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Fulvio Melia*

The Friedmann–Lemaître–Robertson–Walker metric and the principle of equivalence

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Abstract: The evidence in favor of a Universe expanding at a constant rate, in contrast to the various episodes of deceleration and acceleration expected in the standard model, has been accumulating for over a decade now. In recent years, this inference has been strengthened by a study of the Friedmann–Lemaître–Robertson–Walker (FLRW) metric in relation to Einstein's principle of equivalence. This earlier work concluded that the choice of lapse function $g_{tt} = 1$ characterizing the FLRW solution to Einstein's equations is inconsistent with any kind of accelerated cosmic expansion. In this paper, we demonstrate and confirm this important result by directly testing the self-consistency of four well-known FLRW cosmologies. These include the Milne universe, de Sitter space, the Lanczos universe, and the $R_{\rm h} = ct$ model. We show that only the constantly expanding models (Milne and $R_{\rm h} = ct$) are consistent with the principle of equivalence, while de Sitter and Lanczos fail the test. We discuss some of the many consequences of this conclusion.

Keywords: cosmological models; general relativity; principle of equivalence; spacetime metric.

1 Introduction

Modern cosmology is based on the Cosmological principle, whose symmetries inform the Friedmann–Lemaître–Robertson–Walker (FLRW) metric [12, 24], one of the most influential solutions to Einstein's equations. FLRW is a special member of the class of spherically-symmetric spacetimes often used in problems of gravitational collapse or expansion [10, 20, 21, 23].

But a principal difference between FLRW and the other solutions in this category is that the dynamical equations describing the Universe's expansion are derived by first using all of the possible symmetries, such as homogeneity and isotropy, to greatly simplify the coefficients prior to introducing the metric into Einstein's equations. FLRW is conventionally written in the form

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\, d\Omega^2\right), \qquad (1.1)$$

where $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\, d\phi^2$. The expansion factor $a(t)$ is a function of cosmic time t, and the spatial coordinates (r, θ, ϕ) are defined in the co-moving frame and remain "fixed" for all particles. The spatial curvature constant k is $+1$ for a closed universe, 0 for a flat, open universe, and -1 for an open universe. Quite remarkably, this approach assumes *free-fall* conditions throughout the cosmos by setting the lapse function equal to a constant (i.e., $g_{tt} = 1$), without confirming whether this assumption is consistent with the time dilation arising from an accelerated expansion when $\ddot{a} \neq 0$. Yet the Hubble flow is not inertial in ΛCDM, so it is unclear why FLRW, with a constant g_{tt}, should adequately account for the dynamics in standard cosmology.

In several recent papers, we have carefully studied this question on the basis of Einstein's principle of equivalence (PoE) and have concluded that the choice $g_{tt} = 1$ is generally **not** consistent with arbitrary equations-of-state. It must be emphasized that the field equations themselves are fully consistent with the PoE, since they were derived within the mathematical framework founded on the equality of the inertial and gravitational masses. But this does not ensure that any given solution must also be consistent with the PoE if the symmetries used to simplify the metric coefficients are in conflict with the choice of stress-energy tensor.

A mathematical formulation of the PoE [24] states that there exists—at every spacetime point x^μ—a local, inertial (i.e., free-falling) frame $\xi^\mu(x)$, with respect to which one may 'measure' the spacetime curvature in the accelerated frame. The coordinates ξ^μ fulfill the role required of the local, free-falling (inertial) frame if they satisfy the equations

$$\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu} = \Gamma^\lambda{}_{\mu\nu}\frac{\partial \xi^\alpha}{\partial x^\lambda}, \qquad (1.2)$$

where the Christoffel symbols,

$$\Gamma^\lambda{}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\lambda}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\alpha}\right\}, \qquad (1.3)$$

***Corresponding author: Fulvio Melia**, Department of Physics, The Applied Math Program, and Department of Astronomy, Then University of Arizona, AZ 85721, Tucson, USA, E-mail: fmelia@arizona.edu. https://orcid.org/0000-0002-8014-0593

characterize the spacetime curvature. Alternatively, one may demonstrate consistency of the metric coefficients with the PoE via the transformation equation,

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}, \tag{1.4}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ is the corresponding metric tensor in flat spacetime.

We must also point out that there are, of course, an infinite number of local inertial frames at each point within the spacetime, each distinguished from the others by a constant non-zero velocity. Nevertheless, there is only one, unique local free-falling frame at that point, whose coordinates, $\xi^\mu(x)$, are the ones that must satisfy Equation (1.2). These are the coordinates we shall derive for each of the four FLRW solutions we consider in this paper, as given in Equations (2.6), (3.15), (4.10) and (5.2).

