k}\sum_{\mu=0}^{k}\binom{k}{\mu}(-1)^\mu\frac{(m-\mu-1)!}{(m+k-\mu)!}T_{m+k-2\mu}(x), \tag{4.20}$$

and let $w^{(s)}$ *exist, then,*

$$\begin{aligned}\int_{-1}^{1} & w(x)T_m(x)\,dx \\ &= \sum_{\nu=0}^{s-1}(-1)^\nu\Big(C_{\nu+1}(1)w^{(\nu)}(1)-C_{\nu+1}(-1)w^{(\nu)}(-1)\Big) \\ &\qquad + (-1)^s\int_{-1}^{1}C_s(x)w^{(s)}(x)\,dx \\ &= -m\sum_{\nu=0}^{s-1}\frac{(2\nu+1)!(m-\nu-2)!}{2^\nu\nu!(m+\nu+1)!}\Big(w^{(\nu)}(1)+(-1)^{m+\nu}w^{(\nu)}(-1)\Big) \\ &\qquad + (-1)^s\int_{-1}^{1}C_s(x)w^{(s)}(x)\,dx.\end{aligned} \tag{4.21}$$

Proof of the Lemma: We can show by induction that

$$\frac{d^s}{dx^s}C_s(x) = T_m(x) \tag{4.22}$$

and that

$$(-1)^{\nu+1}C_{\nu+1}(1) = (-1)^m C_{\nu+1}(-1) = \frac{m(2\nu+1)!(m-\nu-2)!}{2^\nu\nu!(m+\nu+1)!}. \tag{4.23}$$

(4.21) is therefore readily proved by partial integration. □

$|Q_n^{CC}[T_\nu]|$ is obviously bounded above by $2\|w\|_1$. For the estimation of the integral in (4.21), we substitute $x=\cos t$ and then apply Lemma 1 of Braß [1], p. 170, from which it results that

$$\int_{-1}^{1}C_{s,m}(x)w^{(s)}(x)\,dx = o(m^{-s}). \tag{4.24}$$

For fixed k, we therefore have

$$\begin{aligned}R_n[T_{n-1+k}] = \sum_{\nu=r}^{s-1}(-1)^\nu\Big\{&\big(C_{\nu+1,n-1+k}(1)-C_{\nu+1,n-1-k}(1)\big)w^{(\nu)}(1) \\ &-\big(C_{\nu+1,n-1+k}(-1)-C_{\nu+1,n-1-k}(-1)\big)w^{(\nu)}(-1)\Big\}+o(n^{-s}).\end{aligned} \tag{4.25}$$

Using

$$C_{\nu+1,n-1+k}(1) - C_{\nu+1,n-1-k}(1) = (-1)^\nu \frac{k(2\nu+2)!}{2^{\nu-1}\nu! n^{2\nu+3}}(1+O(1/n)), \quad \text{as } n \to \infty, \tag{4.26}$$

we see that, under the assumptions of the theorem,

$$R_n^{CC}[T_{n+\mu}] = O\left(n^{-(2r+3)}\right) \tag{4.27}$$

for fixed μ. Hence, $\int_{-1}^{1} |K_{n,1}(x)|\,dx$ is the main term of $\varrho_n(R_n^{CC}, w)$. We furthermore observe that

$$\begin{aligned} |R_n^{CC}[T_n]| &= |\int_{-1}^{1} (T_n(x) - T_{n-2}(x))w(x)\,dx| \\ &= |(C_{r+1,n}(1) - C_{r+1,n-2}(1))w^{(r)}(1) \\ &\quad - (C_{r+1,n}(-1) - C_{r+1,n-2}(-1))w^{(r)}(-1)| \\ &\quad + O\left(n^{-(2r+3)}\right) + o(n^{-s}), \end{aligned} \tag{4.28}$$

which yields the theorem. □

Proof of Theorem 3: The proof is essentially the same as that of Theorem 1. Now, the functionals $L_s^{(\mu,\kappa)}$, which shall be estimated and which annihilate the respective polynomial spaces are of the form

$$L_s^{(\mu,\kappa)}[f] = \int_0^\pi \sin\mu x \frac{\sin\kappa x}{\sin x} f(\cos x)\,dx - Q_r^{(\mu,\kappa)}[f]. \tag{4.29}$$

In the following, an asterisk at a sum indicates that each summand with a zero denominator has to be omitted. By the identity

$$\sin\mu x \frac{\sin\kappa x}{\sin x} = \sum_{\nu=1}^{\kappa} \sin(\mu - \kappa - 1 + 2\nu)x, \tag{4.30}$$

the terms of the form $(\mu \pm \kappa - s)^{-1}$, $s = 0, \pm 1, \ldots$, in (4.3) ff. will be replaced by the sums

$$\lambda^{(\mu,\kappa)} = \sum_{\nu=1}^{\kappa}{}^{*} \frac{1}{\mu - \kappa - 1 + 2\nu}. \tag{4.31}$$

We take now care of choosing the nodes of the respective quadrature formulae at the boundary, since the function W vanishes at ± 1. Hence, the quadrature formula $Q_2^{(\mu,\kappa)}$ corresponding to $L_0^{(\mu,\kappa)}$ is given by

$$Q_2^{(\mu,\kappa)}[f] = \lambda^{(\mu,\kappa)} \left\{ f(1) + (-1)^{\mu-\kappa} f(-1) \right\}. \tag{4.32}$$

The moments again satisfy the recurrence relation (4.2), but with the initialization

$$m_0^{(\mu,\kappa)} = (1 + (-1)^{\mu-\kappa}) \lambda^{(\mu,\kappa)}. \tag{4.33}$$

Analogously to the proof of Theorem 1 we obtain

$$\begin{aligned} \|K_1(L_0^{(\mu,\kappa)}, \cdot)\|_\infty &\le \lambda^{(\mu,\kappa)} \le \lambda^{([(\mu+\kappa)/2],[(\mu+\kappa)/2])} \\ &\le \frac{1}{2} \left\{ \ln(\mu+\kappa) + C + \ln 2 + \frac{1}{6(\mu+\kappa)^2} \right\}, \end{aligned} \tag{4.34}$$

where $C = 0.577\ldots$ is Euler's constant (for the last inequality, see Ostrowski [4], eq. (III,11), who referred an earlier result on the remainder in Euler-Maclaurin's summation formula of Malmstén). Thus, we can estimate the occuring Peano kernels by

$$\|K_1(L_0^{(n-1,k)}, \cdot)\|_\infty \le \lambda^{(n-1,k)} \le \frac{4k}{3n} \left(1 + \frac{1}{n} + \frac{1}{9n^2} \right) \qquad \text{for } k \le n/2, \tag{4.35}$$

and

$$\|K_1(L_0^{(n-1,1)}, \cdot)\|_\infty \le \frac{1}{n-1}. \tag{4.36}$$

Furthermore, we have

$$\|K_2(L_1^{(\mu,\kappa)}, \cdot)\|_\infty \le \left| \left(\frac{1}{\mu-\kappa} - \frac{1}{\mu-\kappa+1} \right) * \right|. \tag{4.37}$$

Let

$$\begin{aligned} k^{(\mu,\kappa)} := \frac{1}{4} \Bigg| \Bigg(& \frac{1}{\mu-\kappa-1} - \frac{1}{\mu-\kappa} - \frac{1}{\mu-\kappa+1} + \frac{1}{\mu-\kappa+2} \\ & + \frac{1}{\mu+\kappa-2} - \frac{1}{\mu+\kappa-1} - \frac{1}{\mu+\kappa} + \frac{1}{\mu+\kappa+1} \\ & + \lambda^{(\mu,\kappa-2)} - 3\lambda^{(\mu+1,\kappa-2)} + 4\lambda^{(\mu+2,\kappa-2)} \\ & - 3\lambda^{(\mu+3,\kappa-2)} + \lambda^{(\mu+4,\kappa-2)} \Bigg) * \Bigg|, \end{aligned} \tag{4.38}$$

then,

$$\|K_3(L_2^{(\mu,\kappa)},\cdot)\|_\infty = \|K_3(L_2^{(\kappa,\mu)},\cdot)\|_\infty$$

$$(4.39)\qquad \leq \begin{cases} \frac{1}{4}\left(\frac{1}{\mu-2}-\frac{1}{\mu-1}-\frac{1}{\mu+1}+\frac{1}{\mu+2}\right) & \text{for } \kappa=1, \\ k^{(\mu,\kappa)} & \text{for } 2\leq\kappa\leq\mu-2 \text{ and} \\ k^{(\mu,\kappa)}+7/12 & \text{for } \kappa\in\{\mu-1,\mu\}. \end{cases}$$

For the sum of the $\lambda^{(\mu,\nu)}$ in (4.38), we apply Ostrowski's equation as in (4.34). (4.39) may be estimated for all $\mu \geq 4$ and arbitrary κ by

$$(4.40)\qquad \|K_3(L_2^{(\mu,\kappa)},\cdot)\|_\infty \leq 1.1,$$

as well as for $\mu = n-1$ and $k \leq (2n-3)/3$ by

$$(4.41)\qquad \|K_3(L_2^{(\mu,\kappa)},\cdot)\|_\infty \leq \frac{81k}{50(n-1)^3}.$$

Inserting the obtained bounds for $R_n^{CC}[T_\nu]$ into the estimate of Braß and Förster, (2.5), we have (3.8)–(3.10). □

Proof of Theorem 4: We need the following Lemma.

LEMMA. *Under the same assumptions as in Theorem 4, we have*

$$(4.42)\qquad \begin{aligned} |K_n(R_n^{CC},x)| \leq & \frac{r^2}{\pi(r^2-1)2^{n+1}}|\partial\mathcal{C}_r| \max_{z\in\partial\mathcal{C}_r}|W(z)|\cdot \\ & \cdot\left\{\frac{n+3}{n-1}\frac{\Gamma(1/2)}{\Gamma(n-1/2)}+\left(1+\frac{1}{(n-1)(r-1)}\right)\frac{3\Gamma((n-1)/2)}{2\Gamma((3n-3)/2)}\right\}. \end{aligned}$$

The lemma is proved as follows. For each $z \in \partial\mathcal{C}_r$, define $v(z)$ such that $z = (v(z)+v^{-1}(z))/2$ and $|v(z)| = r^{-1}$. Since $R_n^{CC}[T_\nu]$ may be written as a difference of expressions of the form $I[T_\mu]$, we use the representation (cf. Braß, [1], p.71)

$$(4.43)\qquad I[T_\mu] = \int_{-1}^{1} T_\mu(x)w(x)\,dx = \frac{1}{2\pi i}\int_{\partial\mathcal{C}_r} W(z)k(z)\,dz,$$

where

$$k(z) = \int_{-1}^{1} \frac{T_\mu(x)}{(z-x)\sqrt{1-x^2}}\,dx = \frac{\pi v^\mu(z)}{1-v(z)^2}. \tag{4.44}$$

Using the functions

$$b_\mu(x) := b_{n,\mu}(x) := \frac{2}{\pi}\frac{n!2^n}{(2n)!}\left(1-x^2\right)^{n-1/2}\frac{P_\mu^{(n)}(x)}{P_\mu^{(n)}(1)}, \tag{4.45}$$

we have, according to Braß and Förster [2], that

$$\begin{aligned} K_n(R_n^{CC},x) &= \sum_{\mu=n}^{\infty} R_n^{CC}[T_\mu]\cdot b_{\mu-n}(x) \\ &= \frac{1}{2\pi i}\int_{\partial C_r} W(z)\cdot\frac{\pi}{1-v^2(z)}\Bigg\{v^n(z)\sum_{\mu=0}^{\infty}v^\mu(z)b_\mu(z) \\ &\quad+\sum_{l=1}^{\infty}b_{(2l-1)(n-1)-1}(x)+v^{n-1}(z)\sum_{l=1}^{\infty}b_{2l(n-1)-1}(x) \\ &\quad+\sum_{\mu=1}^{n-2}v^\mu(z)\Big(\sum_{l=1}^{\infty}b_{2l(n-1)+\mu-1}(x)+\sum_{l=1}^{\infty}b_{2l(n-1)-\mu-1}(x)\Big)\Bigg\}\,dz. \end{aligned} \tag{4.46}$$

Braß and Förster ([2], Lemma 1) proved the inequality

$$\beta_\mu := \frac{2}{\pi(\mu+1)(\mu+3)\cdots(\mu+2n-1)} \ge \|b_\mu\|_\infty, \tag{4.47}$$

and that ([2], eq. (29))

$$\sum_{l=0}^{\infty}\beta_{\varrho+2l(n-1)} \le \sum_{l=0}^{\infty}\beta_{\varrho+2l} = \frac{1}{\pi(n-1)}\frac{\Gamma((\mu+1)/2)}{\Gamma(n-1+(\mu+1)/2)}. \tag{4.48}$$

Inserting these bounds in (2.5), we obtain

$$\begin{aligned} &|K_n(R_n^{CC},x)| \\ &\le\frac{1}{2\pi}\int_{\partial C_r}|W(z)|\cdot\frac{r^2}{(r^2-1)(n-1)2^{n-1}}\Bigg\{\frac{1}{r^n}\Big(\frac{\Gamma(1/2)}{\Gamma(n-1/2)}+\frac{1}{(n-1)!}\Big) \\ &\qquad+\frac{\Gamma((n-1)/2)}{\Gamma((3n-3)/2)}+\frac{1}{r^{n-1}}\frac{\Gamma(n-1)}{\Gamma(2n-2)} \\ &\qquad+\sum_{\mu=1}^{n-2}\frac{1}{r^\mu}\Big(\frac{\Gamma((n-1+\mu)/2)}{\Gamma((3n-3+\mu)/2)}+\frac{\Gamma((n-1-\mu)/2)}{\Gamma((3n-3-\mu)/2)}\Big)\Bigg\}\,dz. \end{aligned} \tag{4.49}$$

Let $\gamma_\mu = \Gamma((n-1+\mu)/2)/\Gamma((3n-3+\mu)/2)$, then we can show that the sequence of the values $\gamma_\mu/\gamma_{\mu-1}$ is monotonely increasing with limit 1 as $\mu \to \infty$. Hence, the sequence (γ_μ) is decreasing. We can also show that, defining $\delta_\mu = r^{-\mu}\Gamma((n-1-\mu)/2)/\Gamma((3n-3-\mu)/2)$, the sequence $(\delta_\mu\delta_{\mu-2}/\delta_{\mu-1}^2)$ is monotonely decreasing with limit 1 as μ is decreasing to $-\infty$, such that $1 < \delta_\mu\delta_{\mu-2}/\delta_{\mu-1}^2 \le (\delta_\mu+\delta_{\mu-2})^2/4\delta_{\mu-1}^2$ yields the convexity of the sequence (δ_μ) . We therefore have

$$\delta_\mu \le \frac{\mu}{n-2}\delta_{n-2} + \frac{n-2-\mu}{n-2}\delta_0. \tag{4.50}$$

The lemma now directly follows, and Theorem 3 is an immediate consequence of the lemma and Stirling's formula. □

Proof of Theorem 5: Let the constant on the right-hand side of (3.13) be less than $1/2$. Using the notation of Theorem 4 and 5, we have

$$\begin{aligned} 2^{n-1}n!\varrho_n(R_n^{CC},x) &\ge |R_n^{CC}[T_n]| = |I[T_n - T_{n-2}]| \\ &= |\int_{-1}^{1} W(x)\frac{T_n(x)-T_{n-2}(x)}{\sqrt{1-x^2}}\,dx| \\ &= \frac{1}{2}|\int_{-\pi}^{\pi} v(t)(\cos nt - \cos(n-2)t)\,dt|. \end{aligned} \tag{4.51}$$

v is integrable and even, such that the sine coefficients of the Fourier series equal zero while the cosine coefficients a_ν tend to zero. The assumption is equivalent with

$$|a_n - a_{n-2}| \le c\cdot q^n \qquad \text{for all } n \tag{4.52}$$

with appropriate constants c and $q<1$. This implies

$$\begin{aligned} |a_n| &= |\sum_{k=0}^{\infty}(a_{n+2k} - a_{n+2k+2})| \\ &\le \sum_{k=0}^{\infty}|a_{n+2k+2} - a_{n+2k}| \\ &\le \sum_{k=0}^{\infty} c\cdot q^{n+2k+2} = \frac{c\cdot q^{n+2}}{1-q^2}. \end{aligned} \tag{4.53}$$

This estimate of the Fourier coefficients implies that v is analytic.

If we assume the analyticity of v , we obtain an estimate $|a_n| \le d\cdot r^n$ with $r<1$ for all n and thus

$$|a_{n-1+k} - a_{n-1-k}| \le 2dr^{n-1-k}, \qquad k = 1,2,\ldots,n-1. \tag{4.54}$$

We can therefore estimate the error constant in the same way as in the proof of Theorem 4. □

Acknowledgement: I am grateful to the referee for his helpful comments.

REFERENCES

1. BRASS, H.: *Quadraturverfahren*, Vandenhoeck & Ruprecht, Göttingen, 1977.
2. H. BRASS, H., FÖRSTER, K.-J.: On the estimation of linear functionals, *Analysis* **7** (1987), 237-258.
3. FIEDLER, H.: Das asymptotische Verhalten der Peanokerne einiger interpolatorischer Quadraturverfahren, *Numer. Math.* **240** (1987), 571-581.
4. OSTROWSKI, A.M.: On the remainder term of the Euler-Maclaurin Formula, *J. Reine Angew. Math.* **240** (1969), 268-286.
5. SLOAN, I.H., SMITH, W.E.: Product integration with the Clenshaw-Curtis points: implementation and error estimates, *Numer. Math.* **34** (1980), 387-401.

Analysis 14, 19 - 41 (1994)

Analysis

ON THE GENERAL ASYMPTOTIC EXPANSIONS OF $H_v^{(1)}$-TRANSFORM AND RELATED BESSEL TRANSFORMS

Hui-Hui Dai

Received

Abstract Theorems are established to give the asymptotic expansions of $H_v^{(1)}$ - transforms and related Bessel transforms of functions which have general asymptotic expansions near the origin. The results are thus the extensions to those of the existing literature (e.g. Wong 1976, Soni 1982 and Soni&Soni 1985) which dealt with the cases that the functions have power expansions near the origin. Our approach uses a lemma proved in an earlier paper (Dai 1992) on the general asymptotic expansions of K_v - transforms and adopts a novel method to overcome divergent integrals.