We showed earlier [11] that, instead of $g_{tt} = 1$ in Equation (1.1), the FLRW lapse function must satisfy the constraint

$$\int^{ct} \sqrt{g_{tt}(t')}\, d(ct') = c g_{tt}(t)\frac{a}{\dot{a}}, \tag{1.5}$$

for an arbitrary expansion factor $a(t)$. (Note that we have corrected the missing square-root sign in the original expression.) This is the critical condition that relates the time dilation, measured in the accelerated frame, to the expansion factor $a(t)$ and the equation-of-state. Clearly, the choice $g_{tt} = 1$ is consistent only with the expansion profile $a(t) \propto t$ (and the much less relevant Minkowski space solution with $a =$ constant).

The goal of this paper is to demonstrate and affirm this result by reversing this procedure. Instead of beginning the derivation with an arbitrary lapse function and showing that the choice $g_{tt} = 1$ necessarily forces $a(t)$ to be linear in t, we take several well-known FLRW models, including the Milne universe, the $R_\mathrm{h} = ct$ universe, de Sitter space, and the Lanczos universe, and explicitly demonstrate that the former two, with $\ddot{a} = 0$, satisfy Equations (1.2) and (1.4), while the latter, with $\ddot{a} \neq 0$, do not.

Besides providing an important confirmation of this result using an alternative approach, the steps we undertake in this paper also address any possible concern one may have that beginning with $g_{tt} \neq 1$ in Equation (1.1) may be unnecessarily shifting the FLRW metric out of the free-falling frame, thereby creating an artificial need for a time dilation. We shall instead take the FLRW solutions *as they are*, with $g_{tt} = 1$, and prove that they are inconsistent with the PoE unless their expansion is linear in t.

In § 2, we begin this process with the Milne universe to establish the fact that the FLRW metric in Equation (1.1) with $g_{tt} = 1$ does in fact satisfy Equations (1.2) and (1.4) for a non-accelerating cosmology. We then show in § 3 that the outcome for de Sitter space is completely different. And we amplify this result in § 4 by showing that the Lanczos cosmology, another accelerated universe, also fails this test. Whereas Milne and de Sitter are open universes, Lanczos is closed, so we rule out any possibility that this inconsistency may somehow be related to spatial curvature. Finally, in § 5, we confirm that FLRW cosmologies with $\ddot{a} = 0$ satisfy the PoE by also considering the $R_\mathrm{h} = ct$ universe, which is based on the zero active mass condition ($\rho + 3p = 0$), producing an expansion with $a(t) = (t/t_0)$, in terms of the age of the Universe today. We end with our conclusions in § 6.

2 The Milne universe

The Milne universe is empty, with $\rho = 0$ and a negative spatial curvature constant, $k = -1$. Its expansion factor may be written

$$a(t) = ct. \tag{2.1}$$

This FLRW model is thus one of the special cases discussed above, for which the lapse function g_{tt} should be set equal to one. First introduced by ref. [19], this model has zero acceleration, i.e., $\ddot{a}(t) = 0$, and may be viewed as a re-parametrization of Minkowski space. To find the corresponding Cartesian coordinates $\xi^\alpha = (\xi^0, \xi^1, \xi^2, \xi^3)$ in a local inertial frame at any spacetime point $x^\mu = (ct, r, \theta, \phi)$, we shall follow a procedure described in refs. [1, 2, 14].

We first introduce the co-moving distance variable χ, defined according to

$$r = \sinh\,\chi, \tag{2.2}$$

and re-write the FLRW metric (Eq. 1.1) as

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - (ct)^2\Big(d\chi^2 + \sinh^2\,\chi\, d\Omega^2\Big). \tag{2.3}$$

With the subsequent transformation,

$$T = t\cosh\,\chi$$
$$R = ct\sinh\,\chi, \tag{2.4}$$

we may then recast this metric into the form

$$\mathrm{d}s^2 = c^2 dT^2 - \mathrm{d}R^2 - R^2 d\Omega^2. \tag{2.5}$$

And with the definitions

$$\xi^0 = cT$$
$$\xi^1 = R\sin\,\theta\cos\,\phi$$

$$\xi^2 = R \sin\theta \sin\phi$$
$$\xi^3 = R \cos\theta, \tag{2.6}$$

we arrive at its final Minkowski form,

$$ds^2 = (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2. \tag{2.7}$$

The coordinates ξ^α therefore correspond to the local free-falling (inertial) frame, evaluated via Equation (2.6), anywhere in the Milne universe.

From Equation (1.3), we find that the only non-zero Christoffel symbols in this case are

$$\begin{aligned}
&\Gamma^t{}_{rr} = \frac{ct}{1+r^2} \qquad \Gamma^t{}_{\theta\theta} = ctr^2 \\
&\Gamma^t{}_{\phi\phi} = ctr^2 \sin^2\theta \\
&\Gamma^r{}_{rt} = \Gamma^r{}_{tr} = \frac{1}{ct} \qquad \Gamma^\theta{}_{\theta t} = \Gamma^\theta{}_{t\theta} = \frac{1}{ct} \\
&\Gamma^\phi{}_{\phi t} = \Gamma^\phi{}_{t\phi} = \frac{1}{ct} \\
&\Gamma^r{}_{rr} = -\frac{r}{1+r^2} \qquad \Gamma^r{}_{\theta\theta} = -r(1+r^2) \\
&\Gamma^r{}_{\phi\phi} = -r(1+r^2)\sin^2\theta \\
&\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r} \\
&\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \\
&\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r} \\
&\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta.
\end{aligned} \tag{2.8}$$

Turning our attention to Equation (1.2), we can now consider several illustrative values of the indices. For $\alpha = \mu = \nu = 0$, the left-hand side gives

$$\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\big(\cosh\chi\big) = 0. \tag{2.9}$$

By comparison, the right-hand side of this equation is

$$\Gamma^\lambda{}_{\mu\nu}\frac{\partial\xi^\alpha}{\partial x^\lambda} = \Gamma^\lambda{}_{tt}\frac{\partial\xi^t}{\partial x^\lambda} = 0, \tag{2.10}$$

an exact match. With $\alpha = 0$ and $\mu = \nu = 1$, the left-hand side becomes

$$\frac{\partial^2(cT)}{\partial r^2} = \frac{ct}{(1+r^2)^{3/2}}. \tag{2.11}$$

Correspondingly,

$$\begin{aligned}
\Gamma^\lambda{}_{rr}\frac{\partial\xi^t}{\partial x^\lambda} &= \frac{ct}{1+r^2}\frac{\partial(cT)}{\partial(ct)} - \frac{r}{1+r^2}\frac{\partial(cT)}{\partial r} \\
&= \frac{ct}{(1+r^2)^{3/2}},
\end{aligned} \tag{2.12}$$

another exact match.

It is easy to verify that such an exact match emerges for all the other indices in Equation (1.2) as well. Alternatively, we may wish to check that the coordinates x^μ and ξ^α satisfy the transformation of the metric coefficients in Equation (1.4). For example,

$$g_{rr} = \frac{\partial\xi^\alpha}{\partial r}\frac{\partial\xi^\beta}{\partial r}\eta_{\alpha\beta}, \tag{2.13}$$

which gives

$$\begin{aligned}
g_{rr} &= \left(ct\sinh\chi\frac{\partial\chi}{\partial r}\right)^2 - \big(ct\sin\theta\cos\phi\big)^2 \\
&\quad - \big(ct\sin\theta\sin\phi\big)^2 - \big(ct\cos\theta\big)^2 \\
&= -\frac{(ct)^2}{1+r^2},
\end{aligned} \tag{2.14}$$

the correct FLRW coefficient for the rr-component of the Milne metric in Equation (1.1). We thus confirm that the constraint in Equation (1.5) derived in ref. [11] is satisfied by the non-accelerating Milne universe, whose metric with $g_{tt} = 1$ and $a(t) = ct$ is completely consistent with both Equations (1.2) and (1.4).

3 de Sitter space

Let us next consider the perpetually accelerating de Sitter universe, which also assumes $g_{tt} = 1$ in Equation (1.1), but instead has $\ddot{a} \neq 0$. Its metric (with $k = 0$) may be written

$$ds^2 = c^2 dt^2 - e^{2Ht}\Big(dr^2 + r^2\,d\Omega^2\Big), \tag{3.1}$$

where we have put $a(t) = e^{Ht}$. As we shall see, de Sitter space contrasts sharply with Milne because—unlike the latter—it does not at all satisfy the PoE.