AMS subject classifications: 41A60, 44A15, 33A40.

1.Introduction

Olver (1974) obtained a formula for the asymptotic expansion of the Fourier transform

(1.1) $$F(s)=\int_0^\infty e^{ist}f(t)dt$$

when the parameter $s \to +\infty$ and where the behaviour of the function $f(t)$ near the origin is given by an asymptotic expansion of the form

(1.2) $$f(t)\sim\sum_{n=0}^{\infty} b_n t^{n+\lambda-1}\ ,\quad t\to o^+$$

where $0 < \lambda \leq 1$. The method adopted by Olver entails writing

$$(1.3) \qquad f(t)=\sum_{n=0}^{N-1} b_n t^{n+\lambda-1} + f_N(t)$$

As the individual terms on the right hand side of (1.3) do not possess Fourier transforms, this equation is first multiplied by $e^{-\epsilon t}$, and then, Fourier transform is taken for each side of the equation. The summability factor $e^{-\epsilon t}$ evades the appearance of divergent integrals. The parameter ϵ is positive and later made to tend to zero. By this procedure, and by treating the herein appeared Fourier transform of $e^{-\epsilon t} f_N(t)$ by integrating by parts N times, the desired asymptotic expansion of $F(s)$ can be obtained.

Olver's method was consequently adopted by Wong (1976) to obtain the asymptotic expansion of the Hankel transform

$$(1.4) \qquad I(s)=\int_0^{\infty} J_\nu(st) f(t)dt \ , \quad s\to+\infty$$

by introducing the summability factor $e^{-\epsilon t^2}$, where the behaviour of $f(t)$ near to the origin is given by (1.2). Also see the paper by Soni and Soni (1985) which gave a detailed discussion of the conditions under which the summability methods applicable to this class of problems. Another method in connection with the asymptotic expansion of Hankel transforms was given by Soni, which is an extension of the technique due to Handelsman and Lew (1969) by using the Parseval relation for the Mellin transform. For details, we refer to Soni (1982).

In all above-mentioned methods, the function is assumed to have an algebraic power expansion near the origin. However, in many practical cases (for example, see Naylor 1990), the form of the function near $t = 0$ is normally given by the general asymptotic expansion

$$(1.5) \qquad f(t)\sim\sum_{n=0}^{\infty} a_n \phi_n(t) \ , \quad t\to 0^+$$

where $\{\phi_n(t)\}$ is an asymptotic sequence. If $\phi_n(t)$ can not in turn be expanded in power series, generally speaking, neither Olver's method or Soni's method is applicable. Even if $\phi_n(t)$ can be expanded in a power series

$$(1.6) \qquad \phi_n(t)=\sum_{n=0}^{\infty} b_n t^{n+\lambda-1} \ , \quad t\to o^+$$

and then the above-mentioned methods are applicable, the resulting expansions of integral transforms would be expressed in terms of coefficients which are related to a_n and b_n by somewhat complicated formulas. Thus it would be desirable to obtain expansions involving only coefficients a_n.

In a previous paper (Dai and Naylor 1992), we have obtained the asymptotic expansion of the Fourier transform $F(s)$ when the parameter $s \to +\infty$ and where the behaviour of the function $f(t)$ near the origin is given by (1.5). In this paper we shall give the asymptotic expansions of the following Bessel transforms

$$\int_0^\infty H_\nu^{(1)}(st)f(t)dt\ ,\quad \int_0^\infty H_\nu^{(2)}(st)f(t)dt\ ,\quad \int_0^\infty Y_\nu(st)f(t)dt\ ,\quad \int_0^\infty J_\nu(st)f(t)dt$$

when $s \to +\infty$.

We shall concentrate on the case of a $H_\nu^{(1)}$ - transform, and the other transforms are then treated similarly. Difficulty in this class of problems, as discussed in the above, is the divergence of the integral herein appeared. Here, the difficulty is overcome by introducing a factor similar to a neutralizer (c.f. (3.3)), which was first used in a earlier paper (Dai and Naylor 1992). The $H_\nu^{(1)}$ - transform can then be transformed into a K_ν - transform. A lemma proved in a previous paper (Dai 1992) on the general asymptotic expansions of K_ν - transforms (c.f. lemma 1) is used to handle the difficuly caused by the fact that the function has a general expansion near origin. Our method seems to be novel, and the results are extensions to those in the existing literature.

For simplicity, we shall assume that ν is real and $\nu \geq 0$. For $\nu < 0$, similar results can be obtained by standard continuation formulas, given, for example, by Watson (1958, pp74-75).

2.Preliminaries

In the following, we use $\Omega_\nu(t)$ to denote functions $H_\nu^{(1)}(t)$, $H_\nu^{(2)}(t)$, or $Y_\nu(t)$, and write

(2.1) $$\nu = K + \mu$$

where $K \geq 0$ is a positive integer and $0 \leq \mu < 1$. We make the following assumptions:

(i)The behaviour of $f(t)$ near the origin is given by (1.5), which is asymptotic. So that, if we write

(2.2) $$f(t) = \sum_{n=0}^{N-1} a_n \phi_n(t) + f_N(t)$$

then

(2.3) $$f_N(t) = O(\phi_N(t))\ ,\quad t \to 0^+$$

(ii)In $(0, \infty)$ $f(t)$ is continuous, M times differentiable and let asymptotic series for the derivatives $f^{1)}, f^{2)}, \ldots, f^{M)}$ are obtainable from (1.5) by successive differentiation, and

$$(2.4) \qquad f_N^{(n)}(t)=O(\phi_N^{(n)}(t)) \ , \quad n=1,2,\ldots,M, \quad t\to 0^+$$

(iii)$f^{n)}(t) = o(t^{-1/2}), \ t \to \infty, \ n = 0, 1, \ldots, M-1$; and

$$\int_1^\infty f(t)\Omega_p(st)dt \ , \quad \int_1^\infty f^{(n)}(t)t^{n-M}\Omega_p(st)dt$$

are assumed to be uniformly convergent for s large enough for each $n = 0, 1, \ldots, M$, where $p = \nu - M$ if $M \le K + 1$ or $p = \mu$ if $M = K + 2k$ or $p = \mu - 1$ if $M = K + 2k + 1$, where $k \ge 1$ is a positive integer (see (2.1) for the definitions of K and μ).
(iv)$\{\phi_n(\pm it)\}^*$ is an asymptotic sequence; and $\phi_n(\pm it)$ $(n=0,1,\ldots,N-1)$ does not spiral into zero (so that equations (2.15) and (2.16) hold).
(*Herein and after, the sign $\pm$ means that, the sign + should be taken if $\Omega_\nu(t) = H_\nu^{(1)}(t)$, the sign - should be taken if $\Omega_\nu(t) = H_\nu^{(2)}(t)$, and both signs + and - should be taken if $\Omega_\nu(t) = Y_\nu(t)$).
(v)$\phi_n(\pm z)$ $(n = 0, 1, \ldots, N-1)$ is analytic and regular in the first quarter plane except at $z = 0$, and there exists a positive constant c (which may depend on N) such that

$$(2.5) \qquad |\phi_n(\pm z)|<A|e^{cz}| \ , \quad 0\le \arg z<\frac{\pi}{2} \ , \quad |z|\to+\infty$$

$$(2.6) \qquad |\phi_n(\pm it)|<A|e^{ct}| \ , \quad t\to+\infty$$

where A is a positive constant.
(vi)

$$(2.7) \qquad \phi_0(t)=O(t^\alpha) \ , \quad \phi_0(\pm it)=O(t^\alpha) \ , \quad t\to 0^+$$

where $\alpha > -1 + \nu = K + \mu - 1$.
(vii)

$$(2.8) \qquad \phi_N(t)=O(t^\beta) \ , \quad \phi_N(\pm it)=O(t^\beta) \ , \quad t\to 0^+$$

$$(2.9) \qquad \phi_N^{(j)}(t)=O(t^{\beta-j}) \ , \quad \phi_N^{(j)}(\pm it)=O(t^{\beta-j}) \ , \quad j=1,\ldots,M.$$

where $\beta > \alpha > K + \mu - 1$, and if further $M \ge K + 1$, we further require that

(2.10) $$\beta > M-1+\mu \ , if \ M=K+2k$$

(2.11) $$\beta > M-\mu \ , \ if \ M=K+2k+1$$

where $k \geq 0$ is a positive integer.

As we allow the function $f(t)$ to have a general asymptotic expansion near the origin, it is necessary to impose the additional conditions (iv) to (vii) on $\phi_n(t)$. But we would point out that these further conditions are automatically satisfied for a large class of functions.

To prove our main results presented in sections 3 and 4, we require the following three lemmas.

Lemma 1 Suppose that real functions $g(t)$ and $h(t)$ have K_ν - transforms

(2.12) $$G(s)=\int_0^\infty K_\nu(st)g(t)dt$$

(2.13) $$H(s)=\int_0^\infty K_\nu(st)h(t)dt$$

for sufficiently large s, and $h(t) > 0$ for $0 < t < t_1$, where t_1 is a positive constant. Then, $H(s) > 0$ for sufficiently large s, and

(2.14) $$\overline{\lim_{s\to+\infty}}\frac{|G(s)|}{H(s)} \leq \overline{\lim_{t\to 0^+}}\frac{|g(t)|}{h(t)}$$

This lemma is a special case of a lemma given in a previous work (Dai 1992), and for proof, we refer to that paper.

Lemma 2 $\{\Phi_n^{\,i}(s)\}$ $(0 \leq n \leq N-1)$ is an asymptotic sequence as $s \to +\infty$, where $\Phi_n^{\,i}(s)$ denotes the K_ν - transform of $\phi_n(it)$.

Proof

From the assumptions listed above it is easy to see that the K_ν - transform for $\phi_n(it)$ exists. Noting that $Re\phi_n(it)$ and $Im\phi_n(it)$ is continuous for $t > 0$ and we can put factors ± 1 and $\pm i$ into a_n in (1.5) (also see assumption (iv)), we may assume that

(2.15) $$Re\phi_n(it)>0 \quad for \quad 0<t<t_1$$

and

(2.16) $$\overline{\lim_{t\to 0^+}}\frac{|Im\phi_n(it)|}{Re\phi_n(it)}\le 1$$

Then, we have

(2.17) $$|\phi_n(it)|\le BRe\phi_n(it)\ ,\quad t\to 0^+$$

where B is a positive constant. For a positive s, we have

(2.18) $$|\Phi_n^i(s)|\ge Re\Phi_n^i(s)=\int_0^\infty K_\nu(st)Re\phi_n(it)dt$$

From Lemma 1, we know that $Re\Phi_n^i(s) > 0$, using (2.17) and (2.18), we have

$$\overline{\lim_{s\to+\infty}}\frac{|\Phi_{n+1}^i(s)|}{|\Phi_n^i(s)|}\le\overline{\lim_{s\to+\infty}}\frac{|\Phi_{n+1}^i(s)|}{Re\Phi_n^i(s)}\le\overline{\lim_{t\to 0^+}}\frac{\phi_{n+1}(it)}{Re\phi_n(it)}\le\overline{\lim_{t\to 0^+}}\frac{\phi_{n+1}(it)}{\frac{1}{B}\phi_n(it)}=0$$

Thus

$$\Phi_{n+1}^i(s)=o(\Phi_n^i(s))$$

and $\{\Phi_n^i(s)\}$ $(n = 0, 1, \ldots, N-1)$ is an asymptotic sequence.

Lemma 3 $\{\Phi_n^{-i}(s)\}$ $(0 \le n \le N-1)$ is an asymptotic sequence as $s \to +\infty$, where $\{\Phi_n^{-i}(s)\}$ denotes the K_ν - transform of $\phi_n(-it)$.

The proof of this lemma is very similar to that of lemma 2, and we omit the details.

3. The asymptotic expansions for $H_\nu^{(1)}$ - transforms, $H_\nu^{(2)}$-transforms and Y_ν-transforms

Theorem 1 Let the functions $f(t)$, $\phi_n(t)$ satisfy the conditions enumerated in the section 2 (in present case, $\Omega_\nu(t) = H_\nu^{(1)}(t)$, and for the sign $\pm$, we only take the sign $+$), then

(3.1) $$F_1(s)=\int_0^\infty H_\nu^{(1)}(st)f(t)dt=\frac{2}{\pi}e^{-\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1}a_n\Phi_n^i(s)\ +\ o(\Phi_{N-1}^i(s))\ +\ O(s^{-M})\ ,\quad s\to+\infty$$

If further $\phi_{N-1}(it) \sim C_1 t^{M_0-1}$, $t \to 0^+$, where C_1 and $M_0 < M$ are constants, then

(3.2) $$F_1(s)=\frac{2}{\pi}e^{-\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1} a_n\Phi_n^i(s) + o(\Phi_{N-1}^i(s))\ , \quad s\to+\infty$$

We point out that, for the formula (3.1) to be effective one or more terms of the series appeared in (3.1) must dominate the $O(s^{-M})$ term. Alternatively, as in the second part of the above theorem, we make a suitable assumption concerning the magnitude of the function $\phi_{N-1}(it)$ for small value of t, then magnitude of the $O(s^{-M})$ term can be related to that of the other terms.

Proof

Consider the function

(3.3) $$g(t)=(1-e^{-bt})^{M_1}\sum_{n=0}^{N-1} a_n\phi_n(t)$$

where $b > c$ (for the definition of c see assumption (v)) and $M_1 > 0$ is an integer taken as large as desired. We then write

(3.4) $$\begin{aligned} f(t)&=\sum_{n=0}^{N-1} a_n\phi_n(t) - g(t) + f_N(t) + g(t) \\ &=G(t) + L(t) \end{aligned}$$

where

(3.5) $$\begin{aligned} G(t)&=\sum_{n=0}^{N-1} a_n\phi_n(t) - g(t) \\ &=\sum_{n=0}^{N-1} a_n\phi_n(t) - (1-e^{-bt})^{M_1}\sum_{n=0}^{N-1} a_n\phi_n(t) \end{aligned}$$

(3.6) $$\begin{aligned} L(t)&=f_N(t) + g(t) \\ &=f_N(t) + (1-e^{-bt})^{M_1}\sum_{n=0}^{N-1} a_n\phi_n(t) \end{aligned}$$

As

$$\phi_n(t)=O(e^{ct}) \ , \quad t\to+\infty$$
$$\phi_0(t)=O(t^{\alpha}) \ , \quad t\to 0^+$$

it follows that

$$G(t)=O(e^{-(b-c)t}) \ , \quad t\to+\infty \tag{3.7}$$

$$G(t)=O(t^{\alpha}) \ , \quad t\to 0^+ \tag{3.8}$$

Thus the $H_\nu^{(1)}$ - transform of $G(t)$ exists. Since $L(t) = f(t) - G(t)$, the $H_\nu^{(1)}$ - transform of $L(t)$ also exists. So we can write

$$\begin{aligned} F(s) &= \int_0^\infty H_\nu^{(1)}(st)f(t)dt \\ &= \int_0^\infty H_\nu^{(1)}(st)G(t)dt + \int_0^\infty H_\nu^{(1)}(st)L(t)dt \end{aligned} \tag{3.9}$$

For the first integral, rotating the integration path to the upper-half imaginary axis, which is permitted by the Cauchy's theorem, for sufficient large s, and noting that

$$K_\nu(z)=\frac{1}{2}\pi i e^{\frac{i}{2}\nu\pi} H_\nu^{(1)}(ze^{\frac{i}{2}\pi}) \ , \tag{3.10}$$

we have

$$\begin{aligned} \int_0^\infty H_\nu^{(1)}(st)G(t)dt &= \frac{2}{\pi}e^{-\frac{i}{2}\nu\pi}\int_0^\infty K_\nu(st)G(it)dt \\ &= \frac{2}{\pi}e^{-\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1} a_n \Phi_n^i(s) - \frac{2}{\pi}e^{-\frac{i}{2}\nu\pi}\int_0^\infty K_\nu(st)g(it)dt \end{aligned} \tag{3.11}$$

The last term in (3.11) can be shown to be $o(\Phi_{N-1}^i(s))$ by using Lemma 1. For this purpose, we may assume that $\phi_{N-1}(it)$ satisfies equations (2.15), (2.16) and (2.17) with $n = N-1$. Then, applying lemma 1, we have

$$(3.12)\qquad \overline{\lim_{s\to+\infty}}\frac{\left|\int_0^\infty K_\nu(st)g(it)dt\right|}{\Phi^i_{N-1}(s)}\leq\overline{\lim_{t\to0^+}}\frac{|g(it)|}{Re\phi_{N-1}(it)}$$

From equation (3.3) we have

$$(3.13)\qquad g(it)=O(t^{M_1}\phi_0(it))=O(t^{M_1+\alpha})\ ,\quad t\to0^+$$

and from (2.17), assumptions (iv) and (vii), we have

$$(3.15)\qquad BRe\phi_{N-1}(it)\geq|\phi_{N-1}(it)|>|\phi_N(it)|=O(t^\beta)\ ,\quad t\to0^+$$

As M_1 can be taken as large as desired, the right hand side of the inequavility (3.12) equals zero, so that

$$(3.16)\qquad \left|\int_0^\infty K_\nu(st)g(it)dt\right|=o(\Phi^i_{N-1}(s))\ ,\quad s\to+\infty$$

Consequently, we have

$$(3.17)\qquad \int_0^\infty H^{(1)}_\nu(st)G(t)dt=\frac{2}{\pi}e^{-\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1}a_n\Phi^i_n(s)\ +\ o(\Phi^i_{N-1}(s))\ ,\quad s\to+\infty$$

We shall treat the second integral on the right hand side of (3.9) by integration by parts. There are two formulas available for this purpose (Abamowitz&Stegun 1964, p361):

$$(3.18)\qquad H^{(1)}_\nu(z)=(\nu-1)z^{-1}H^{(1)}_{\nu-1}(z)\ -\ \frac{d}{dz}H^{(1)}_{\nu-1}(z)$$

$$(3.19)\qquad H^{(1)}_{\nu-1}(z)=\nu z^{-1}H^{(1)}_\nu(z)\ +\ \frac{d}{dz}H^{(1)}_\nu(z)$$