Both the Milne universe (as we have seen) and de Sitter space have constant spacetime curvature. Using fixed observer coordinates, one may therefore find a transformation that renders all of their coefficients independent of time (see, e.g., ref. [14]). For de Sitter, we put

$$\begin{aligned}
cT &= ct - \frac{1}{2}R_h \ln\Phi \\
R &= a(t)r,
\end{aligned} \tag{3.2}$$

where R_h is the gravitational (or Hubble) radius [17],

$$R_h \equiv \frac{c}{H}, \tag{3.3}$$

and

$$\Phi \equiv 1 - \left(\frac{R}{R_h}\right)^2. \tag{3.4}$$

Under this transformation, the de Sitter interval becomes

$$ds^2 = \Phi c^2 dT^2 - \Phi^{-1} dR^2 - R^2\, d\Omega^2. \tag{3.5}$$

As one can confirm, all the metric coefficients in Equation (3.5) are independent of the new time coordinate T. For future reference, we also define the Cartesian coordinates

$$\begin{aligned} X^1 &= R \sin\theta \cos\phi \\ X^2 &= R \sin\theta \sin\phi \\ X^3 &= R \cos\theta. \end{aligned} \tag{3.6}$$

To bring this metric into its Cartesian isotropic form, we introduce an additional coordinate transformation,

$$R = \sigma \left[1 + \left(\frac{\sigma}{\sigma_{\rm h}}\right)^2\right]^{-1}, \tag{3.7}$$

where

$$\sigma_{\rm h} \equiv 2R_{\rm h}. \tag{3.8}$$

Therefore,

$$dR = d\sigma\, \frac{P}{Q^2}, \tag{3.9}$$

and

$$\Phi = \frac{P^2}{Q^2}, \tag{3.10}$$

in terms of the newly defined quantities

$$\begin{aligned} P &\equiv \left[1 - \left(\frac{\sigma}{\sigma_{\rm h}}\right)^2\right] \\ Q &\equiv \left[1 + \left(\frac{\sigma}{\sigma_{\rm h}}\right)^2\right]. \end{aligned} \tag{3.11}$$

The metric for de Sitter space may thus also be written

$$ds^2 = \left(\frac{P}{Q}\right)^2 c^2 dT^2 - \frac{1}{Q^2}\left(d\sigma^2 + \sigma^2 d\Omega^2\right), \tag{3.12}$$

and we arrive at its final Cartesian isotropic form,

$$ds^2 = \left(\frac{P}{Q}\right)^2 c^2 dT^2 - \frac{1}{Q^2}\left[(d\sigma^1)^2 + (d\sigma^2)^2 + (d\sigma^3)^2\right], \tag{3.13}$$

with the introduction of the Cartesian coordinates corresponding to σ:

$$\begin{aligned} \sigma^1 &= \sigma \sin\theta \cos\phi \\ \sigma^2 &= \sigma \sin\theta \sin\phi \\ \sigma^3 &= \sigma \cos\theta. \end{aligned} \tag{3.14}$$

The PoE requires the coordinates in the local inertial frame to satisfy Equation (1.2) only in the vicinity of each selected spacetime point x^μ. Thus, one may assume that R and σ are approximately constant wherever they appear inside the metric coefficients, allowing us to write $P \approx P(x^\mu)$ and $Q \approx Q(x^\mu)$ in the vicinity of x^μ. The local free-falling (inertial) frame coordinates may thus be defined as

$$\begin{aligned} \xi^0 &= \frac{P(x^\mu)}{Q(x^\mu)}\, cT \\ \xi^1 &= \frac{1}{Q(x^\mu)}\sigma^1 \\ \xi^2 &= \frac{1}{Q(x^\mu)}\sigma^2 \\ \xi^3 &= \frac{1}{Q(x^\mu)}\sigma^3, \end{aligned} \tag{3.15}$$

allowing us to write the de Sitter metric in its Minkowski form consistent with Equation (2.7).

To see if these coordinates satisfy the PoE, we shall need the corresponding non-zero Christoffel symbols, written in spherical coordinates:

$$\begin{aligned} \Gamma^t{}_{rr} &= \frac{e^{2Ht}}{R_{\rm h}} \\ \Gamma^t{}_{\theta\theta} &= \frac{e^{2Ht}}{R_{\rm h}} r^2 \\ \Gamma^t{}_{\phi\phi} &= \frac{e^{2Ht}}{R_{\rm h}} r^2 \sin^2\theta \\ \Gamma^r{}_{rt} = \Gamma^r{}_{tr} &= \frac{1}{R_{\rm h}} \\ \Gamma^\theta{}_{\theta t} = \Gamma^\theta{}_{t\theta} &= \frac{1}{R_{\rm h}} \\ \Gamma^\phi{}_{\phi t} = \Gamma^\phi{}_{t\phi} &= \frac{1}{R_{\rm h}} \\ \Gamma^r{}_{\theta\theta} &= r \\ \Gamma^r{}_{\phi\phi} &= r \sin^2\theta \\ \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{1}{r} \\ \Gamma^\theta{}_{\phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} &= \frac{1}{r} \\ \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} &= \cot\theta. \end{aligned} \tag{3.16}$$

We may now examine whether the coordinates x^μ and ξ^α (in Eq. 3.15) satisfy Equation (1.2). In fact, they do not. For example, the $\alpha = \mu = \nu = 0$ component yields

$$\frac{P}{Q}\frac{2}{R_{\rm h}}\left(\frac{R}{R_{\rm h}}\right)^2 \Phi^{-2} = 0, \tag{3.17}$$

which cannot be consistent for arbitrary values of R (or r). Similarly, we find for the $\alpha = \mu = \nu = 1$ component that

$$0 = \frac{e^{2Ht}}{R_h} \frac{Q}{P} \frac{X^1}{R_h} \tag{3.18}$$

which, again, is not correct for arbitrary values of X^1 and t. Notice, e.g., that $e^{2Ht} X^1$ may become arbitrarily large compared to R_h, which is fixed in this cosmology.

Unlike the Milne universe, de Sitter space is therefore not consistent with Einstein's PoE. Both of these models are FLRW cosmologies and both assume a lapse function $g_{tt} = 1$. But whereas the Milne universe expands at a constant rate, with an implied zero time dilation with respect to the local free-falling (inertial) frame, de Sitter space expands at an accelerated rate. A lapse function $g_{tt} = 1$ for this model therefore cannot adequately account for the spacetime curvature, reflected in this spacetime's inconsistency with Equations (3.17) and (3.18).

4 The Lanczos universe

In the previous two sections, we considered a non-accelerating cosmology (Milne) and the perpetually accelerating de Sitter model, both open universes with $k \le 0$. To round out the discussion, we here consider another accelerating cosmology—the Lanczos universe [6]—but this time with $k > 0$, implying a finite, closed spacetime. The Lanczos metric is

$$ds^2 = c^2 dt^2 - (cb)^2 \cosh^2(t/b) \times \left(\frac{dr^2}{1 - r^2} + r^2\, d\Omega^2 \right), \tag{4.1}$$

where b is a constant (though not the Hubble constant $H \equiv \dot{a}/a$) and $k = +1$. The expansion factor is $a(t) = (cb) \cosh(t/b)$, so $H = (1/b) \tanh(t/b)$. The physical interpretation of this spacetime is that it represents the gravitational field of a rigidly rotating dust cylinder coupled to a cosmological constant.

To find a local inertial frame at any location x^μ in this spacetime, we begin with the following transformation [3] that brings the metric into its static form:

$$\begin{aligned} R &= a(t) r \equiv cbr \cosh(t/b) \\ \tanh(T/b) &\equiv \left(1 - r^2\right)^{-1/2} \tanh(t/b). \end{aligned} \tag{4.2}$$

In terms of the new coordinates (cT, R, θ, ϕ), the line element becomes

$$ds^2 = \left[1 - \left(\frac{R}{cb}\right)^2\right] c^2 dT^2 - \left[1 - \left(\frac{R}{cb}\right)^2\right]^{-1} dR^2 - R^2 d\Omega^2. \tag{4.3}$$

Equation (4.3) is actually identical to Equation (3.5) for de Sitter, except that the gravitational (or apparent) horizon is now $R_h = cb$ instead of c/H. But both of these radii are constant, so it is not surprising to find that the same kind of transformation we used in Equations (3.7)–(3.11) may be used here as well.

We introduce a new radial coordinate ϖ via the definition

$$R = \varpi \left[1 + \left(\frac{\varpi}{\varpi_h}\right)^2\right]^{-1}, \tag{4.4}$$

where

$$\varpi_h \equiv 2R_h = 2cb. \tag{4.5}$$

Then,

$$dR = d\varpi \frac{U}{V^2}, \tag{4.6}$$

in terms of the quantities

$$\begin{aligned} U &\equiv \left[1 - \left(\frac{\varpi}{\varpi_h}\right)^2\right] \\ V &\equiv \left[1 + \left(\frac{\varpi}{\varpi_h}\right)^2\right]. \end{aligned} \tag{4.7}$$

The Lanczos metric written in Cartesian isotropic form is thus

$$ds^2 = \left(\frac{U}{V}\right)^2 c^2 dT^2 - \frac{1}{V^2}\left[(d\varpi^1)^2 + (d\varpi^2)^2 + (d\varpi^3)^2\right], \tag{4.8}$$

where the Cartesian coordinates corresponding to ϖ are

$$\begin{aligned} \varpi^1 &= \varpi \sin\theta \cos\phi \\ \varpi^2 &= \varpi \sin\theta \sin\phi \\ \varpi^3 &= \varpi \cos\theta. \end{aligned} \tag{4.9}$$

The local free-falling (inertial) frame coordinates for the Lanczos universe are therefore

$$\begin{aligned} \xi^0 &= \frac{U(x^\mu)}{V(x^\mu)} cT \\ \xi^1 &= \frac{1}{V(x^\mu)} \varpi^1 \\ \xi^2 &= \frac{1}{V(x^\mu)} \varpi^2 \\ \xi^3 &= \frac{1}{V(x^\mu)} \varpi^3, \end{aligned} \tag{4.10}$$

and we may use these to write its line element in Minkowski form, analogous to Equation (2.7).