We introduce the following functions

$$(3.20)\qquad L_0(t)=L(t)$$

(3.21) $$L_j(t)=\frac{d}{dt}L_{j-1}(t)\ +\ (\nu-j)t^{-1}L_{j-1}(t)\ ,\quad (j=1,\ldots,K+1)$$

and if $M > K + 1$, then

(3.22) $$L_{K+2j}(t)=-\frac{d}{dt}L_{K+2j-1}(t)\ +\ (\nu-K)t^{-1}L_{K+2j-1}$$

(3.23) $$L_{K+2j+1}(t)=\frac{d}{dt}L_{K+2j}(t)\ +\ (\nu-K-1)t^{-1}L_{K+2j}\ ,\quad (j=1,\ldots,\frac{M-K-1}{2}\ \text{or}\ \frac{M-K}{2})$$

From assumptions (i), (vii) and (3.6), it is easy to see that

(3.24) $$L_0(t)=O(t^{\beta})\ ,\quad t\to 0^+$$

From assumptions (ii) and (vii) and equations (3.6) and (3.20) to (3.23), it follows that

(3.25) $$L_j(t)=O(t^{\beta-j})\ ,\quad t\to 0^+\ ,\quad j=1,\ldots,M$$

By appealling to Ritt's Theorem, it is known that $\phi_n^{(m)}(z) = O(e^{cz}),\ z \to \infty$. Thus, from equation (3.5), we have

(3.26) $$G^{(m)}(t)=O(e^{-(b-c)t}),\quad t\to +\infty$$

As

(3.27) $$f^{(m)}(t)=o(t^{-1/2})\ ,\quad m=0,1,\ldots,M-1\ ,\quad t\to +\infty$$

and

(3.28) $$L_0(t)=f(t)\ -\ G(t)\ ,$$

from equation (3.20) to (3.23), it is easy to see that

(3.29) $$L_j(t)=o(t^{-1/2})\ ,\quad j=0,1,\ldots,M-1\ ,\quad t\to +\infty$$

After these preparations we now can evaluate the second integral on the right hand side of equation (3.9). We have

$$\int_0^\infty L(t)H_\nu^{(1)}(st)dt=\lim_{a\to\infty}\int_0^a L(t)H_\nu^{(1)}(st)dt \tag{3.30}$$

Using equation (3.18) and integrating by parts once, it follows that

$$\int_0^\infty L(t)H_\nu^{(1)}(st)dt=-\frac{1}{S}\lim_{a\to\infty}L(t)H_{\nu-1}^{(1)}(st)|_0^a+\frac{1}{S}\lim_{a\to\infty}\int_0^a L_1(t)H_{\nu-1}^{(1)}(st)dt \tag{3.31}$$

Noting that

$$H_\nu^{(1)}(z)\sim\sqrt{\frac{2}{\pi z}}e^{i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}\ ,\quad z\to\infty\ , \tag{3.32}$$

$$H_\nu^{(1)}(z)\sim-\frac{i}{\pi}\Gamma(\nu)(\frac{1}{2}z)^{-\nu}\ ,\quad z\to 0\ , \tag{3.33}$$

and from the asymptotic properties of $L_j(t)$ (see equations (3.24), (3.25) and (3.29)), it easy to see that the integrated term vanishes. If $M \leq K+1$, we shall continue this procedure $M-1$ times (all the integrated terms shall vanish), then

$$\int_0^\infty L(t)H_\nu^{(1)}(st)dt=(\frac{1}{S})^M\lim_{a\to\infty}\int_0^a L_M(t)H_p^{(1)}(st)dt \tag{3.34}$$

where $p=\nu-M$ (for $M\leq K+1$). If $M > K+1$, we use above procedure $K+1$ times, i.e. we have

$$\begin{aligned}\int_0^\infty L(t)H_\nu^{(1)}(st)dt&=(\frac{1}{S})^{K+1}\lim_{a\to\infty}\int_0^a L_{K+1}(t)H_{\nu-K-1}^{(1)}(st)dt\\&=(\frac{1}{S})^{K+1}\lim_{a\to\infty}\int_0^a L_{K+1}(t)H_{\mu-1}(st)dt\end{aligned} \tag{3.35}$$

In the following, we shall use equations (3.19) and (3.18) in turn, and noting that all the integrated terms vanish as β satisfies (2.10) and (2.11), we have

(3.36) $$\int_0^\infty L(t)H_\nu^{(1)}(st)dt=(\frac{1}{s})^M \lim_{a\to\infty}\int_0^a L_M(t)H_p^{(1)}(st)dt$$

where $p = \mu$ if $M = K + 2k$ or $p = \mu - 1$ if $M = K + 2k + 1$ ($k \geq 1$ is a positive integer).

Now we wish to show that

$$\int_0^\infty L_M(t)H_p^{(1)}(st)dt$$

exists. We write

(3.37) $$\int_0^\infty L_M(t)H_p^{(1)}(st)dt=\int_0^1 L_M(t)H_p^{(1)}(st)dt + \int_1^\infty L_M(t)H_p^{(1)}(st)dt$$

By substituting (3.28) into (3.20) to (3.23), it can be seen that

(3.38) $$L_M(t)=\sum_{n=0}^{M} d_n t^{n-M}f^{(n)}(t) - \sum_{n=0}^{M} d_n t^{n-M}G^{(n)}(t)$$

where d_n are constant coefficients. Substituting (3.38) into (3.40), we have

(3.39) $$\int_0^\infty L_M(t)H_p^{(1)}(st)dt=\int_0^1 L_M(t)H_p^{(1)}(st)dt+\sum_{n=0}^{M} d_n\int_1^\infty t^{n-M}f^{(n)}(t)H_p^{(1)}(st)dt -\sum_{n=0}^{M} d_n\int_1^\infty t^{n-M}G^{(n)}(t)H_p^{(1)}(st)dt$$

From the asymptotic properties of $G^{(n)}(t)$ (see (3.26)) and $L_M(t)$ (see (3.25)), and assumption (iii), it is easy to see that all the integrals on the right hand side of (3.39) exists. Thus

$$\int_0^\infty L_M(t)H_p^{(1)}(st)dt$$

exists. Consequently, from (3.36) (or (3.34)), we have

(3.40) $$\int_0^\infty L(t)H_\nu^{(1)}(st)dt=(1/s)^M\int_0^\infty L_M(t)H_p^{(1)}(st)dt=O(s^{-M}) , \quad s\to+\infty$$

Combining equations (3.40), (3.17) and (3.9) gives equation (3.1).

For the second part of theorem 1, we only require to prove that $O(s^{-M}) = o(\Psi^i_{N,l}(s))$. For this purpose we may assume that $\phi_{N-1}(it)$ satisfies equations (2.15), (2.16) and (2.17) with $n = N-1$. Applying lemma 1, and noting that $\phi_{N-1}(it) \sim C_1 t^{M_0 - 1}$, $M_0 < M$, we have

$$\overline{\lim_{s\to+\infty}} \frac{\left|\int_0^\infty K_\nu(st)t^{M-1}dt\right|}{|\Phi^i_{N-1}(s)|} \le \overline{\lim_{s\to+\infty}} \frac{\left|\int_0^\infty K_\nu(st)t^{M-1}dt\right|}{Re\Phi^i_{N-1}(s)} \le \overline{\lim_{t\to 0^+}} \frac{t^{M-1}}{Re\phi_{N-1}(it)} \le \overline{\lim_{t\to 0^+}} \frac{t^{M-1}}{\frac{1}{B}|\phi_{N-1}(it)|} = 0$$

As

$$\int_0^\infty K_\nu(st)t^{M-1}dt = 2^{M-2}s^{-M}\Gamma(\frac{1}{2}M+\frac{1}{2}\nu)\Gamma(\frac{1}{2}M-\frac{1}{2}\nu)$$

it follows that

$$O(s^{-M}) = o(\Phi^i_{N-1}(s)).$$

This completes the proof of the second part of theorem 1.

By noting that

$$K_\nu(z) = -\frac{i}{2}\pi e^{-\frac{i}{2}\nu\pi} H^{(2)}_\nu(ze^{-\frac{i}{2}\pi}) \,, \tag{3.41}$$

it is easy to see that we can establish the similar results for the asymptotic expansions of $H_\nu^{(2)}$ - transforms. This is consequently the following theorem:

Theorem 2 Let the functions $f(t)$ and $\phi_n(t)$ satisfy the conditions enumerated in section 2 (in present case, $\Omega_\nu(t) = H_\nu^{(2)}(t)$, and for the sign $\pm$, we only take the sign -), then

$$F_2(s) = \int_0^\infty H^{(2)}_\nu(st)f(t)dt = \frac{2}{\pi}e^{\frac{i}{2}\nu\pi} \sum_{n=0}^{N-1} a_n \Phi_n^{-i}(s) + o(\Phi^{-i}_{N-1}(s)) + O(s^{-M}) \,, \quad s\to+\infty \tag{3.42}$$

If further $\phi_{N-1}(-it) \sim C_2 t^{M_2 - 1}$, $t \to 0^+$, where C_2 and $M_2 < M$ are constants, then

(3.43) $$F_2(s)=\frac{2}{\pi}e^{\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1}a_n\Phi_n^{-i}(s) + o(\Phi_{N-1}^{-i}(s)) \ , \quad s\to+\infty$$

To prove this theorem we write

(3.44) $$F_2(s)=\int_0^\infty H_\nu^{(2)}(st)G(t)dt + \int_0^\infty H_\nu^{(2)}(st)L(t)dt$$

For the first integral on the right hand side of (3.44), by rotating the integration path to the lower-half imaginary axis, it can be shown that

(3.45) $$\int_0^\infty H_\nu^{(2)}(st)G(t)dt=\frac{2}{\pi}e^{\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1}a_n\Phi_n^{-i}(s) + o(\Phi_{N-1}^{-i}(s)) \ , \quad s\to+\infty$$

For the second integral on the right hand side of (3.44), by integrating by parts, it can be shown that

(3.46) $$\int_0^\infty L(t)H_\nu^{(2)}(st)dt=(1/s)^M\int_0^\infty L_M(t)H_p^{(2)}(st)dt=O(s^{-M}) \ , \quad s\to+\infty$$

The detailed procedure used to derive (3.45) and (3.46) is very similar to that used to derive (3.17) and (3.40), we thus omit the details. Combining (3.44), (3.45) and (3.46) gives (3.42). The second part of theorem 2 can be closely proved by the method used to prove the second part of theorem 1.

Noting that

(3.47) $$Y_\nu(z)=-\frac{i}{2}(H_\nu^{(1)}(z)-H_\nu^{(2)}(z))$$

We see that the asymptotic expansions of Y_ν - transforms can be obtained by combining the results of theorem 1 and theorem 2. However, as there are both $H_\nu^{(1)}$ - transforms and $H_\nu^{(2)}$ - involved, we have to impose conditions on both $\phi_n(it)$ and $\phi_n(-it)$.

<u>**Theorem 3**</u> Let the function $f(t)$ and $\phi_n(t)$ satisfy the conditions enumerated in section 2 (in present case, $\Omega_\nu(t) = Y_\nu(t)$, and for the sign $\pm$, we take both the sign + and the sign -), then

(3.48)
$$F_3(s)=\int_0^\infty Y_\nu(st)f(t)dt=-\frac{i}{\pi}e^{-\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1}a_n\Phi_n^i(s)+\frac{i}{\pi}e^{\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1}a_n\Phi_n^{-i}(s)$$
$$+\ o(\Phi_{N-1}^i(s))\ +\ o(\Phi_{N-1}^{-i}(s))\ +\ O(s^{-M})\ ,\quad s\to+\infty$$

If further $\phi_{N-1}(it)\ \sim\ C_1t^{M_1-1}$ or $\phi_{N-1}(-it)\ \sim\ C_2t^{M_2-1}$, $t\to 0^+$, where C_1, C_2, $M_1 < M$ and $M_2 < M$ are constants, then

(3.49)
$$F_3(s)=-\frac{i}{\pi}e^{-\frac{i}{2}\frac{\nu}{\pi}}\sum_{n=0}^{N-1}a_n\Phi_n^i(s)+\frac{i}{\pi}e^{\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1}a_n\Phi_n^{-i}(s)$$
$$+\ o(\Phi_{N-1}^i(s))\ +\ o(\Phi_{N-1}^{-i}(s))\ ,\quad s\to+\infty$$

To prove this theorem we write

(3.50)
$$F_3(s)=\int_0^\infty Y_\nu(st)f(t)dt=-\frac{i}{2}\int_0^\infty H_\nu^{(1)}(st)G(t)dt+\frac{i}{2}\int_0^\infty H_\nu^{(2)}(st)G(t)dt+\int_0^\infty Y_\nu(st)L(t)dt$$

The first two integrals on the right hand side of (3.50) have been evaluated previously (see (3.17) and (3.45)). The last integral can be treated by the same method used to handle the last integral on the right hand side of (3.9). The second part of theorem 3 can be closely proved by the method used to prove the second part of theorem 1. Here, we omit the details.

4. The asymptotic expansions of J_ν - transforms

The asymptotic expansions of J_ν - transforms (Hankel transforms) can be obtained similarly by noting that

(4.1)
$$J_\nu(z)=\frac{1}{2}(H_\nu^{(1)}(z)+H_\nu^{(2)}(z)).$$

However,

(4.2)
$$J_\nu(z)\sim\frac{(z/2)^\nu}{\Gamma(\nu+1)}\ ,\quad z\to 0$$

which is different from the asymptotic properties of $H_\nu^{(1)}(z)$ ($= O(z^{-\nu})$, $z\to 0$) and $H_\nu^{(2)}(z)$

$(= O(z^{\nu}),\ z \to 0)$. Consequently, the condition imposed on $f(t)$ and $\phi_n(t)$ are somewhat different. We modify the assumptions enumerated in section 2 as follows:

(i), (ii), (iv), (v) and (vi) are same as those for Y_ν - transforms.

(iii) $f^{(n)}(t) = o(t^{-1/2}),\ t \to \infty,\ n=0,\ 1,\ \ldots,\ M-1;$ and

$$\int_1^\infty f(t)J_\nu(st)dt\ ,\quad \int_1^\infty f^{(n)}(t)t^{n-M}J_{\nu+M}(st)dt$$

are assumed to be uniformly convergent for s large enough for each $n = 0,\ 1,\ \ldots,\ M.$

(vii)

$$\phi_N(t)=O(t^\beta)\ ,\quad \phi_N(\pm it)=O(t^\beta)\ ,\quad t\to 0^+ \tag{4.3}$$

$$\phi_N^{(j)}(t)=O(t^{\beta-j})\ ,\quad \phi_N^{(j)}(\pm it)=O(t^{\beta-j})\ ,\quad j=1,\ldots,M \tag{4.4}$$

where $\beta > \alpha$.

<u>Theorem 4</u> Let the functions $f(t)$ and $\phi_n(t)$ satisfy the conditions enumerated above, then

$$\begin{aligned} F_4(s)=\int_0^\infty J_\nu(st)f(t)dt= \frac{1}{\pi}e^{-\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1} a_n\Phi_n^{i}(s)+\frac{1}{\pi}e^{\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1} a_n\Phi_n^{-i}(s) \\ + o(\Phi_{N-1}^{i}(s)) + o(\Phi_{N-1}^{-i}(s)) + O(s^{-M})\ ,\quad s\to+\infty \end{aligned} \tag{4.5}$$

If further $\phi_{N-1}(it) \sim C_1 t^{M_0 l}$ or $\phi_{N-1}(-it) \sim C_2 t^{M_2 l}$, $t \to 0^+$, where C_1, C_2, $M_0 < M$ and $M_2 < M$ are constants, then

$$\begin{aligned} F_4(s)= \frac{1}{\pi}e^{-\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1} a_n\Phi_n^{i}(s)+\frac{1}{\pi}e^{\frac{i}{2}\nu\pi}\sum_{n=0}^{N-1} a_n\Phi_n^{-i}(s) \\ + o(\Phi_{N-1}^{i}(s)) + o(\Phi_{N-1}^{-i}(s))\ ,\quad s\to+\infty \end{aligned} \tag{4.6}$$

<u>Proof</u>

We write

$$\begin{aligned}(4.7)\qquad \int_0^\infty J_\nu(st)f(t)dt &= \int_0^\infty J_\nu(st)G(t)dt + \int_0^\infty J_\nu(st)L(t)dt \\ &= \frac{1}{2}\int_0^\infty H_\nu^{(1)}(st)G(t)dt + \frac{1}{2}\int_0^\infty H_\nu^{(2)}(st)G(t)dt + \int_0^\infty J_\nu(st)L(t)dt\end{aligned}$$

where $G(t)$ and $L(t)$ are defined by (3.5) and (3.6). The first two integrals on the right hand side of (4.7) have been evaluated before (see (3.17) and (3.45)). To evaluated the last integral, we introduce the following functions defined by:

$$(4.8)\qquad P_0(t) = L(t) = f(t) - G(t)$$

$$(4.9)\qquad P_j(t) = -\frac{d}{dt}P_{j-1}(t) + (\nu + j)t^{-1}P_{j-1}(t)\ ,\quad j=1,\dots,M.$$

It can be seen that

$$(4.10)\qquad P_0(t) = O(\phi_N(t)) = O(t^\beta)\ ,\quad t \to 0^+$$

From assumptions (ii) and (vii) and equations (3.6), (4.8) and (4.9), it follows that

$$(4.11)\qquad P_j(t) = O(t^{\beta - j})\ ,\quad j=1,\dots,M.$$

By substituting (4.8) into (4.9), we have

$$(4.12)\qquad P_j(t) = \sum_{n=0}^{j} \omega_n^j t^{n-j} f^{(n)}(t) - \sum_{n=0}^{j} \omega_n^j t^{n-j} G^{(n)}(t)\ ,\quad j=1,..M.$$

where ω_n^j are constant coefficients. From assumption (ii) and equation (3.26), it can be seen that

$$(4.13)\qquad P_j(t) = o(t^{-1/2})\ ,\quad t \to +\infty\ ,\quad j=0,1,\dots,M-1.$$

Now we can treat the third integral on the right hand side of (4.7). We write

$$(4.14)\qquad \int_0^\infty J_\nu(st)dt = \lim_{a\to+\infty}\int_0^a J_\nu(st)L(t)dt$$

Using the formula

(4.15) $$J_\nu(z)=\frac{d}{dz}J_{\nu+1}(z)+(\nu+1)z^{-1}J_{\nu+1}(z)$$

and integrating by parts once, we have

(4.16) $$\int_0^\infty J_\nu(st)L(t)dt=\lim_{a\to+\infty}\frac{1}{s}L(t)J_{\nu+1}(st)\,|_0^a+\frac{1}{s}\lim_{a\to+\infty}\int_0^a P_1(t)J_{\nu+1}(st)dt\ .$$

Noting that

(4.17) $$J_\nu(z)\sim\sqrt{\frac{2}{\pi z}}\cos(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)\ ,\quad z\to\infty\ ,\quad |\arg z|<\pi$$

and using equations (4.2), (4.10) and (4.13), it can be seen that the integrated term vanishes. Continuing this procedure *M-1* times (all the integrated terms vanish), we obtain

(4.18) $$\int_0^\infty L(t)J_\nu(st)dt=(\frac{1}{s})^M\lim_{a\to+\infty}\int_0^a P_M(t)H_p^{(1)}(st)dt$$

It can be shown that

$$\int_0^\infty P_M(t)J_{\nu+M}(st)dt$$

exists. Indeed, by using equation (4.12), we have

(4.19) $$\int_0^\infty P_M(t)J_{\nu+M}(st)dt=\int_0^1 P_M(t)J_{\nu+M}(st)dt+\sum_{n=0}^{M}\omega_n^M\int_1^\infty t^{n-M}f^{(n)}(t)J_{\nu+M}(st)dt$$
$$-\sum_{n=0}^{M}\omega_n^M\int_1^\infty t^{n-M}G^{(n)}(t)J_{\nu+M}(st)dt$$

By assumption (iii) (in section 4) and the asymptotic properties of $P_M(t)$ and $G^{(m)}(t)$, it is easy to see that all the integrals on the right hand side of (4.19) exists. Thus

(4.20) $$\int_0^\infty J_\nu(st)L(t)dt=(1/s)^M\int_0^\infty P_M(t)J_{\nu+M}(st)dt=O(s^{-M})\ ,\quad s\to+\infty$$

Combining equations (4.7), (4.20), (3.17) and (3.45) gives equation (4.5).