As we did for Milne and de Sitter, we begin to examine whether these coordinates satisfy the PoE by first deriving the non-zero Christoffel symbols:

$$\begin{aligned}
{\Gamma^t}_{\phi\phi} &= cbr^2 \sin^2\theta \cosh\frac{t}{b}\sinh\frac{t}{b}\\
{\Gamma^t}_{rr} &= \frac{cb}{1-r^2}\cosh\frac{t}{b}\sinh\frac{t}{b}\\
{\Gamma^t}_{\theta\theta} &= cbr^2\cosh\frac{t}{b}\sinh\frac{t}{b}\\
{\Gamma^r}_{rt} = {\Gamma^r}_{tr} &= \frac{1}{cb}\tanh\frac{t}{b}\\
{\Gamma^\theta}_{\theta t} = {\Gamma^\theta}_{t\theta} &= \frac{1}{cb}\tanh\frac{t}{b}\\
{\Gamma^\phi}_{\phi t} = {\Gamma^\phi}_{t\phi} &= \frac{1}{cb}\tanh\frac{t}{b}\\
{\Gamma^r}_{rr} &= \frac{r}{1-r^2}\\
{\Gamma^r}_{\theta\theta} &= -(1-r^2)r\\
{\Gamma^r}_{\phi\phi} &= -(1-r^2)r\sin^2\theta\\
{\Gamma^\theta}_{r\theta} = {\Gamma^\theta}_{\theta r} &= \frac{1}{r}\\
{\Gamma^\theta}_{\phi\phi} &= -\sin\theta\cos\theta\\
{\Gamma^\phi}_{r\phi} = {\Gamma^\phi}_{\phi r} &= \frac{1}{r}\\
{\Gamma^\phi}_{\theta\phi} = {\Gamma^\phi}_{\phi\theta} &= \cot\theta. \qquad (4.11)
\end{aligned}$$

Then, for the illustrative set of indices $\alpha = \mu = \nu = 0$ and $\alpha = \mu = \nu = 1$, we find that

$$\frac{r^2\sqrt{1-r^2}}{cb}\,\frac{2\tanh(t/b)[1-\tanh^2(t/b)]}{[1-r^2-\tanh^2(t/b)]^2} = 0, \qquad (4.12)$$

and

$$0 = \frac{cbr}{1-r^2}\,\frac{V}{U}\cosh^3(t/b)\sin\theta\cos\theta. \qquad (4.13)$$

As was the case for de Sitter, neither of these equations is satisfied for arbitrary values of r and t. With its lapse function $g_{tt} = 1$, the Lanczos universe fails the PoE test. Like de Sitter, Lanczos accelerates, and this failure is an affirmation of our conclusion that nonlinear expansions of the cosmos need to be represented by metrics that allow for possible time dilation relative to local inertial frames.

5 The $R_\mathrm{h} = ct$ universe

We close this discussion with an application of this test to the $R_\mathrm{h} = ct$ universe, another FLRW cosmology with an expansion parameter $a(t) \propto t$ [12, 13], but with $k = 0$. Unlike the Milne universe, however, this model is not empty; it contains the same constituents in the cosmic fluid as the standard model does, though it incorporates an additional constraint from general relativity—an overall equation-of-state given by the zero active mass condition, i.e., $\rho + 3p = 0$, in terms of the total energy density ρ and pressure p. It is not difficult to recognize from the Raychaudhuri equation [22] that this constraint produces zero acceleration (i.e., $\ddot{a} = 0$), and therefore implies an expansion with $a(t) = t/t_0$. This normalization in terms of the age (t_0) of the Universe is consistent with zero spatial flatness.

The line element for the $R_\mathrm{h} = ct$ universe may therefore be written

$$ds^2 = c^2dt^2 - \left(\frac{t}{t_0}\right)^2\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] \qquad (5.1)$$

where, as usually defined, $x^\mu = (ct, x^1, x^2, x^3)$ are the coordinates in the co-moving frame. A distinguishing feature of the $R_\mathrm{h} = ct$ universe, in comparison with Milne, de Sitter and Lanczos, is that its spacetime curvature is not static. Like ΛCDM, it therefore does not offer us the possibility of first finding a coordinate-transformation (like Eqs. 2.4, 3.2 and 4.2) that permits its metric to be written in a time-independent form.

Nevertheless, pursuant to the aforementioned requirement that the local free-falling (inertial) frame satisfying Equation (1.2) need only be defined in the vicinity of each spacetime point in the observer's frame, we may use the following coordinate transformation at x^μ:

$$\begin{aligned}
\xi^0 &= ct\,\eta(x)\\
\xi^i &= a(t)x^i = \left(\frac{t}{t_0}\right)x^i, \qquad (5.2)
\end{aligned}$$

where $\eta(x)$ is approximately constant for points in the neighborhood of x^μ. Whereas one finds a global inertial frame for Milne (corresponding to Eq. 2.6), the local inertial frames in $R_\mathrm{h} = ct$ need to be found point by point, each with its own value of η.

It is not difficult to see that the line element in Equation (2.7), written in terms of Equation (5.2), will match Equation (5.1) as long as

$$\eta(x)^2 \equiv 1 + \frac{1}{(ct_0)^2}\frac{d}{dt}(tr^2). \qquad (5.3)$$

Of course, $dr/dt = 0$ if the inertial frame coincides with the Hubble flow, in which case

$$\eta(x)^2 \equiv 1 + \frac{r^2}{(ct_0)^2}, \qquad (5.4)$$

but we do not need to consider such details. The key point is that $\eta(x)$ is always of order unity. This coordinate transformation therefore provides us with a reasonably accurate representation of the local free-falling (inertial) frames we are seeking in this spacetime.

Then, the non-zero Christoffel symbols corresponding to the $R_h = ct$ metric (Eq. 5.1) written in Cartesian coordinates are

$$\begin{aligned}\Gamma^t{}_{11} &= \Gamma^t{}_{22} = \Gamma^t{}_{33} = \frac{1}{ct_0}\frac{t}{t_0}\\ \Gamma^1{}_{t1} &= \Gamma^1{}_{1t} = \frac{1}{ct}\\ \Gamma^2{}_{t2} &= \Gamma^2{}_{2t} = \frac{1}{ct}\\ \Gamma^3{}_{t3} &= \Gamma^3{}_{3t} = \frac{1}{ct}.\end{aligned} \tag{5.5}$$

With these, we can now examine whether the coordinates x^μ and ξ^α for the $R_h = ct$ universe are consistent with Equation (1.2).

First, we confirm that x^μ and ξ^α for the $R_h = ct$ universe satisfy the coordinate transformation Equation (1.4). We have

$$\begin{aligned}g_{tt} &= \left(\frac{\partial\xi^0}{\partial ct}\right)^2 - \sum_i\left(\frac{\partial\xi^i}{\partial ct}\right)^2\\ &= \eta^2 - \frac{r^2}{(ct_0)^2}\\ &= 1.\end{aligned} \tag{5.6}$$

In addition,

$$\begin{aligned}g_{ii} &= \left(\frac{\partial\xi^0}{\partial x^i}\right)^2 - \sum_j\left(\frac{\partial\xi^j}{\partial x^i}\right)^2\\ &= 0 - a^2\\ &= a^2,\end{aligned} \tag{5.7}$$

as required.

In Equation (1.2), one has for $\alpha = \mu = \nu = 0$,

$$\frac{\partial^2\xi^0}{\partial(ct)^2} = 0, \tag{5.8}$$

and

$$\Gamma^\lambda{}_{00}\frac{\partial\xi^0}{\partial x^\lambda} = 0, \tag{5.9}$$

an exact match. Similarly, for $\alpha = \mu = i$ and $\nu = 0$, one gets

$$\frac{\partial^2\xi^i}{\partial x^i\,\partial ct} = \frac{\partial}{\partial x^i}\left(\frac{x^i}{ct_0}\right) = \frac{1}{ct_0}, \tag{5.10}$$

while the right-hand side gives

$$\Gamma^\lambda{}_{i0}\frac{\partial\xi^i}{\partial x^\lambda} = \frac{1}{ct}\frac{\partial\xi^i}{\partial x^i} = \frac{1}{ct_0}, \tag{5.11}$$

an equally precise match. As a third illustrative example, consider the case $\alpha = \mu = \nu = i$, for which

$$\frac{\partial^2\xi^i}{\partial(x^i)^2} = \frac{\partial(t/t_0)}{\partial x^i} = 0. \tag{5.12}$$

The right-hand side yields

$$\Gamma^\lambda{}_{ii}\frac{\partial\xi^i}{\partial x^\lambda} = \frac{1}{ct_0}\frac{t}{t_0}\frac{\partial\xi^i}{\partial ct} = \frac{1}{ct_0}\frac{x^i}{ct_0}\frac{t}{t_0} \approx 0, \tag{5.13}$$

given that $|x^i/ct_0| \leq 1$ and $t/t_0 \leq 1$, while $ct_0 \gg 1$. All the other components are similarly satisfied.