For the second part of theorem 4, we only require to prove that $s^{-M} = o(\Phi_{N-1}{}^{i}(s))$ or $s^{-M} = o(\Phi_{N-1}{}^{-i}(s))$. The proof is very similar to that of the second part of theorem 1. We omit the details here.

We point out that the condition $\alpha - \nu > -1$ in assumption (vi) is stronger than necessary as $J_\nu(t) = O(t^\nu)$ $(\nu \geq 0)$, $t \to o^+$. What is actually required is that $\alpha + \nu > -1$. In practice we could make certain modifications to include the case $\alpha + \nu > -1$. For instance, we may introduce two functions $f_1(t)$ and $f_2(t)$ and write

(4.21) $$f(t)=f_1(t)+f_2(t)$$

such that $f_1(t)$ satisfies all the requirement of theorem 4 and the J_ν - transform of $f_2(t)$ exists and can be explicitly evaluated. Then

(4.22) $$\int_0^\infty J_\nu(st)f(t)dt=\int_0^\infty J_\nu(st)f_1(t)dt+\int_0^\infty J_\nu(st)f_2(t)dt\ .$$

By applying theorem 4 to the J_ν - transform of $f_1(t)$ we can consequently obtain the asymptotic expansion of the J_ν - transform of $f(t)$.

Example The asymptotic expansions of the $H_1^{(1)}$ -, $H_1^{(2)}$ -, Y_1 - and J_1 - transform of the following function

$$f(t)=\frac{t^\gamma Y_1(t)}{(1+t)^\mu},\quad \mu>\gamma>1\ .$$

(i)$H_1^{(1)}$ *- transform case*

We have

$$f(t)=\frac{t^\gamma Y_1(t)}{(1+t)^\mu}\sim Y_1(t)\sum_{n=0}^\infty \binom{-\mu}{n}t^{n+\gamma}=\sum_{n=0}^\infty \binom{-\mu}{n}\phi_n(t)\ ,\quad t\to0^+$$

where

$$\phi_n(t)=Y_1(t)t^{n+\gamma} \ .$$

We note that, due to the asymptotic behaviour of $Y_1(t)$, $\phi_n(t)$ cannot be expanded in the form of (1.6), thus, the methods in literature (Wong 1976, Soni 1982) are not applicable to this example. Here we apply our results.

It is easy to check that all the conditions of theorem 1 are satisfied (we take $M = N = 2k+1$ ($k \geq 1$ is an integer) first, and then let k tend to infinity). So we have

$$\int_0^\infty f(t)H_0^{(1)}(st)dt=-\frac{2i}{\pi}\sum_{n=0}^{N-1}\binom{-\mu}{n}\Phi_n^i(s)+o(\Phi_N^i(s)) \ , \quad s\to+\infty$$

where

$$\Phi_n^i(s)=\int_0^\infty\phi_n(it)K_1(st)dt=-e^{i\frac{n+\gamma}{2}\pi}\int_0^\infty t^{n+\gamma}I_1(t)K_1(st)dt \ + \ \frac{2}{\pi}e^{i\frac{n+\gamma+1}{2}\pi}\int_0^\infty t^{n+\gamma}K_1(t)K_1(st)dt$$

In obtaining the above equation, use has been made of the following relation

$$Y_1(it)=-I_1(t)+i\frac{2}{\pi}K_1(t) \ .$$

Using the fomulas given by Oberhettinger (1974, p124), we have

$$\int_0^\infty t^{n+\gamma}I_1(t)K_1(st)dt=2^{n+\gamma-1}\Gamma(\frac{n+\gamma+3}{2})\Gamma(\frac{n+\gamma+1}{2})(s^2-1)^{-\frac{n+\gamma+1}{2}}P^{-1}_{-\frac{n+\gamma+1}{2}}(\frac{s^2+1}{s^2-1}) \ ,$$

$$\int_0^\infty t^{n+\gamma}K_1(t)K_1(st)dt=\pi^{1/2}2^{n+\gamma-2}\Gamma(\frac{n+\gamma+1}{2})\Gamma(\frac{n+\gamma+3}{2})\Gamma(\frac{n+\gamma-1}{2})s^{-1/2}(s^2-1)^{-\frac{n+\gamma}{2}}P_{\frac{1}{2}}^{-\frac{n+\gamma}{2}}(\frac{s^2+1}{2s}) \ .$$

Where $p^v_y(s)$ is the Legendre function. Thus, letting k tend to infinity, we have

$$\int_0^\infty\frac{t^\gamma Y_1(t)}{(1+t)^\mu}H_1^{(1)}(st)dt\sim\frac{1}{\pi}\sum_{n=0}^\infty\binom{-\mu}{n}2^{n+\gamma}e^{i\frac{n+\gamma+1}{2}\pi}\Gamma(\frac{n+\gamma+3}{2})\Gamma(\frac{n+\gamma+1}{2})(s^2-1)^{-\frac{n+\gamma+1}{2}}P^{-1}_{-\frac{n+\gamma+1}{2}}(\frac{s^2+1}{s^2-1})$$
$$+\frac{s^{-1/2}}{\pi^{3/2}}\sum_{n=0}^\infty\binom{-\mu}{n}2^{n+\gamma}e^{i\frac{n+\gamma}{2}}\Gamma(\frac{n+\gamma+1}{2})\Gamma(\frac{n+\gamma+3}{2})\Gamma(\frac{n+\gamma-1}{2})(s^2-1)^{-\frac{n+\gamma}{2}}P_{\frac{1}{2}}^{-\frac{n+\gamma}{2}}(\frac{s^2+1}{2s}) \ ,$$

as $s \to +\infty$.

(ii)$H_l^{(2)}$ - transform case

Applying theorem 2, we have

$$\int_0^\infty e^{-t} t^{-1/2}(1+t)^{1/2} H_0^{(2)}(st)dt \sim \frac{2i}{\pi}\sum_{n=0}^{\infty} \binom{1/2}{n} \Phi_n^{-i}(s), \quad s \to +\infty$$

where

$$\Phi_n^{-i}(s) = \int_0^\infty \phi_n(-it) K_1(st)dt = -e^{-i\frac{n+\gamma}{2}\pi}\int_0^\infty t^{n+\gamma} I_1(t)K_1(st)dt - \frac{2}{\pi}e^{-i\frac{n+\gamma-1}{2}\pi}\int_0^\infty t^{n+\gamma}K_1(t)K_1(st)dt.$$

Using the above-mentioned formula given by Oberhettinger (1974), we have

$$\int_0^\infty \frac{t^\gamma Y_1(t)}{(1+t)^\mu} H_1^{(2)}(st)dt \sim \frac{1}{\pi}\sum_{n=0}^{\infty}\binom{-\mu}{n} 2^{n+\gamma} e^{-i\frac{n+\gamma+1}{2}\pi}\Gamma\left(\frac{n+\gamma+3}{2}\right)\Gamma\left(\frac{n+\gamma+1}{2}\right)(s^2-1)^{-\frac{n+\gamma+1}{2}} P^{-1}_{-\frac{n+\gamma+1}{2}}\left(\frac{s^2+1}{s^2-1}\right)$$
$$+\frac{s^{-1/2}}{\pi^{3/2}}\sum_{n=0}^{\infty}\binom{-\mu}{n} 2^{n+\gamma} e^{-i\frac{n+\gamma+1}{2}}\Gamma\left(\frac{n+\gamma+1}{2}\right)\Gamma\left(\frac{n+\gamma+3}{2}\right)\Gamma\left(\frac{n+\gamma-1}{2}\right)(s^2-1)^{-\frac{n+\gamma}{2}} P_{\frac{1}{2}}^{-\frac{n+\gamma}{2}}\left(\frac{s^2+1}{2s}\right),$$

as $s \to +\infty$.

(iii)Y_l - transform case

From theorem 3, we have

$$\int_0^\infty \frac{t^\gamma Y_1(t)}{(1+t)^\mu} Y_1(st)dt \sim \frac{1}{\pi}\sum_{n=0}^{\infty}\binom{-\mu}{n}\Phi_n^{i}(s) - \frac{1}{\pi}\sum_{n=0}^{\infty}\binom{-\mu}{n}\Phi_n^{-i}(s), \quad s \to +\infty$$

Using the above-obtained results, we have

$$\int_0^\infty \frac{t^\gamma Y_1(t)}{(1+t)^\mu} Y_1(st)dt \sim \frac{-i}{\pi}\sum_{n=0}^{\infty}\binom{-\mu}{n} 2^{n+\gamma}\sin\left(\frac{n+\gamma}{2}\pi\right)\Gamma\left(\frac{n+\gamma+3}{2}\right)\Gamma\left(\frac{n+\gamma+1}{2}\right)(s^2-1)^{-\frac{n+\gamma+1}{2}} P^{-1}_{-\frac{n+\gamma+1}{2}}\left(\frac{s^2+1}{s^2-1}\right)$$
$$+\frac{is^{-1/2}}{\pi^{3/2}}\sum_{n=0}^{\infty}\binom{-\mu}{n} 2^{n+\gamma}\cos\left(\frac{n+\gamma}{2}\pi\right)\Gamma\left(\frac{n+\gamma+1}{2}\right)\Gamma\left(\frac{n+\gamma+3}{2}\right)\Gamma\left(\frac{n+\gamma-1}{2}\right)(s^2-1)^{-\frac{n+\gamma}{2}} P_{\frac{1}{2}}^{-\frac{n+\gamma}{2}}\left(\frac{s^2+1}{2s}\right),$$

as $s \to +\infty$.

(iv)J_l - transform case

From theorem 4, and using the above-obtained results, we have

$$\int_0^\infty \frac{t^\gamma Y_1(t)}{(1+t)^\mu} J_1(st)dt \sim \frac{-i}{\pi} \sum_{n=0}^\infty \binom{-\mu}{n} 2^{n+\gamma} \cos\left(\frac{n+\gamma}{2}\pi\right) \Gamma\left(\frac{n+\gamma+3}{2}\right) \Gamma\left(\frac{n+\gamma+1}{2}\right) (s^2-1)^{-\frac{n+\gamma+1}{2}} P^{-1}_{-\frac{n+\gamma+1}{2}}\left(\frac{s^2+1}{s^2-1}\right)$$

$$-\frac{is^{-1/2}}{\pi^{3/2}} \sum_{n=0}^\infty \binom{-\mu}{n} 2^{n+\gamma} \sin\left(\frac{n+\gamma}{2}\pi\right) \Gamma\left(\frac{n+\gamma+1}{2}\right) \Gamma\left(\frac{n+\gamma+3}{2}\right) \Gamma\left(\frac{n+\gamma-1}{2}\right) (s^2-1)^{-\frac{n+\gamma}{2}} P_{\frac{1}{2}}^{-\frac{n+\gamma}{2}}\left(\frac{s^2+1}{2s}\right),$$

as $s \to +\infty$.

Acknowledgement

The author would like to thank Professor D. Naylor for his critical reading of the first manuscript and valuable comments. The paper was written when the author was a postdoctoral fellow in the Department of Applied Mathematics of the University of Western Ontario and was supported by NSERC funds. He is also grateful to the refree for valuable comments.

References

1. Abramowitz, M. and Stegun, I. A.: Handbook of Mathematical Functions, Applied Mathematics Series 55, New York: National Bureau of Standards, 1974.
2. Dai, H.-H.: On the general asymptotic expansions of K_ν - transforms, *Applicable Analysis,* 46, 157-174(1992).
3. Dai, H.-H. and Naylor, D.: On an asymptotic expansions of Fourier integrals, *Proc. R. Soc. Lond.* A., 436, 109-120(1992).
4. Handelsman, R. A. and Lew, J. S.: Asymptotic expansion of a class of integral transform via Mellin transforms, *Arch. Rational Mech. Anal.*, **35**, 382-396(1969).
5. Naylor, D.: On an asymptotic expansion of the Kontonrovich-Lebedev transform, *Applicable Analysis*, **39**, 249-263(1990).
6. Oberhettinger, F.: Tables of Mellin Transforms, New York: Springer-Verlag, 1974.
7. Olver, F. W. J.: Error bounds for stationary phase approximations, *SIAM J. Math. Anal.*, 5, 19-29(1974).
8. Soni, K.: Asymptotic expansion of the Hankel transform with explicit remainder term, *Quart. Appl. Math.*, **50**, 1-14(1982).
9. Soni, K. and Soni, R. P.: A note on summability and asymptotics, *SIAM J. Math. Anal.* **16**, 392-404(1985).
10. Watson, G. N.: Theory of Bessel Functions, 2nd ed., Cambridge: Cambridge University

Press, 1958.

11. Wong, R.: Error bounds for asymptotic expansions of Hankel transforms, *SIAM J. Math. Anal.*, 7, 767-770(1976).

Department of Applied Mathematics*
The University of Western Ontario
London, Ontario
Canada N6A 5B9

*Current address:
Department of Applied Mathematics
University of Manitoba
Winnipeg, Manitoba
Canada R3T 2N2

Analysis 14, 43 - 56 (1994)

Analysis

OSCILLATORY BEHAVIOUR OF SOLUTIONS OF COUPLED HYPERBOLIC DIFFERENTIAL EQUATIONS

N. PARHI and M. KIRANE

Received : January 4, 1993; revised: July 15, 1993

Abstract: Sufficient conditions have been obtained for oscillation of all solutions of a class of coupled hyperbolic differential equations of neutral type.

Subject Classification: (AMS 1991)

1. Oscillatory behaviour of solutions of hyperbolic differential equations of neutral type has been studied by several authors in recent years (see [2, 4, 5, 8, 9] and the references therein). As coupled hyperbolic equations occur in many mathematical models in physics (see [6]), it seems interesting to study the oscillation of solutions of such equations. It appears to the present authors that the study of oscillatory behaviour of solutions of coupled hyperbolic differential equations of neutral type has not been undertaken before. It is interesting to note that the present study is applicable to a class of coupled nonlinear Klein–Gordon equations ([3, 6]) which describe the motion of charged mesons in an electromagnetic field.

We consider coupled hyperbolic differential equations of neutral type of the form

$$\begin{aligned} & u_{tt}(x,t) + \delta_1\, u_{tt}(x,t-\rho_1) + \gamma_1\, u_t(x,t-\theta_1) \\ & \quad - [\alpha_1 \Delta u(x,t) + \alpha_2 \Delta u(x,t-\tau_1) + \alpha_3 \Delta v(x,t) + \alpha_4 \Delta v(x,t-\tau_2)] \\ & \quad + c_1(x,t,u(x,t),u(x,t-\sigma_1),v(x,t),v(x,t-\sigma_2)) = f_1(x,t) \end{aligned} \tag{1}$$

and

$$\begin{aligned} & v_{tt}(x,t) + \delta_2\, v_{tt}(x,t-\rho_2) + \gamma_2\, v_t(x,t-\theta_2) \\ & \quad - [\beta_1 \Delta u(x,t) + \beta_2 \Delta u(x,t-\tau_3) + \beta_3 \Delta v(x,t) + \beta_4 \Delta v(x,t-\tau_4)] \\ & \quad + c_2(x,t,u(x,t),u(x,t-\sigma_3),v(x,t),v(x,t-\sigma_4)) = f_2(x,t) \end{aligned} \tag{2}$$

$(x,t) \in Q := \Omega \times (0,\infty)$, where Ω is a bounded domain in R^n with smooth boundary

$\Gamma := \partial\Omega$ and Δ is the Laplacian in R^n, with boundary conditions

(B_1) $$\frac{\partial u}{\partial \nu} = \psi_1 \quad \text{and} \quad \frac{\partial v}{\partial \nu} = \psi_2 \quad \text{on} \quad \Gamma \times (0,\infty)$$

(B_2) $$\frac{\partial u}{\partial \nu} + \mu_1(x,t)\, u = \psi_1 \quad \text{and} \quad \frac{\partial v}{\partial \nu} + \mu_2(x,t)\, v = \psi_2 \quad \text{on} \quad \Gamma \times (0,\infty)$$

(B_3) $$u = \tilde{\psi}_1 \quad \text{and} \quad v = \tilde{\psi}_2 \quad \text{on} \quad \Gamma \times (0,\infty)\,,$$

where $\psi_1, \psi_2, \tilde{\psi}_1$ and $\tilde{\psi}_2$ are real–valued continuous functions on $\Gamma \times (0,\infty)$, μ_1 and μ_2 are non–negative continuous functions on $\Gamma \times (0,\infty)$ and ν denotes the unit exterior normal vector to Γ.

A pair of functions $(u(x,t), v(x,t))$ such that each of u and $v \in C^2(\Omega \times (-m_1, \infty)) \cap C^1(\bar{\Omega} \times (-m_2,\infty))$ is said to be a solution of the problem (1,2), (B_i), $i = 1,2,3$, if the equations (1) and (2) and the boundary conditions (B_i), $i = 1,2,3$, are satisfied simultaneously, where $m_1 = \max\{\rho_1,\rho_2,\tau_1,\tau_2,\tau_3,\tau_4\}$ and $m_2 = \max\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\theta_1,\theta_2\}$. A real–valued continuous function $\omega(x,t)$ in Q is said to oscillate in Q if for every $t_0 \geq 0$ there exists a point $(x_1,t_1) \in Q_{t_0} \equiv \Omega \times (t_0,\infty)$ such that $\omega(x_1,t_1) = 0$. A solution $(u(x,t), v(x,t))$ of the problem (1,2), (B_i), $i = 1,2,3$, is said to oscillate in Q if $u(x,t)$ or $v(x,t)$ oscillates in Q. It is said to oscillate strongly in Q if each of $u(x,t)$ and $v(x,t)$ oscillates in Q.

The following assumptions are made for the work in this paper:

(A_1) $c_1(x,t,\xi_1,\xi_2,\eta_1,\eta_2)$ and $c_2(x,t,\xi_1,\xi_2,\eta_1,\eta_2)$ are real–valued continuous functions in $\bar{Q} \times \mathbb{R}^4$ such that

(i) $$c_1(x,t,\xi_1,\xi_2,\eta_1,\eta_2) \begin{cases} \geq 0 & \text{if } \xi_1 \text{ and } \xi_2 \in (0,\infty) \\ \leq 0 & \text{if } \xi_1 \text{ and } \xi_2 \in (-\infty,0) \end{cases}$$

(ii) $$c_2(x,t,\xi_1,\xi_2,\eta_1,\eta_2) \begin{cases} \geq 0 & \text{if } \eta_1 \text{ and } \eta_2 \in (0,\infty) \\ \leq 0 & \text{if } \eta_1 \text{ and } \eta_2 \in (-\infty,0) \end{cases}$$

(A_2) $f_1(x,t)$ and $f_2(x,t)$ are real–valued continuous functions in $\bar{Q}$.

(A_3) $\alpha_2,\alpha_3,\alpha_4,\beta_4,\delta_1,\delta_2,\gamma_1,\gamma_2,\rho_1,\rho_2,\theta_1,\theta_2,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\tau_1,\tau_2,\tau_3,\tau_4$ are non–negative constants, β_1,β_2 are non–positive constants, α_1,β_3 are postive constants.

(A_4) $\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_4,\delta_1,\delta_2,\gamma_1,\gamma_2,\rho_1,\rho_2,\theta_1,\theta_2,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\tau_1,\tau_2,\tau_3,\tau_4$ are non–negative constants, α_1,β_3 are postive constants.

(A_5) $T_0 = \max\{\rho_1,\rho_2,\theta_1,\theta_2,\tau_1,\tau_2,\tau_3,\tau_4,\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$.

It is well-known (see [1]) that the first eigenvalue λ_1 of the eigenvalue problem $-\Delta\omega = \lambda\omega$ in Ω, $\omega = 0$ on $\partial\Omega$ is positive and the associated eigenfunction $\varphi(x)$ is of constant sign in Ω and hence may be chosen positive in Ω. We shall assume:

(i) $(\alpha_1 + \beta_3)^2 \geq 4(\alpha_1\beta_3 - \alpha_3\beta_1)$,

(ii) $\alpha_1\beta_3 > \beta_1\alpha_3$

which ensures the total hyperbolicity of Eqs. (1) and (2) (see [7]).

We use the following notations in the sequel: For u and $v \in C^2(Q) \cap C^1(\bar{Q})$, we denote

$$U(t) = \int_\Omega u(x,t)dx, \quad t > 0, \quad V(t) = \int_\Omega v(x,t)dx, \quad t > 0$$
$$\tilde{U}(t) = \int_\Omega \varphi(x)\ u(x,t)dx, \quad t > 0, \quad \tilde{V}(t) = \int_\Omega \varphi(x)\ v(x,t)dx, \quad t > 0\ .$$

Further we denote

$$\Psi_i(t) = \int_\Gamma \psi_i(x,t)ds, \quad i = 1,2, \quad t > 0$$
$$\tilde{\Psi}_i(t) = \int_\Gamma \tilde{\psi}_i(x,t)\ \frac{\partial\varphi(x)}{\partial\nu}\ ds, \quad i = 1,2, \quad t > 0$$
$$F_i(t) = \int_\Omega f_i(x,t)dx, \quad i = 1,2, \quad t > 0$$
$$\tilde{F}_i(t) = \int_\Omega \varphi(x)\ f_i(x,t)dx, \quad i = 1,2, \quad t > 0\ .$$

We obtain the following results in this work.

THEOREM I Suppose that $(A_1),(A_2),(A_4),(A_5)$ hold. If

$$\liminf_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^t (t-s)[F_1(s) + \alpha_1\Psi_1(s) + \alpha_2\Psi_1(s-\tau_1) + \alpha_3\Psi_2(s) + \alpha_4\Psi_2(s-\tau_2)]ds$$
(H_1) $$= -\infty\ ,$$

and

$$\limsup_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^t (t-s)[F_1(s) + \alpha_1\Psi_1(s) + \alpha_2\Psi_1(s-\tau_1) + \alpha_3\Psi_2(s) + \alpha_4\Psi_2(s-\tau_2)]ds$$
(H_2) $$= \infty\ ,$$

or, if

$$\liminf_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^t (t-s)[F_2(s) + \beta_1\Psi_1(s) + \beta_2\Psi_1(s-\tau_3) + \beta_3\Psi_2(s) + \beta_4\Psi_2(s-\tau_4)]ds$$
(H_3) $$= -\infty\ ,$$

and

$$\limsup_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)[F_2(s)+\beta_1\Psi_1(s)+\beta_2\Psi_1(s-\tau_3)+\beta_3\Psi_2(s)+\beta_4\Psi_2(s-\tau_4)]ds$$
$$(H_4) \qquad = \infty \,,$$

for any $t_0 \geq 0$, then all solutions of the problem (1,2), (B_1) oscillate in Q.

THEOREM II Suppose that $(A_1),(A_2),(A_4),(A_5),(H_1)-(H_4)$ hold. Then all solutions of the problem (1,2), (B_1) oscillate strongly in Q.

THEOREM III Let the conditions $(A_1),(A_2),(A_3)$ and (A_5) be satisfied. If $(H_1)-(H_4)$ hold, then all solutions of the problem (1,2), (B_2) oscillate in Q.

THEOREM IV Let $(A_1),(A_2),(A_3)$ and (A_5) hold. If

$$\liminf_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)[\tilde{F}_1(s)-\alpha_1\tilde{\Psi}_1(s)-\alpha_2\tilde{\Psi}_1(s-\tau_1)-\alpha_3\tilde{\Psi}_2(s)-\alpha_4\tilde{\Psi}_2(s-\tau_2)]ds$$
$$= -\infty \,,$$
$$\liminf_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)[\tilde{F}_2(s)-\beta_1\tilde{\Psi}_1(s)-\beta_2\tilde{\Psi}_1(s-\tau_3)-\beta_3\tilde{\Psi}_2(s)-\beta_4\tilde{\Psi}_2(s-\tau_4)]ds$$
$$= -\infty \,,$$
$$\limsup_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)[\tilde{F}_1(s)-\alpha_1\tilde{\Psi}_1(s)-\alpha_2\tilde{\Psi}_1(s-\tau_1)-\alpha_3\tilde{\Psi}_2(s)-\alpha_4\tilde{\Psi}_2(s-\tau_2)]ds$$
$$= \infty \,,$$

and

$$\limsup_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t}(t-s)[\tilde{F}_2(s)-\beta_1\tilde{\Psi}_1(s)-\beta_2\tilde{\Psi}_1(s-\tau_3)-\beta_3\tilde{\Psi}_2(s)-\beta_4\tilde{\Psi}_2(s-\tau_4)]ds$$
$$= \infty \,,$$

for any $t_0 \geq 0$, then all solutions of the problem (1,2), (B_3) oscillate in Q.

2. This section deals with the formulation of the problem. We need the following lemmas in the sequel.

LEMMA 2.1. Suppose that $(A_1)(i),(A_2),(A_4),(A_5)$ are satisfied. If $(u(x,t),v(x,t))$ is a solution of the problem (1,2), (B_1) such that $u(x,t)>0$ in Q_{t_0} for some $t_0 \geq 0$, then the function $U(t)$ satisfies the differential inequality of neutral type

$$\begin{aligned}(3) \qquad & y''(t)+\delta_1\, y''(t-\rho_1)+\gamma_1\, y'(t-\theta_1)\\ & \leq F_1(t)+\alpha_1\Psi_1(t)+\alpha_2\Psi_1(t-\tau_1)+\alpha_3\Psi_2(t)+\alpha_4\Psi_2(t-\tau_2)\end{aligned}$$

for $t > t_0 + T_0$.

Proof. Let $t > t_0 + T_0$. Integrating (1) with respect to x over the domain Ω we obtain

$$\begin{aligned} &U''(t) + \delta_1\, U''(t-\rho_1) + \gamma_1\, U'(t-\theta_1) \\ &\le F_1(t) + \alpha_1 \int_\Omega \Delta u(x,t)dx + \alpha_2 \int_\Omega \Delta u(x,t-\tau_1)dx \\ &\quad + \alpha_3 \int_\Omega \Delta v(x,t)dx + \alpha_4 \int_\Omega \Delta v(x,t-\tau_2)dx\ . \end{aligned}$$

Green's formula yields

$$\int_\Omega \Delta u(x,t)dx = \int_\Gamma \frac{\partial u}{\partial \nu}\, ds = \int_\Gamma \psi_1(x,t)ds = \Psi_1(t)$$

and

$$\int_\Omega \Delta u(x,t-\tau_1)dx = \int_\Gamma \frac{\partial u}{\partial \nu}\,(x,t-\tau_1)ds = \int_\Gamma \psi_1(x,t-\tau_1)ds = \Psi_1(t-\tau_1)\ .$$

Thus we have

$$\begin{aligned} &U''(t) + \delta_1\, U''(t-\rho_1) + \gamma_1\, U'(t-\theta_1) \\ &\le F_1(t) + \alpha_1\, \Psi_1(t) + \alpha_2\, \Psi_1(t-\tau_1) + \alpha_3\, \Psi_2(t) + \alpha_4\, \Psi_2(t-\tau_2) \end{aligned}$$

for $t > t_0 + T_0$.

Hence the lemma is proved.

LEMMA 2.2. Let the assumptions $(A_1)(ii), (A_2), (A_4), (A_5)$ hold. If $(u(x,t), v(x,t))$ is a solution of the problem (1,2), (B_1) such that $v(x,t) > 0$ in Q_{t_0} for some $t_0 \ge 0$, then the function $V(t)$ satisfies the differential inequality of neutral type

$$\begin{aligned} &y''(t) + \delta_2\, y''(t-\rho_2) + \gamma_2\, y'(t-\theta_2) \\ &\le F_2(t) + \beta_1\Psi_1(t) + \beta_2\Psi_1(t-\tau_3) + \beta_3\Psi_2(t) + \beta_4\Psi_2(t-\tau_4) \end{aligned} \tag{4}$$

for $t > t_0 + T_0$.

The proof proceeds in the lines of that of Lemma 2.1 and hence is omitted.

LEMMA 2.3. Let the assumptions $(A_1)(i), (A_2), (A_3), (A_5)$ hold. Let $(u(x,t), v(x,t))$ be a solution of the problem (1,2), (B_2). If $u(x,t) > 0$ and $v(x,t) > 0$ in Q_{t_0} for some $t_0 \ge 0$, then the function $U(t)$ satisfies the inequality (3) for $t > t_0 + T_0$. If $u(x,t) < 0$ and $v(x,t) < 0$ in Q_{t_0} for some $t_0 \ge 0$, then the function $-U(t)$ satisfies the differential inequality of neutral type

$$\begin{aligned} &y''(t) + \delta_1\, y''(t-\rho_1) + \gamma_1\, y'(t-\theta_1) \\ &\le -[F_1(t) + \alpha_1\Psi_1(t) + \alpha_2\Psi_1(t-\tau_1) + \alpha_3\Psi_2(t) + \alpha_4\Psi_2(t-\tau_2)] \end{aligned} \tag{5}$$

for $t > t_0 + T_0$.

LEMMA 2.4. Suppose that $(A_1)(ii), (A_2), (A_3), (A_5)$ hold. Let $(u(x,t), v(x,t))$ be a solution of the problem (1,2) (B_2). If $u(x,t) < 0$ and $v(x,t) > 0$ in Q_{t_0} for some $t_0 \geq 0$, then the function $V(t)$ satisfies the inequality (4) for $t > t_0 + T_0$. If $u(x,t) > 0$ and $v(x,t) < 0$ in Q_{t_0} for some $t_0 \geq 0$, then the function $-V(t)$ satisfies the inequality

$$\begin{aligned} & y''(t) + \delta_2\, y''(t-\rho_2) + \gamma_2\, y'(t-\theta_2) \\ & \quad \leq -[F_2(t) + \beta_1 \Psi_1(t) + \beta_2 \Psi_1(t-\tau_3) + \beta_3 \Psi_2(t) + \beta_4 \Psi_2(t-\tau_4)] \end{aligned} \tag{6}$$

for $t > t_0 + T_0$.

The proof of each of the Lemmas 2.3 and 2.4 is similar to that of Lemma 2.1 and hence is omitted.

LEMMA 2.5. Let the assumptions (A_1)(i), $(A_2), (A_3), (A_5)$ hold. Let $(u(x,t), v(x,t))$ be a solution of the problem (1,2), (B_3). If $u(x,t) > 0$ and $v(x,t) > 0$ in Q_{t_0} for some $t_0 \geq 0$, then the function $\tilde{U}(t)$ satisfies the inequality

$$\begin{aligned} & y''(t) + \delta_1\, y''(t-\rho_1) + \gamma_1\, y'(t-\theta_1) \\ & \quad \leq \tilde{F}_1(t) - \alpha_1 \tilde{\Psi}_1(t) - \alpha_2 \tilde{\Psi}_1(t-\tau_1) - \alpha_3 \tilde{\Psi}_2(t) - \alpha_4 \tilde{\Psi}_2(t-\tau_2) \end{aligned} \tag{7}$$

for $t > t_0 + T_0$. If $u(x,t) < 0$ and $v(x,t) < 0$ in Q_{t_0} for some $t_0 \geq 0$, then the function $-\tilde{U}(t)$ satisfies the inequality

$$\begin{aligned} & y''(t) + \delta_1\, y''(t-\rho_1) + \gamma_1\, y'(t-\theta_1) \\ & \quad \leq -[\tilde{F}_1(t) - \alpha_1 \tilde{\Psi}_1(t) - \alpha_2 \tilde{\Psi}_1(t-\tau_1) - \alpha_3 \tilde{\Psi}_2(t) - \alpha_4 \tilde{\Psi}_2(t-\tau_2)] \end{aligned} \tag{8}$$

for $t > t_0 + T_0$.

Proof. Let $t > t_0 + T_0$. Multiplying (1) through by $\varphi(x)$ and integrating the resulting identity with respect to x over the domain Ω, we obtain

$$\begin{aligned} & \tilde{U}''(t) + \delta_1\, \tilde{U}''(t-\rho_1) + \gamma_1\, \tilde{U}'(t-\theta_1) \\ & \leq \tilde{F}_1(t) + \alpha_1 \int_\Omega \varphi(x)\, \Delta u(x,t)dx + \alpha_2 \int_\Omega \varphi(x)\, \Delta u(x,t-\tau_1)dx \\ & \qquad + \alpha_3 \int_\Omega \varphi(x)\, \Delta v(x,t)dx + \alpha_4 \int_\Omega \varphi(x)\, \Delta v(x,t-\tau_2)dx\ . \end{aligned}$$

Applying Green's formula we get

$$\begin{aligned} \int_\Omega \varphi(x)\, \Delta u(x,t)dx &= -\int_\Gamma \tilde{\psi}_1(x,t)\, \frac{\partial \varphi}{\partial \nu}\, ds - \lambda_1 \int_\Omega u(x,t)\, \varphi(x)dx \\ &= -\tilde{\Psi}_1(t) - \lambda_1\, \tilde{U}(t) \end{aligned}$$

and

$$\int_\Omega \varphi(x)\ \Delta u(x,t-\tau_1)dx = -\int_\Gamma \tilde{\psi}_1(x,t-\tau_1)\ \frac{\partial\varphi}{\partial\nu}\ ds - \lambda_1 \int_\Omega u(x,t-\tau_1)\ \varphi(x)dx$$
$$= -\tilde{\Psi}_1(t-\tau_1) - \lambda_1\ \tilde{U}(t-\tau_1)\ .$$

Thus we have

$$\tilde{U}''(t) + \delta_1\ \tilde{U}''(t-\rho_1) + \gamma_1\ \tilde{U}'(t-\theta_1)$$
$$\le \tilde{F}_1(t) - \alpha_1\tilde{\Psi}_1(t) - \alpha_2\tilde{\Psi}_1(t-\tau_1) - \alpha_3\tilde{\Psi}_2(t) - \alpha_4\tilde{\Psi}_2(t-\tau_2)\ .$$

Thus the first part of the lemma is proved. The proof of the second part of the lemma proceeds as above.

Hence the lemma is proved.