6 Conclusion

The four cosmological models we have considered in this paper span a broad range of possible applications of the FLRW metric. Three of them, i.e., Milne, de Sitter, and Lanczos, have constant spacetime curvature, while the fourth ($R_h = ct$) does not. Two of these (Milne and $R_h = ct$) expand at a constant rate, while the other two accelerate. And de Sitter, Lanczos, and $R_h = ct$ are spatially flat, while Milne is negatively curved. Therefore, our results cannot be attributed to some unknown selection bias.

We have shown by direct application of the PoE equations that the two non-accelerating models (Milne and $R_h = ct$) are completely consistent with Einstein's principle, while de Sitter and Lanczos do not satisfy Equation (1.2). This is an important confirmation of the result in Equation (1.5) [11, 12], that constrains the range of possible lapse functions (g_{tt}) consistent with a given expansion factor $a(t)$. In that earlier work, we showed that the choice $g_{tt} = 1$ is consistent only with two equations-of-state that lead to $a(t) \propto t$ and $a(t) =$ constant. Here, we have affirmed this conclusion by demonstrating that Milne and $R_h = ct$ are consistent with the PoE, while de Sitter and Lanczos are not. Ultimately, the reason for this disparity is that one cannot ignore the gravitationally-induced time dilation in the accelerated frame (i.e., the Hubble flow) relative to an underlying local inertial frame.

As discussed in these earlier publications, the FLRW metric is unique among the various solutions to Einstein's equations, in part because it is written to comply with the cosmological principle, which includes the assumption of homogeneity throughout the Universe. The lapse function g_{tt} therefore cannot be a function of the spatial coordinates; at most, it can depend only on t. This is the peculiarity that allows one to begin with the FLRW metric written in terms of the co-moving coordinates, (ct, r, θ, ϕ), force the condition $g_{tt} = 1$, and then choose a stress-energy tensor in

Einstein's equations that results in an accelerated expansion. But by keeping g_{tt} equal to 1, one is effectively carrying out a gauge transformation that shifts the original time coordinate in the co-moving frame to the new coordinate, $\mathrm{d}t' \equiv {g_{tt}}^{1/2}\,\mathrm{d}t$, that belongs to the observer in the local free-falling frame. Because of the cosmological principle, this can happen without affecting any of the other metric coefficients, an outcome that is clearly a contradiction because one is using the co-moving coordinates in a frame where the lapse function is 1 (zero time dilation), i.e., the local free-falling frame.

Needless to say, the consequences of this conclusion are rather significant. An extensive discussion of the various issues raised by the inconsistencies we have highlighted in this paper have appeared elsewhere, including refs. [9, 11, 12, 18]. For example, slow-roll inflation has become an indispensable component of the standard model [4, 7]. This brief period of accelerated expansion shortly after the Big Bang is believed to have solved the horizon problem and seeded the quantum fluctuations that eventually classicalized and grew to form the large-scale structure we see today. But slow-roll inflation relies critically on the expansion profile provided by de Sitter. Thus, if the de Sitter cosmology is inconsistent with the PoE, there doesn't appear to be a viable framework for describing the inflationary expansion in the context of FLRW.

Moreover, the inconsistency of an accelerated cosmic expansion with the PoE, as one finds within the various evolutionary phases of the standard model, begins to provide an explanation for the growing tension seen between ΛCDM and the high-precision measurements carried out today [18]. By now, comparative tests between ΛCDM and $R_\mathrm{h} = ct$ have been completed using over 27 different kinds of data (see, e.g., a recent summary in Table 2 of ref. [16]). In each and every case, the FLRW cosmology with a constant expansion rate accounts for the observations at least as well—and often even better—than the standard model with $\ddot{a}(t) \neq 0$.

Fortunately, a confirmation (or rejection) of these ideas and conclusions will be available in the near future. One of the most exciting and informative campaigns will measure the real-time redshift drift of distant quasars [5,8]. This measurement will provide a simple yes/no answer: the redshift drift is expected to be zero throughout the cosmos if $a(t) \propto t$, and non-zero otherwise [15]. A confidence level of $\sim 3\sigma$ will be achievable after only 5 years of observation, while $\sim 5\sigma$ should be reached in about 20 years. Very interestingly, an FLRW cosmology with $a(t) \propto t$ does not have any horizon problems, so a confirmation of the work reported in this paper may completely obviate the need for inflation anyway. The stakes could not be higher.

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