<u>LEMMA 2.6.</u> Suppose that the conditions $(A_1)(ii), (A_2), (A_3)$ and (A_5) are satisfied. Let $(u(x,t), v(x,t))$ be a solution of the problem (1,2), (B_3). If $u(x,t) < 0$ and $v(x,t) > 0$ in Q_{t_0} for some $t_0 \ge 0$, then the function $\tilde{V}(t)$ satisfies the inequality

$$y''(t) + \delta_2\ y''(t-\rho_2) + \gamma_2\ y'(t-\theta_2)$$
$$\le \tilde{F}_2(t) - \beta_1\tilde{\Psi}_1(t) - \beta_2\tilde{\Psi}_1(t-\tau_3) - \beta_3\tilde{\Psi}_2(t) - \beta_4\tilde{\Psi}_2(t-\tau_4) \tag{9}$$

for $t > t_0 + T_0$. If $u(x,t) > 0$ and $v(x,t) < 0$ in Q_{t_0} for some $t_0 \ge 0$, then the function $-\tilde{V}(t)$ satisfies the inequality

$$y''(t) + \delta_2\ y''(t-\rho_2) + \gamma_2\ y'(t-\theta_2)$$
$$\le -[\tilde{F}_2(t) - \beta_1\tilde{\Psi}_1(t) - \beta_2\tilde{\Psi}_1(t-\tau_3) - \beta_3\tilde{\Psi}_2(t) - \beta_4\tilde{\Psi}_2(t-\tau_4)] \tag{10}$$

for $t > t_0 + T_0$.

The proof is similar to that of Lemma 2.5 and hence is omitted.

<u>THEOREM 2.7.</u> Let the conditions $(A_1), (A_2), (A_4), (A_5)$ hold. If the differential inequalities (3) and (5) or if the differential inequalities (4) and (6) do not admit positive solutions for large t, then all solutions of the problem (1,2), (B_1) oscillate in Q.

Proof. Let $(u(x,t), v(x,t))$ be a solution of the problem (1,2), (B_1) such that it does not oscillate in Q. Then there exists a $t_0 > 0$ such that $u(x,t) \ne 0$ and $v(x,t) \ne 0$ in Q_{t_0}.

Suppose that the differential inequalities (3) and (5) do not admit positive solutions for large t. Clearly, $u(x,t) \ne 0$ in Q_{t_0} implies that $u(x,t) > 0$ or < 0 in Q_{t_0}. If $u(x,t) > 0$ in Q_{t_0}, then $U(t)$ is a positive solution of (3) for $t > t_0 + T_0$ by Lemma 2.1, a contradiction. If $u(x,t) < 0$ in Q_{t_0}, then we set $\hat{u}(x,t) = -u(x,t)$ for $(x,t) \in Q$. Hence $\hat{u}(x,t) > 0$ in Q_{t_0} and $(\hat{u}(x,t), v(x,t))$ is a solution of the problem

$$u_{tt}(x,t) + \delta_1\ u_{tt}(x,t-\rho_1) + \gamma_1\ u_t(x,t-\theta_1)$$
$$- [\alpha_1\Delta u(x,t) + \alpha_2\Delta u(x,t-\tau_1) - \alpha_3\Delta v(x,t) - \alpha_4\Delta v(x,t-\tau_2)]$$
$$- c_1(x,t,-u(x,t),-u(x,t-\sigma_1), v(x,t), v(x,t-\sigma_2)) = -f_1(x,t)$$

and

$$
\begin{aligned}
&v_{tt}(x,t) + \delta_2\, v_{tt}(x,t-\rho_2) + \gamma_2\, v_t(x,t-\theta_2) \\
&\quad - [-\beta_1 \Delta u(x,t) - \beta_2 \Delta u(x,t-\tau_3) + \beta_3 \Delta v(x,t) + \beta_4 \Delta v(x,t-\tau_4)] \\
&\quad + c_2(x,t,-u(x,t),-u(x,t-\sigma_3),v(x,t),v(x,t-\sigma_4)) = f_2(x,t)
\end{aligned}
$$

$(x,t) \in Q$ with boundary conditions

$$\frac{\partial u}{\partial \nu} = -\psi_1 \quad \text{and} \quad \frac{\partial v}{\partial \nu} = \psi_2 \quad \text{on} \quad \Gamma \times (0,\infty)\ .$$

Proceeding as in Lemma 2.1 one may show that $\hat{U}(t)$ is a positive solution of (5), where

$$\hat{U}(t) = \int_\Omega \hat{u}(x,t)dx, \quad t > 0\ , \tag{11}$$

a contradiction.

If the differential inequalities (4) and (6) do not admit positive solutions for large t, then we proceed as above considering $v(x,t) \neq 0$ in Q_{t_0} to arrive at necessary contradictions.

Hence the theorem is proved.

<u>**THEOREM 2.8.**</u> Let the conditions $(A_1),(A_2),(A_4),(A_5)$ hold. Suppose that none of the inequalities (3), (4), (5) and (6) admit a positive solution for large t. Then all solutions of the problem (1,2), (B_1) oscillate strongly in Q.

Proof. Assume the contrary. So there exists a solution $(u(x,t),v(x,t))$ of the problem (1,2), (B_1) which does not oscillate strongly in Q. Thus $u(x,t)$ or $v(x,t)$ does not oscillate in Q. If $u(x,t)$ does not oscillate in Q, then there exists a $t_0 > 0$ such that $u(x,t) > 0$ or < 0 in Q_{t_0}. If $u(x,t) > 0$ in Q_{t_0}, then from Lemma 2.1 it follows that $U(t)$ is a positive solution of (3), a contradiction. If $u(x,t) < 0$ in Q_{t_0}, then setting $\hat{u}(x,t) = -u(x,t)$ and proceeding as in Lemma 2.1 it may be shown that $\hat{U}(t)$, given by (11), is a positive solution of (5), a contradiction. Similar contradictions may be obtained with the help of Lemma 2.2 if $v(x,t)$ does not oscillate in Q.

Thus the proof of the theorem is complete.

<u>**THEOREM 2.9.**</u> Let the assumptions $(A_1),(A_2),(A_3),(A_5)$ hold. If the inequalities (3), (4), (5) and (6) do not admit positive solutions for large t then every solution of the problem (1,2), (B_2) oscillates in Q.

Proof. If possible, suppose that $(u(x,t),v(x,t))$ is a solution of the problem (1,2), (B_2) which does not oscillate in Q. So there exists a $t_0 > 0$ such that $u(x,t) \neq 0$ and $v(x,t) \neq 0$ in Q_{t_0}. If $u(x,t) > 0$ and $v(x,t) > 0$ in Q_{t_0}, then $U(t)$ is a positive solution of (3) for $t > t_0 + T_0$ by Lemma 2.3, a contradiction. If $u(x,t) > 0$ and $v(x,t) < 0$ in Q_{t_0}, then $-V(t)$ is a positive solution of (6) for $t > t_0 + T_0$ by Lemma 2.4, a contradiction. If

$u(x,t) < 0$ and $v(x,t) > 0$ in Q_{t_0}, then from Lemma 2.4 it follows that $V(t)$ is a positive solution of (4) for $t > t_0 + T_0$, a contradiction. If $u(x,t) < 0$ and $v(x,t) < 0$ in Q_{t_0}, then from Lemma 2.3 we conclude that $-U(t)$ is a positive solution of (5) for $t > t_0 + T_0$, a contradiction.

Thus the theorem is proved.

THEOREM 2.10. Suppose that $(A_1), (A_2), (A_3), (A_5)$ hold. If the inequalities (7), (8), (9), (10) do not admit postive solutions for large t, then every solution of the problem (1,2), (B_3) oscillates in Q.

The proof follows from Lemmas 2.5 and 2.6.

3. In this section we obtain sufficient conditions so that differential inequality of neutral type

$$y''(t) + \lambda_1\, y''(t-\rho) + \lambda_2\, y'(t-\theta) \le g(t)\ , \tag{12}$$

where $\lambda_1 \ge 0, \lambda_2 \ge 0, \rho > 0, \theta > 0$ and $g(t)$ is a real-valued continuous function on $(0,\infty)$, does not admit a postive solution for large t. We also prove Theorems I–IV.

LEMMA 3.1. If

$$\liminf_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} (t-s)\; g(s)ds = -\infty$$

for every $t_0 > 0$, then (12) does not admit a positive solution for large t.

Proof. If possible, let $y(t)$ be a solution of (12) such that $y(t) > 0$ for $t > t_0 > 0$. Integrating (12) from t_1 to t, where $t > t_1 > t_0 + \max\{\rho, \theta\}$, we obtain

$$\begin{aligned} y'(t) + \lambda_1\, y'(t-\rho) &\le y'(t) + \lambda_1\, y'(t-\rho) + \lambda_2\, y(t-\theta) \\ &< c_1 + \int_{t_1}^{t} g(s)ds\ , \end{aligned}$$

where c_1 is a constant. Further integration from t_1 to t yields

$$0 < y(t) + \lambda_1\, y(t-\rho) < c_2 + c_1(t-t_1) + \int_{t_1}^{t} (t-s)\; g(s)ds\ ,$$

where c_2 is a constant. Thus

$$0 \le \liminf_{t\to\infty} \frac{1}{t-t_1}\, [y(t) + \lambda_1\, y(t-\rho)] \le -\infty\ ,$$

a contradiction. Hence the lemma is proved.

Proof of Theorem I. It follows from Lemma 3.1 and Theorem 2.7.

Proof of Theorem II. It follows from Lemma 3.1 and Theorem 2.8.

Proof of Theorem III. It follows from Lemma 3.1 and Theorem 2.9.

Proof of Theorem IV. It follows from Lemma 3.1 and Theorem 2.10.

Remark. We may note that Theorems I–IV are applicable to nonlinear Klein–Gordon equations

$$u_{tt} - \Delta u + \alpha^2 u + g^2\ v^2 u = 0$$
$$v_{tt} - \Delta v + \beta^2 v + h^2\ u^2 v = 0$$

$(x,t) \in Q$, where α, β, g and h are non–zero constants, with boundary conditions (B_i), $i = 1,2,3$, given in Section 1.

The following remark is useful for our examples.

Remark. If

$$\liminf_{t\to+\infty} \frac{1}{t} \int_0^t (t-s)\ G(s)\ ds = -\infty\ ,$$

then

$$\liminf_{t\to+\infty} \frac{1}{t-t_0} \int_{t_0}^t (t-s)\ G(s)\ ds = -\infty \quad \text{for any large} \quad t_0 > 0\ .$$

Indeed, for large $t_0 > 0$, we can write

$$\frac{1}{t-t_0} \int_{t_0}^t (t-s)\ G(s)\ ds = \frac{1}{t-t_0} \int_0^t (t-s)\ G(s)\ ds - \frac{1}{t-t_0} \int_0^{t_0} (t-s)\ G(s)\ ds =$$
$$= \frac{t}{t-t_0} \cdot \frac{1}{t} \int_0^t (t-s)\ G(s)\ ds - \frac{1}{t-t_0} \int_0^{t_0} (t-s)\ G(s)\ ds\ .$$

We may notice that

$$\lim_{t\to+\infty} \frac{1}{t-t_0} \int_0^{t_0} (t-s)\ G(s)\ ds = \int_0^{t_0} G(s)\ ds\ ,$$

and

$$\liminf_{t\to+\infty} \frac{1}{t-t_0} \int_0^t (t-s)\ G(s)\ ds = \liminf_{t\to+\infty} \frac{t}{t-t_0} \cdot \frac{1}{t} \int_0^t (t-s)\ G(s)\ ds = -\infty$$

since

$$\frac{t}{t-t_0} > 0 \quad \text{and} \quad \liminf_{t\to+\infty} \frac{1}{t} \int_0^t (t-s)\ G(s)\ ds = -\infty\ .$$

Thus

$$-\infty = \liminf_{t\to+\infty} \frac{1}{t-t_0} \int_0^t (t-s)\ G(s)\ ds =$$
$$= \liminf_{t\to+\infty} \left\{ \frac{1}{t-t_0} \int_{t_0}^t (t-s)\ G(s)\ ds + \frac{1}{t-t_0} \int_0^{t_0} (t-s)\ G(s)\ ds \right\}$$
$$\geq \liminf_{t\to+\infty} \frac{1}{t-t_0} \int_{t_0}^t (t-s)\ G(s)\ ds + \liminf_{t\to+\infty} \frac{1}{t-t_0} \int_0^{t_0} (t-s)\ G(s)\ ds\ .$$

The following examples illustrate our results.

Example 1. Consider the problem

$$u_{tt}(x,t) + \frac{1}{2}\, u_{tt}(x,t-\pi) + u_t(x,t-\pi)$$
$$- \left[\Delta u(x,t) + \Delta u(x,t-\pi) + \Delta v(x,t) + \Delta v\left(x, t-\frac{\pi}{2}\right)\right]$$
$$+ u(x,t) + u(x,t-\pi)$$
$$= 2(e^{-\pi}-1)\ e^t \sin t \sin x + (2-3e^{-\pi})\ e^t \cos t \sin x$$
$$+ e^t \sin t \cos x - e^{-\frac{\pi}{2}}\ e^t \cos t \cos x \tag{13}$$

and

$$v_{tt}(x,t) + \frac{1}{2}\, v_{tt}(x,t-\pi) + v_t(x,t-\pi)$$
$$- \left[\Delta u(x,t) + \Delta u(x,t-\pi) + \Delta v(x,t) + \Delta v\left(x, t-\frac{\pi}{2}\right)\right]$$
$$+ v(x,t) + v\left(x, t-\frac{\pi}{2}\right)$$
$$= 2(1-e^{-\pi}-e^{-\frac{\pi}{2}})\ e^t \cos t \cos x + (2-e^{-\pi})\ e^t \sin t \cos x$$
$$+ (1-e^{-\pi})\ e^t \cos t \sin x \tag{14}$$

$(x,t) \in (0,\pi)$ with boundary conditions

$$-u_x(0,t) = u_x(\pi,t) = -e^t \cos t$$

and

$$-v_x(0,t) = v_x(\pi,t) = 0\ . \tag{15}$$

Thus $\Omega = (0,\pi), \psi_1(x,t) = -e^t\cos t$ and $\psi_2(x,t) = 0$. Consequently, $\Psi_1(t) = -2e^t \cos t$ and $\Psi_2(t) = 0, t > 0$. Further,

$$F_1(t) = \int_0^\pi f_1(x,t)dx$$
$$= 4(e^{-\pi}-1)\ e^t \sin t + 2(2-3e^{-\pi})\ e^t \cos t$$

and

$$F_2(t) = \int_0^\pi f_2(x,t)dx = 2(1-e^{-\pi})\ e^t \cos t\ .$$

Thus

$$I_1(t) = \frac{1}{t}\int_0^t (t-s)[F_1(s) + \Psi_1(s) + \Psi_1(s-\pi)]ds$$
$$= \frac{1}{t}\int_0^t (t-s)[4(e^{-\pi}-1)\ e^s \sin s + 2(1-2e^{-\pi})\ e^s \cos s]ds$$
$$= 2(e^{-\pi}-1)\ \frac{1}{t}\ (1+t-e^t\cos t)$$
$$+ (1-2e^{-\pi})\ \frac{1}{t}\ (-t + e^t \sin t)$$

and

$$I_2(t) = \frac{1}{t}\int_0^t (t-s)[F_2(s) + \Psi_1(s) + \Psi_1(s-\pi)]ds = 0 .$$

Clearly, $\liminf_{t\to\infty} I_1(t) = -\infty$ and $\limsup_{t\to\infty} I_1(t) = \infty$. From Theorem I it follows that all solutions of the problem (13, 14), (15) oscillate in $(0,\pi)\times(0,\infty)$. In particular, $(u(x,t), v(x,t)) \equiv (e^t \cos t \sin x, e^t \sin t \cos x)$ is an oscillatory solution of the problem.

Example 2. Consider the problem

$$\begin{aligned} & u_{tt}(x,t) + u_{tt}(x,t-\pi) + u_t\left(x, t-\frac{\pi}{2}\right) \\ & \quad - [\Delta u(x,t) + \Delta u(x,t-\pi) + \Delta v(x,t) + \Delta v(x,t-2\pi)] \\ & \quad + u(x,t) + u\left(x,t-\frac{\pi}{2}\right) \\ & = 2\left(-1 + e^{-\pi} + e^{-\frac{\pi}{2}}\right)\ e^t \sin t \sin x \\ & \quad + \left(2 + e^{-\frac{\pi}{2}} - e^{-\pi}\right)\ e^t \cos t \sin x + 2(t-\pi)\sin x \end{aligned} \tag{16}$$

and

$$\begin{aligned} & v_{tt}(x,t) + v_{tt}\left(x,t-\frac{\pi}{2}\right) + v_t(x,t-2\pi) \\ & \quad - \left[-\Delta u(x,t) - \Delta u(x,t-\pi) + \Delta v(x,t) + \Delta v\left(x,t-\frac{\pi}{2}\right)\right] \\ & \quad + v(x,t) + v(x,t-2\pi) \\ & = \left(\frac{2-5\pi}{2}\right)\sin x + 4t\sin x + (e^{-\pi}-1)\ e^t\cos t\sin x \end{aligned} \tag{17}$$

$(x,t) \in (0,\pi)\times(0,\infty)$ with boundary conditions

$$u = 0 \quad \text{and} \quad v = 0 \quad \text{on} \quad \{0,\pi\}\times(0,\infty) . \tag{18}$$

Thus $\Omega = (0,\pi)$, $\tilde{\psi}_1(x,t) = 0$ and $\tilde{\psi}_2(x,t) = 0$. Consequently, $\tilde{\Psi}_1(t) = 0$ and $\tilde{\Psi}_2(t) = 0$ for $t > 0$. Here $\varphi(x) = \sin x$ and $\lambda_1 = 1$. Hence

$$\begin{aligned} \tilde{F}_1(t) &= \int_0^\pi f_1(x,t)\sin x\ dx \\ &= a_1\ e^t \sin t + a_2\ e^t\cos t + \pi t - \pi^2 \end{aligned}$$

where

$$a_1 = \pi\left(-1 + e^{-\pi} + e^{-\frac{\pi}{2}}\right), \quad a_2 = \frac{\pi}{2}\left(2 + e^{-\frac{\pi}{2}} - e^{-\pi}\right)$$

and

$$\begin{aligned} \tilde{F}_2(t) &= \int_0^\pi f_2(x,t)\sin x\ dx \\ &= (2-5\pi)\ \frac{\pi}{4} + 2\pi t + \frac{\pi}{2}\ (e^{-\pi}-1)\ e^t\cos t . \end{aligned}$$

Thus

$$
\begin{aligned}
I_1(t) &= \frac{1}{t}\int_0^t (t-s)\, \tilde{F}_1(s)ds \\
&= \frac{e^t}{2t}\Big[a_1(e^{-t} + t\ e^{-t} + \cos t) + a_2(-t\ e^{-t} + \sin t) \\
&\quad + \frac{\pi}{3}\, t^3\ e^{-t} - \pi^2\, t^2\ e^{-t}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(t) &= \frac{e^t}{t}\Big[\frac{\pi}{3}\, t^3\ e^{-t} + (2-5\pi)\,\frac{\pi}{8}\, t^2\ e^{-t} \\
&\quad + (e^{-\pi}-1)\,\frac{\pi}{4}\,(-t\ e^{-t} + \sin t)\Big]\ .
\end{aligned}
$$

Clearly, $\liminf_{t\to\infty} I_1(t) = -\infty$, $\limsup_{t\to\infty} I_1(t) = \infty$, $\liminf_{t\to\infty} I_2(t) = -\infty$ and $\limsup_{t\to\infty} I_2(t) = \infty$. Hence by Theorem IV all solutions of the problem oscillate in $(0,\pi)\times(0,\infty)$. In particular, $(u(x,t), v(x,t)) \equiv (e^t \cos t \sin x, t\sin x)$ is an oscillatory solution of the problem.

Acknowledgments

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. They would also like to thank the referee for pointing out some misprints.

References

1. Courant, R., Hilbert, D.: Methods of Mathematical Physics. Vol.I, John Wiley and Sons: New York 1989.
2. Kirane, M., Parhi, N.: On the oscillation of a class of hyperbolic equations of neutral type. Communicated.
3. Miranda, M.M., Medeiros, L.A.: On the existence of global solutions of a coupled nonlinear Klein–Gordon equations. Funkc. Ekvac. 30, 147–161 (1987).
4. Mishev, D.P., Bainov, D.D.: Oscillation properties of the solutions of hyperbolic equations of neutral type. Proceedings of the Colloquium on Qualitative Theory of Differential Equations (eds. B.S. Nagy and L. Hatvani) pp.771–780 (1984).
5. Mishev, D.P., Bainov, D.D.: Oscillation properties of the solutions of a class of hyperbolic equations of neutral type. Funkc. Ekvac. 29, 213–218 (1986).
6. Segal, I.: Nonlinear partial differential equations in Quantum Field Theory. Proc. Symp. Appl. Math. A.M.S. 17, 210–226 (1965).

7. Williams, W.E.: Partial Differential Equations. Oxford Applied Mathematics and Computing Science Series. Clarendon Press: Oxford 1980.

8. Yoshida, N.: On the zeros of solutions to nonlinear hyperbolic equations. Proc. Roy. Soc. Edinburgh, Sect.A 106, 121–129 (1987).

9. Yoshida, N.: On the zeros of solutions of hyperbolic equations of neutral type. Diff. Integ. Eqns. 3, 155–160 (1990).

N. Parhi
Department of Mathematics
Berhampur University
Berhampur – 76000
India

M. Kirane
Institut de Mathematiques
Universite de Annaba
B.P.12, Annaba 23000
Algeria

Analysis 14, 57-65 (1994)

Analysis

TRANSLATIVITY OF SOME ABSOLUTE SUMMABILITY METHODS

B.Kuttner[1] **B.Thorpe**

Received : February 25, 1993

Abstract
This note gives a simple set of necessary and sufficient conditions for $|\overline{N}, p_n|_k$ and $|C, \alpha, \gamma|_k$ to be translative.

1991 Mathematics Subject Classification 40F05.

1 Introduction

We shall be concerned with two absolute summability methods $|\overline{N}, p_n|_k$ and $|C, \alpha, \gamma|_k$ defined as follows. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with sequence of n-th partial sums $(s_n)_{n\geq 0}$. Let $p = (p_n)_{n\geq 0}$ be a sequence of positive numbers with

$$P_n = \sum_{r=0}^{n} p_r \to \infty \text{ as } n \to \infty,$$

and define A_n^α by the identity

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1}.$$

The sequence to sequence transformations given by

$$t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r s_r \tag{1}$$

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{r=0}^{n} A_{n-r}^{\alpha-1} s_r \tag{2}$$

define the $(\overline{N}, p_n)$ and (C, α) means, respectively, of the sequence (s_n) . The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|\overline{N}, p_n|_k$ where $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} | t_n - t_{n-1} |^k < \infty \tag{3}$$

[1] Professor Kuttner died 2nd Jan. 1992 before Section 3 was completed.

and summable $|C,\alpha,\gamma|_k$ where $\alpha > -1, \gamma$ is any real number and $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k\gamma+k-1} \mid \sigma_n^{\alpha} - \sigma_{n-1}^{\alpha} \mid^k < \infty. \tag{4}$$

We call $|\overline{N},p_n|_k$ left translative if the summability $|\overline{N},p_n|_k$ of the series $\sum_{n=0}^{\infty} a_n$ implies the summability $|\overline{N},p_n|_k$ of the series $\sum_{n=0}^{\infty} a_{n-1}$ where $a_{-1} = 0$. $|\overline{N},p_n|_k$ is called right translative if the converse holds, and translative if it is both left and right translative. Translativity of $|C,\alpha,\gamma|_k$ is defined similarly.

The purpose of this paper is to use some recent results of Bor and Thorpe (see [4],[5]) to obtain simple necessary and sufficient conditions on (p_n) for $|\overline{N},p_n|_k$ to be translative. This extends the results of Al-Madi in [1] to the case $k > 1$. In the final section we use some recent results of Bennett in [2] to investigate the translativity properties of $|C,\alpha,\gamma|_k$.

2 Weighted mean methods

Let $(\overline{s}_n)_{n\geq 0}$ denote the n-th partial sum of the series $\sum_{n=0}^{\infty} a_{n-1}$, so that $\overline{s}_n = s_{n-1}$ where $s_{-1} = 0$. Let $(u_n)_{n\geq 0}$ be the $(\overline{N},p_{n+1})$ transform of $(s_n)_{n\geq 0}$ and $(\overline{t}_n)_{n\geq 0}$ be the $(\overline{N},p_n)$ of $(\overline{s}_n)_{n\geq 0}$. From the definitions we have

$$\overline{t}_n = \frac{1}{P_n}\sum_{r=0}^{n} p_r\overline{s}_r = \frac{1}{P_n}\sum_{r=0}^{n-1} p_{r+1}s_r, \tag{5}$$

$$u_n = \frac{1}{P_{n+1}-p_0}\sum_{r=0}^{n} p_{r+1}s_r = \frac{P_{n+1}}{P_{n+1}-p_0}\overline{t}_{n+1}. \tag{6}$$

We use the following lemma.

Lemma The sequence $(\overline{s}_n)_{n\geq 0}$ is summable $|\overline{N},p_n|_k$ if and only if $(s_n)_{n\geq 0}$ is summable $|\overline{N},p_{n+1}|_k$.

Proof From (6) above we get that

$$u_n - u_{n-1} = \frac{P_{n+1}}{P_{n+1}-p_0}(\overline{t}_{n+1} - \overline{t}_n) - \frac{\overline{t}_n p_0 p_{n+1}}{(P_{n+1}-p_0)(P_n-p_0)}.$$

Assume that $(\overline{s}_n)_{n\geq 0}$ is summable $|\overline{N},p_n|_k$. By Minkowski's inequality it is sufficient to prove that

$$\sum_{n=1}^{\infty}\left(\frac{P_{n+1}-p_0}{p_{n+1}}\right)^{k-1}\left(\frac{P_{n+1}}{P_{n+1}-p_0}\mid \overline{t}_{n+1}-\overline{t}_n\mid\right)^k < \infty, \tag{7}$$

and that

$$\sum_{n=1}^{\infty}\left(\frac{P_{n+1}-p_0}{p_{n+1}}\right)^{k-1}\left(\frac{p_{n+1}\mid \overline{t}_n\mid}{(P_{n+1}-p_0)(P_n-p_0)}\right)^k < \infty. \tag{8}$$

By the assumption, we know that

$$\sum_{n=1}^{\infty}\left(\frac{P_{n+1}}{p_{n+1}}\right)^{k-1} \mid \overline{t}_{n+1}-\overline{t}_n \mid^k < \infty. \tag{9}$$

Taking the limit of the ratio of the terms in the series (7) and (9), since $P_{n+1}/(P_{n+1}-p_0) \to \infty$ as $n \to \infty$, we see that (7) is equivalent to (9) and so (7) holds.

Similarly, (8) holds if and only if

$$\sum_{n=1}^{\infty}\left(\frac{P_{n+1}}{p_{n+1}}\right)^{k-1}\left(\frac{p_{n+1}\mid \overline{t}_n\mid}{P_{n+1}P_n}\right)^k < \infty. \tag{10}$$

By writing $\bar{t}_n = \sum_{r=0}^{n} \bar{b}_r$ we see that (9) is equivalent to

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid \overline{b_n} \mid^k < \infty. \tag{11}$$

To show that (10) holds consider

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{p_{n+1}}{P_{n+1}P_n^k} \mid \bar{t}_n \mid^k &\leq \sum_{n=1}^{\infty} \frac{p_{n+1}}{P_{n+1}P_n^k} \left(\sum_{r=0}^{n} \mid \bar{b}_r \mid\right)^k \\ &\leq \sum_{n=1}^{\infty} \frac{p_{n+1}}{P_{n+1}P_n^k} \left(\sum_{r=0}^{n} p_r\right)^{k/k'} \left(\sum_{r=0}^{n} p_r^{1-k} \mid \bar{b}_r \mid^k\right)\end{aligned}$$

by Hölder's inequality where $\frac{1}{k} + \frac{1}{k'} = 1$,

$$\begin{aligned}&= \sum_{n=1}^{\infty} \frac{p_{n+1}}{P_{n+1}P_n} \sum_{r=0}^{n} p_r^{1-k} \mid \bar{b}_r \mid^k \\ &= \sum_{r=0}^{\infty} p_r^{1-k} \mid \bar{b}_r \mid^k \sum_{n=max(1,r)}^{\infty} \left(\frac{1}{P_n} - \frac{1}{P_{n+1}}\right) \\ &= \frac{p_0^{1-k} \mid \bar{b}_0 \mid^k}{P_1} + \sum_{r=1}^{n} \left(\frac{P_r}{p_r}\right)^{k-1} \mid \bar{b}_r \mid^k \frac{1}{P_r^k} < \infty\end{aligned}$$

by (11), since $P_r > p_0$ for $r \geq 1$.

Conversely, we have

$$\bar{t}_{n+1} - \bar{t}_n = \left(1 - \frac{P_0}{P_{n+1}}\right)(u_n - u_{n-1}) + \frac{p_0 p_{n+1}}{P_n P_{n+1}} u_{n-1}.$$

Assuming that $(s_n)_{n\geq 0}$ is summable $\mid \overline{N}, p_{n+1} \mid_k$ we know that

$$\sum_{n=1}^{\infty} \left(\frac{P_{n+1} - p_0}{p_{n+1}}\right)^{k-1} \mid u_n - u_{n-1} \mid^k < \infty \tag{12}$$

so that by Minkowski's inequality, it is sufficient to prove that

$$\sum_{n=1}^{\infty} \left(\frac{P_{n+1}}{p_{n+1}}\right)^{k-1} \left(\left(1 - \frac{P_0}{P_{n+1}}\right) \mid u_n - u_{n-1} \mid\right)^k < \infty, \tag{13}$$

and that

$$\sum_{n=1}^{\infty} \left(\frac{P_{n+1}}{p_{n+1}}\right)^{k-1} \left(\frac{p_{n+1}}{P_n P_{n+1}} \mid u_{n-1} \mid\right)^k < \infty. \tag{14}$$

As before, since $P_n \to \infty$ as $n \to \infty$, we see that (13) follows from (12).

To show that (14) holds, let $u_n = \sum_{r=0}^{n} b_r$, and write (14) as

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{p_{n+1}}{P_{n+1}P_n^k} \mid u_{n-1} \mid^k &\leq \sum_{n=1}^{\infty} \frac{p_{n+1}}{P_{n+1}P_n^k} \left(\sum_{r=0}^{n-1} \mid b_r \mid\right)^k \\ &\leq \sum_{n=1}^{\infty} \frac{p_{n+1}}{P_{n+1}P_n^k} \left(\sum_{r=0}^{n-1} p_{r+1}\right)^{k/k'} \left(\sum_{r=0}^{n-1} p_{r+1}^{1-k} \mid b_r \mid^k\right)\end{aligned}$$

by Hölder's inequality where $\frac{1}{k}+\frac{1}{k'}=1$,

$$\begin{aligned} &= \sum_{n=1}^{\infty}\frac{p_{n+1}}{P_{n+1}P_n^k}(P_n-p_0)^{k/k'}\sum_{r=0}^{n-1}p_{r+1}^{1-k}\mid b_r\mid^k \\ &\le \sum_{r=0}^{\infty}p_{r+1}^{1-k}\mid b_r\mid^k\sum_{n=r+1}^{\infty}\frac{p_{n+1}}{P_{n+1}P_n} \\ &= \sum_{r=0}^{\infty}\left(\frac{P_{r+1}}{p_{r+1}}\right)^{k-1}\mid b_r\mid^k\frac{1}{P_{r+1}^k} \\ &\le \frac{1}{P_1^k}\sum_{r=0}^{\infty}\left(\frac{P_{r+1}}{p_{r+1}}\right)^{k-1}\mid b_r\mid^k<\infty \end{aligned}$$

by (12). Hence the result.

For two summability methods A,B we write $A\Rightarrow B$ if every series summable A is also summable B.

Theorem 1 (a)$|\overline{N},p_n|_k$ is left translative if and only if $\mid\overline{N},p_n\mid_k\Rightarrow\mid\overline{N},p_{n+1}\mid_k$.

(b) $\mid\overline{N},p_n\mid_k$ is right translative if and only if $\mid\overline{N},p_{n+1}\mid_k\Rightarrow\mid\overline{N},p_n\mid_k$.

Proof (a) Assume that $\mid\overline{N},p_n\mid_k$ is left translative and that $(s_n)_{n\ge0}$ is summable $\mid\overline{N},p_n\mid_k$. Then $(\bar{s}_n)_{n\ge0}$ is summable $\mid\overline{N},p_n\mid_k$ and so by the lemma $(s_n)_{n\ge0}$ is summable $\mid\overline{N},p_{n+1}\mid_k$ i.e. $\mid\overline{N},p_n\mid_k\Rightarrow\mid\overline{N},p_{n+1}\mid_k$.

Conversely, if $(s_n)_{n\ge0}$ is summable $\mid\overline{N},p_n\mid_k$ then by assumption it is summable $\mid\overline{N},p_{n+1}\mid_k$ and so by the lemma $(\bar{s}_n)_{n\ge0}$ is summable $\mid\overline{N},p_n\mid_k$ i.e. $\mid\overline{N},p_n\mid_k$ is left translative.

(b) This is similar to (a) and so is omitted.

Theorem 2 $\mid\overline{N},p_n\mid_k$ is translative if and only if

$$\frac{p_{n+1}}{P_{n+1}}=O\left(\frac{p_n}{P_n}\right)\text{ and }\frac{p_n}{P_n}=O\left(\frac{p_{n+1}}{P_{n+1}}\right).$$

Proof From Theorem 1, $\mid\overline{N},p_n\mid_k$ is translative if and only if $\mid\overline{N},p_n\mid_k$ is equivalent to $\mid\overline{N},p_{n+1}\mid_k$. The result now follows from Theorem 2 in [5] and the fact that $P_{n+1}\sim P_{n+1}-P_0$ since $P_n\to\infty$ as $n\to\infty$.

3 Cesàro Methods If τ_n^α denotes the n-th (C,α)mean of the sequence $(na_n)_{n\ge0}$ then using the well known identity $\tau_n^\alpha=n(\sigma_n^\alpha-\sigma_{n-1}^\alpha)$ (see [6]), (4) is equivalent to

$$\sum_{n=1}^{\infty}n^{k\gamma-1}\mid\tau_n^\alpha\mid^k<\infty. \tag{15}$$

Thus, if $\bar{\tau}_n^\alpha$ denotes the n-th (C,α)mean of the sequence $(na_{n-1})_{n\ge0}$ where $a_{-1}=0$, then $\mid C,\alpha,\gamma\mid_k$ is left translative if (15) implies that

$$\sum_{n=1}^{\infty}n^{k\gamma-1}\mid\bar{\tau}_n^\alpha\mid^k<\infty$$

and right translative if the converse holds. The following simple examples show that some restriction on γ is necessary for $\mid C,\alpha,\gamma\mid_k$ to be translative. The series $1+0+0+\cdots$ is summable $\mid C,\alpha,\gamma\mid_k$ for $\alpha>-1$ and all real γ since $\tau_n^\alpha=0$ for all $n\ge0$. However the series $0+1+0+0+\cdots$ is not summable $\mid C,\alpha,\gamma\mid_k$ for any $\gamma\ge1$ and $\alpha\ne0$ since $\bar{\tau}_n^\alpha=A_{n-1}^{\alpha-1}/A_n^\alpha$. Thus for left translativity we need that $\gamma<1$ if $\alpha\ne0$. In a similar way we see that the series $\sum_{n=0}^{\infty}a_n$ where $a_n=A_n^{-\alpha-1}/(n+1)$ has $\bar{\tau}_n^\alpha=0$ for $n>1$ so that the series $0 + a_0 + a_1 + \cdots$ is summable $| C, \alpha, \gamma |_k$ for $\alpha > -1$ and all real γ. Using the identity $(\alpha+1)A_n^{-\alpha-1} = -(n+1)A_{n+1}^{-\alpha-2}$ we can check that $\sum_{n=0}^{\infty} a_n$ is not summable $| C, \alpha, \gamma |_k$ for any $\gamma \geq 1$ and $\alpha \neq 0$. Thus for right translativity we also need that $\gamma < 1$ if $\alpha \neq 0$. We now show that these necessary conditions for translativity are also sufficient.

Theorem 3 $| C, \alpha, \gamma |_k$ is translative if either (i) $\alpha = 0$ and γ is any real number, or (ii) $\alpha > -1$ and $\gamma < 1$.

Proof Clearly $| C, 0, \gamma |_k$ is translative for all values of γ. We now assume $\alpha \neq 0$ and that (15) holds. For $n \geq 1$

$$\overline{\tau}_n^\alpha = \frac{1}{A_n^\alpha} \sum_{r=0}^{n} A_{n-r}^{\alpha-1} r a_{r-1} = \frac{1}{A_n^\alpha} \sum_{r=0}^{n-1} A_{n-1-r}^{\alpha-1}(r+1)a_r$$

$$= \frac{A_{n-1}^\alpha}{A_n^\alpha}\tau_{n-1}^\alpha + \frac{1}{A_n^\alpha}\sum_{r=0}^{n-1} A_{n-1-r}^{\alpha-1} a_r$$

and so we get the relation

$$\overline{\tau}_n^\alpha = \frac{A_{n-1}^\alpha}{A_n^\alpha}\tau_{n-1}^\alpha + \frac{A_{n-1}^{\alpha-1}}{A_n^\alpha}\sigma_{n-1}^{\alpha-1}.$$

Using Minkowski's inequality if $k > 1$ and trivially if $k = 1$, we see that $| C, \alpha, \gamma |_k$ is left translative if and only if (15) implies that

$$\sum_{n=1}^{\infty} n^{k\gamma-k-1} | \sigma_n^{\alpha-1} |^k < \infty.$$

Expressing $\sigma_n^{\alpha-1}$ in terms of τ_n^α we get that, for $n \geq 1$,

$$\begin{aligned}\sigma_n^{\alpha-1} &= a_0 + \frac{1}{A_n^{\alpha-1}} \sum_{r=1}^{n} A_{n-r}^{\alpha-1} \sum_{d=1}^{r} \frac{A_{r-d}^{-\alpha-1} A_d^\alpha \tau_d^\alpha}{r} \\ &= a_0 + \frac{1}{A_n^{\alpha-1}} \sum_{d=1}^{n} \tau_d^\alpha A_d^\alpha \sum_{r=d}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d}^{-\alpha-1}}{r}.\end{aligned}$$

If we write this transformation in the form

$$\frac{A_n^{\alpha-1}}{n^{\alpha-1}} n^{\gamma-1-1/k} \sigma_n^{\alpha-1} = \frac{A_n^{\alpha-1}}{n^{\alpha-1}} n^{\gamma-1-1/k} a_0 + \sum_{d=1}^{n} b_{nd} \frac{d^\alpha}{A_d^\alpha} d^{\gamma-1/k} \tau_d^\alpha$$

and assume that $\gamma < 1$, we see that $| C, \alpha, \gamma |_k$ is left translative if $B : l^k \to l^k$ where

$$b_{nd} = \left(\frac{d}{n}\right)^{-\gamma+\alpha+1/k} \sum_{r=d}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d}^{-\alpha-1}}{r} \tag{16}$$

for $1 \leq d \leq n$ and $b_{nd} = 0$ otherwise. We now assume that $\alpha > 0$ and use the identity (see p. 419 of [8] for a similar argument)

$$\sum_{r=d}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d}^{-\alpha-1}}{r} = \sum_{r=d}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d}^{-\alpha-1}}{d} + \sum_{r=d}^{n} \frac{(d-r) A_{n-r}^{\alpha-1} A_{r-d}^{-\alpha-1}}{rd}$$

$$= \frac{A_{n-d}^{-1}}{d} + \frac{\alpha}{d} \sum_{r=d+1}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d-1}^{-\alpha}}{r}.$$

The proof now splits into two cases, α integral or not. If α is an integer we repeat this argument a further $(\alpha - 1)$ times to get the identity

$$\sum_{r=d}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d}^{-\alpha-1}}{r} = \sum_{s=0}^{\alpha} \frac{\alpha(\alpha-1)\dots(\alpha-s+1)}{d(d+1)\dots(d+s)} A_{n-d-s}^{s-1} \tag{17}$$

where $\alpha(\alpha-1)\dots(\alpha-s+1)$ is defined to be 1 if $s = 0$. Putting this back in (16) we see that a sufficient condition for $| C, \alpha, \gamma |_k$ to be left translative is that for each integer s where $0 \le s \le \alpha$, $B^{(s)} : l^k \to l^k$ where

$$b_{nd}^{(s)} = \left(\frac{d}{n}\right)^{-\gamma+\alpha+1/k} \frac{A_{n-s-d}^{s-1}}{d(d+1)\dots(d+s)}. \tag{18}$$

If α is positive and $K < \alpha < K+1$ for some integer K then we proceed as above to obtain (16) and the identity (17) is replaced by

$$\begin{aligned} \sum_{r=d}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d}^{-\alpha-1}}{r} &= \sum_{s=0}^{K-1} \frac{\alpha(\alpha-1)\dots(\alpha-s+1)}{d(d+1)\dots(d+s)} A_{n-d-s}^{s-1} + \\ &\quad \frac{\alpha(\alpha-1)\dots(\alpha-K+1)}{d(d+1)\dots(d+K-1)} \sum_{r=d+K}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d-K}^{-\alpha+K-1}}{r}, \end{aligned} \tag{19}$$

if $K > 0$ and we regard (19) as trivial if $K = 0$. The first term on the RHS of (19) is of the form of (17) when $K > 0$. For the second term in (19) we use the inequality

$$\begin{aligned} \left| \sum_{r=d+K}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d-K}^{-\alpha+K-1}}{r} \right| &\le \frac{1}{d+K} \sum_{r=d+K}^{n} A_{n-r}^{\alpha-1} \left| A_{r-d-K}^{-\alpha+K-1} \right| \\ &= \frac{1}{d+K} (2A_{n-K-d}^{\alpha-1} - A_{n-K-d}^{K-1}), \end{aligned}$$

since $-2 < -\alpha + K - 1 < -1$, so that $A_m^{-\alpha+K-1} < 0$ for $m \ge 1$. In this case we see that a sufficient condition for $| C, \alpha, \gamma |_k$ to be left translative is that for each integer s where $0 \le s \le K$, $B^{(s)} : l^k \to l^k$ and that $C : l^k \to l^k$ where

$$c_{nd} = \left(\frac{d}{n}\right)^{-\gamma+\alpha+1/k} \frac{A_{n-K-d}^{\alpha-1}}{d(d+1)\dots(d+K)}. \tag{20}$$

In both cases, if $s = 0$ then $B^{(0)}$ is a diagonal matrix with $b_{nn} = 1/n$ so it clearly maps l^k to l^k.

Suppose that $k = 1$. Then $B^{(s)} : l \to l$ if and only if $\sum_{n=1}^{\infty} | b_{nd}^{(s)} | = O(1)$. In what follows we use M to denote a constant (that may be different at each occurrence.) Thus, for $1 \le s \le [\alpha]$,

$$\begin{aligned} \sum_{n=1}^{\infty} | b_{nd}^{(s)} | &\le M d^{-\gamma+\alpha-s} \sum_{n=d}^{\infty} n^{\gamma-\alpha-1} (n-d)^{s-1} \\ &\le M d^{-\gamma+\alpha-s} \left(\sum_{n=d}^{2d} n^{\gamma-\alpha-1} (n-d)^{s-1} + \sum_{n=2d+1}^{\infty} n^{\gamma-\alpha-1} (n-d)^{s-1} \right) \\ &= d^{-\gamma+\alpha-s} \left(O(d^{\gamma-\alpha-1}) \sum_{n=d}^{2d} (n-d)^{s-1} + O(1) \sum_{n=2d+1}^{\infty} n^{\gamma-\alpha+s-2} \right) \\ &= d^{-\gamma+\alpha-s} \left(O(d^{\gamma-\alpha+s-1}) + O(d^{\gamma-\alpha+s-1}) \right) = O(d^{-1}) \end{aligned}$$

since $\gamma - \alpha + s - 2 < -1$. This gives the result for the integer case. For non-integer α we also need to consider the matrix C. Now

$$\sum_{n=1}^{\infty} | c_{nd} | \le M d^{-\gamma+\alpha-K} \sum_{n=d}^{\infty} n^{\gamma-\alpha-1}(n-d+1)^{\alpha-1}$$

and splitting the sum up in exactly the same way as the previous case we get

$$\sum_{n=1}^{\infty} | c_{nd} | = O(d^{\alpha-K-1}) = O(1)$$

since $\alpha < K+1$.

If $k > 1$ we use the estimates, for $1 \le s \le [\alpha]$ and $1 \le d \le n/2$,

$$A^{s-1}_{n-s-d} \le M n^{s-1} \quad , \quad A^{\alpha-1}_{n-K-d} \le M n^{\alpha-1}$$

which give

$$b^{(s)}_{nd} \le M d^{-\gamma+\alpha-s-1+1/k} n^{\gamma-\alpha+s-1-1/k}$$

$$c_{nd} \le M d^{-\gamma+\alpha-K-1+1/k} n^{\gamma-1-1/k}$$

for $1 \le d \le n/2$. In the range $n/2 < d \le n$, we have the estimates

$$b^{(s)}_{nd} \le M \frac{(n-d)^{s-1}}{n^{s+1}}$$

$$c_{nd} \le M \frac{(n-d+1)^{\alpha-1}}{n^{K+1}}.$$

Since all the terms are non-negative we have that for $1 \le d \le n, 1 \le s \le [\alpha]$

$$b^{(s)}_{nd} \le M \left(d^{-\gamma+\alpha-s-1+1/k} n^{\gamma-\alpha+s-1-1/k} + (n-d)^{s-1} n^{-s-1} \right), \tag{21}$$

$$c_{nd} \le M \left(d^{-\gamma+\alpha-K-1+1/k} n^{\gamma-1-1/k} + (n-d+1)^{\alpha-1} n^{-K-1} \right). \tag{22}$$

The first term on the RHS of (21) and (22) is a factorable matrix of the form $a_{nd} = n^{-X} d^{-Y}$ (see p. 413 of [2].) In this paper, Bennett gives necessary and sufficient conditions for such a matrix to map l^k to l^k. By Corollary 8 of [2] we see that the first term of (21) maps l^k to l^k provided that $-\gamma + \alpha - s + 1 + 1/k > 1/k$, and the first term of (22) maps l^k to l^k provided that $-\gamma + 1 + 1/k > 1/k$. Both these conditions hold since $\gamma < 1$ and $s \le [\alpha]$.

In order to deal with the second term in (21) and (22) we use Hardy's inequality for Cesàro methods (see example (2) on p.275 of [7].) Suppose that $(x_d)_{d\ge 1} \in l^k$. Then, from (21)

$$\left(\sum_{n=1}^{\infty} n^{-k} \left| \sum_{d=1}^{n} \frac{(n-d)^{s-1} x_d}{n^s} \right|^k \right)^{1/k} \le M \left(\sum_{n=1}^{\infty} \left[\sum_{d=1}^{n} \frac{A^{s-1}_{n-d} | x_d |}{A^s_n} \right]^k \right)^{1/k} \le M \left(\sum_{d=1}^{\infty} | x_d |^k \right)^{1/k}$$

by Hardy's inequality for (C, s). Similarly, from (22)

$$\left(\sum_{n=1}^{\infty} n^{(\alpha-K-1)k} \left| \sum_{d=1}^{n} \frac{(n-d+1)^{\alpha-1} x_d}{n^\alpha} \right|^k \right)^{1/k} \le M \left(\sum_{n=1}^{\infty} \left[\sum_{d=1}^{n} \frac{A^{\alpha-1}_{n-d} | x_d |}{A^\alpha_n} \right]^k \right)^{1/k} \le M \left(\sum_{d=1}^{\infty} | x_d |^k \right)^{1/k}$$

by Hardy's inequality for (C,α). Putting these together we see that $B^{(s)}$ and C map l^k to l^k, and so $|\, C,\alpha,\gamma\,|_k$ is left translative if $\alpha > 0$ and $\gamma < 1$.

In the remaining case, where $-1 < \alpha < 0$, we use the identity

$$\sum_{r=d}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d}^{-\alpha-1}}{r} = \frac{A_{n-d}^{-1}}{n} + \frac{\alpha}{n} \sum_{r=d}^{n-1} \frac{A_{n-1-r}^{\alpha} A_{r-d}^{-\alpha-1}}{r},$$

so that (16) becomes

$$b_{nd} = \left(\frac{d}{n}\right)^{-\gamma+\alpha+1/k} \left(\frac{A_{n-d}^{-1}}{n} + \frac{\alpha}{n} \sum_{r=d}^{n-1} \frac{A_{n-1-r}^{\alpha} A_{r-d}^{-\alpha-1}}{r} \right). \tag{23}$$

If $k = 1$ then since $A_m^x \ge 0$ for $x = \alpha, -\alpha-1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\, b_{nd} \,| &= \frac{1}{d} + |\,\alpha\,|\, d^{-\gamma+\alpha+1} \sum_{n=d+1}^{\infty} n^{\gamma-\alpha-2} \sum_{r=d}^{n-1} \frac{A_{n-1-r}^{\alpha} A_{r-d}^{-\alpha-1}}{r} \\
&= \frac{1}{d} + |\,\alpha\,|\, d^{-\gamma+\alpha+1} \sum_{r=d}^{\infty} \frac{A_{r-d}^{-\alpha-1}}{r} \sum_{n=r+1}^{\infty} n^{\gamma-\alpha-2} A_{n-r-1}^{\alpha} \\
&= \frac{1}{d} + |\,\alpha\,|\, d^{-\gamma+\alpha+1} \sum_{r=d}^{\infty} \frac{A_{r-d}^{-\alpha-1}}{r} O(r^{\gamma-1}) \\
&= \frac{1}{d} + d^{-\gamma+\alpha+1} O(d^{\gamma-\alpha-2}) = O\left(\frac{1}{d}\right),
\end{aligned}$$

so that $B : l \to l$.

If $k > 1$ then in order to show that $B : l^k \to l^k$, it is sufficient to show that $E : l^k \to l^k$, where

$$e_{nd} = d^{-\gamma+\alpha+1/k} n^{\gamma-\alpha-1-1/k} \sum_{r=d}^{n} \frac{A_{n-r}^{\alpha} A_{r-d}^{-\alpha-1}}{r} \tag{24}$$

for $1 \le d \le n$ and $e_{nd} = 0$ otherwise. In the range $n/2 < d \le n$, we use the estimate

$$0 \le e_{nd} \le \frac{M}{n^2}.$$

If $1 \le d \le n/2$ then we claim that

$$0 \le \sum_{r=d}^{n} \frac{A_{n-r}^{\alpha} A_{r-d}^{-\alpha-1}}{r} \le M n^{\alpha}.$$

To see this, let λ be chosen so that $1/2 < \lambda < 1$. Then

$$\begin{aligned}
0 \le \sum_{r=d}^{[\lambda n]} \frac{A_{n-r}^{\alpha} A_{r-d}^{-\alpha-1}}{r} &\le M\,(n(1-\lambda))^{\alpha} \sum_{r=d}^{[\lambda n]} \frac{A_{r-d}^{-\alpha-1}}{r} \\
&\le M n^{\alpha} \sum_{r=d}^{\infty} \frac{A_{r-d}^{-\alpha-1}}{r-d+1} \\
&\le M n^{\alpha} \sum_{r=d}^{\infty} |\, A_{r-d+1}^{-\alpha-2} \,| = O(n^{\alpha}),
\end{aligned}$$

since $\sum_{n=0}^{\infty} A_n^{-\alpha-2} = 0$. Also

$$0 \leq \sum_{r=[\lambda n]+1}^{n} \frac{A_{n-r}^{\alpha} A_{r-d}^{-\alpha-1}}{r} \leq M\frac{1}{\lambda n}(n(\lambda - 1/2))^{-\alpha-1} \sum_{r=[\lambda n]+1}^{n} A_{n-r}^{\alpha}$$
$$= O(\frac{1}{n}) = O(n^{\alpha}).$$

Hence, putting this in (24), if $1 \leq d \leq n/2$ then

$$0 \leq e_{nd} \leq Md^{-\gamma+\alpha+1/k} n^{\gamma-1-1/k},$$

so that, if $1 \leq d \leq n$,

$$0 \leq e_{nd} \leq M\left(d^{-\gamma+\alpha+1/k} n^{\gamma-1-1/k} + n^{-2}\right).$$

Thus by Corollary 8 in [2], we see that $E : l^k \to l^k$ provided that $-\gamma + 1 + 1/k > 1/k$ and $\gamma - \alpha - 1/k - \gamma + 1 + 1/k \geq 1$. Thus $| C, \alpha, \gamma |_k$ is left translative if (ii) holds.

For right translativity we need to express $\sigma_n^{\alpha-1}$ in terms of $\overline{\tau}_n^{\alpha}$. Assuming that $\alpha \neq 0$ we get

$$\sigma_n^{\alpha-1} = \frac{1}{A_n^{\alpha-1}} \sum_{d=1}^{n+1} \overline{\tau}_d^{\alpha} A_d^{\alpha} \sum_{r=d-1}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d+1}^{-\alpha-1}}{r+1},$$

and so $| C, \alpha, \gamma |_k$ is right translative if $F : l^k \to l^k$ where

$$f_{nd} = \left(\frac{d}{n}\right)^{-\gamma+\alpha+1/k} \sum_{r=d-1}^{n} \frac{A_{n-r}^{\alpha-1} A_{r-d+1}^{-\alpha-1}}{r+1}.$$

From (16) we see that for $n, d \geq 1$,

$$f_{nd} = \left(\frac{n+1}{n}\right)^{-\gamma+\alpha+1/k} b_{n+1,d}$$

and so from the results proved above for B we have that $F : l^k \to l^k$. Hence the result.

REFERENCES

1. A.K. Al-Madi, *On translativity of some absolute weighted mean methods,* Bull. Cal. Math. Soc. **79** (1987), 235-241.
2. G. Bennett, *Some elementary inequalities,* Quart. J. Math. Oxford **(2) 38** (1987), 401-425.
3. H. Bor, *On $| \overline{N}, p_n |_k$ summability factors of infinite series,* Jour. Univ. Kuwait (Science) **10** (1983), 37-42.
4. H. Bor and B. Thorpe, *On some absolute summability methods,* Analysis **7** (1987), 145-152.
5. H. Bor and B. Thorpe, *A note on two absolute summability methods,* Analysis **12** (1992), 1-3.
6. T.M. Flett, *Some more theorems concerning the absolute summability of Fourier series and power series,* Proc. London Math. Soc. **(3) 8** (1958), 357-387.
7. G.H. Hardy, *Divergent Series,* O.U.P. (1949).
8. A. Peyerimhoff, *Summierbarkeitsfaktoren für absolut Cesàro-summierbare Reihen,* Math. Zeit. **59** (1954), 417-424.

School of Mathematics and Statistics,
The University of Birmingham,
Birmingham B15 2TT
United Kingdom