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Exact Beltrami flows in a spherical shell

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Abstract: Exact flows of an incompressible fluid satisfying the Beltrami equation inside a spherical shell are constructed in the Cartesian coordinates in terms of elementary functions. Two scale-invariant equations defining two infinite series of eigenvalues λ_n and $\tilde{\lambda}_m$ of the operator curl in the shell with the nonpenetration boundary conditions on the boundary spheres are derived. The corresponding eigenfields are presented in explicit form and their symmetries are investigated. Asymptotics of the eigenvalues λ_n and $\tilde{\lambda}_m$ at $n,m \to \infty$ are obtained.

Keywords: Beltrami equation; eigenvalues; Helmholtz equation; scale invariance; spectral problems; topology of eigenfields.

1 Introduction

As known [1], Beltrami equation has many applications to problems of plasma physics and fluid dynamics. Equations of viscous incompressible magnetohydrodynamics have the form [2]

$$\frac{\partial \mathbf{V}}{\partial t} + \operatorname{curl} \mathbf{V} \times \mathbf{V} = -\operatorname{grad} \left(\frac{p}{\rho} + \frac{1}{2} |\mathbf{V}|^2 + \Phi \right) + \frac{1}{\rho} \operatorname{curl} \mathbf{B} \times \mathbf{B} + \nu(t) \Delta \mathbf{V}, \quad (1.1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta(t)\Delta \mathbf{B}, \quad \text{div } \mathbf{V} = 0, \quad \text{div } \mathbf{B} = 0.$$

Here, $\mathbf{V}(\mathbf{x},t)$ is fluid velocity, $\mathbf{B}(\mathbf{x},t)$ is magnetic field, $p(\mathbf{x},t)$ is the pressure, $\Phi(\mathbf{x})$ is the gravitational potential, ρ is constant fluid density, v(t) is viscosity coefficient, $\eta(t)$ is magnetic viscosity, and Δ is the Laplace operator.

Equation (1.1) for $V(\mathbf{x}, t) = 0$ reduce to

curl
$$\mathbf{B} \times \mathbf{B} = \operatorname{grad}(p + \rho \Phi), \quad \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \text{div } \mathbf{B} = 0.$$
 (1.2)

Equation (1.2) describes plasma equilibria and is equivalent to the generalized Beltrami equation [3]:

$$\operatorname{curl}\left(\operatorname{curl}\mathbf{B}\times\mathbf{B}\right)=0. \tag{1.3}$$

The force-free plasma equilibria are defined by condition $p(\mathbf{x}, t) + \Phi(\mathbf{x}) = \text{const}$ and are equivalent to the Beltrami equation curl $\mathbf{B} = \lambda \mathbf{B}$, where $\lambda = \text{const}$ in view of equation div $\mathbf{B} = 0$.

If magnetic field $\mathbf{B} = 0$ and viscosity v(t) = 0, Eq. (1.1) becomes Euler equations for ideal incompressible fluid:

$$\frac{\partial \mathbf{V}}{\partial t} + \operatorname{curl} \mathbf{V} \times \mathbf{V} = -\operatorname{grad} \left(\frac{p}{\rho} + \frac{1}{2} |\mathbf{V}|^2 + \Phi \right), \text{ div } \mathbf{V} = 0.$$

For the steady case and with the additional condition $p/\rho + |\mathbf{V}|^2/2 + \Phi = \text{const}$, the Euler equations reduce to the Beltrami equation

$$\operatorname{curl} \mathbf{V}(\mathbf{x}) = \lambda \mathbf{V}(\mathbf{x}). \tag{1.4}$$

Equation (1.4) implies $\lambda = \text{const}$ in view of div $\mathbf{V}(\mathbf{x}) = 0$. Beltrami Eq. (1.4) evidently is invariant with respect to the scale transformations

$$x_1 \to cx_1, \quad x_2 \to cx_2, \quad x_3 \to cx_3, \quad \lambda \to \frac{1}{c}\lambda,$$
 (1.5)

where $c \neq 0$ is an arbitrary parameter.

In case of collinear vector fields $\mathbf{V}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, Eq. (1.1) with nonzero viscosities $v(t) \neq 0$, $\eta(t) \neq 0$ have the following solutions [4]:

$$\mathbf{V}(\mathbf{x},t) = \exp\left(-\lambda^2 \int_{t_0}^t \nu(\tau) d\tau\right) \mathbf{V}_1(\mathbf{x}), \tag{1.6}$$

$$\mathbf{B}(\mathbf{x},t) = C_1 \exp\left(-\lambda^2 \int_{t_0}^t \eta(\tau) d\tau\right) \mathbf{V}_1(\mathbf{x}), \qquad (1.7)$$

where $\mathbf{V}_1(\mathbf{x})$ is any solution to the Beltrami Eq. (1.4) and pressure p is defined from equation $p/\rho + |\mathbf{V}|^2/2 + \Phi = \text{const.}$

If $\mathbf{B}(\mathbf{x}, t) = 0$ then Eq. (1.1) reduce to the Navier-Stokes equations

$$\frac{\partial \mathbf{V}}{\partial t} + \operatorname{curl} \mathbf{V} \times \mathbf{V} = -\operatorname{grad}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{V}|^2 + \Phi\right) + \nu(t)\Delta\mathbf{V},$$

$$\operatorname{div} \mathbf{V} = 0,$$
(1.8)

that have exact solutions (1.6) [4].

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In this paper, we construct two infinite series of exact Beltrami flows satisfying Eq. (1.4) inside the spherical shells

$$S_{R_2R_1}$$
: $R_2 \le R \le R_1$, $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$. (1.9)

We assume that on the two boundary spheres \mathbb{S}_1^2 : $R = R_1$ and \mathbb{S}_2^2 : $R = R_2$ the nonpenetration boundary condition $\mathbf{n}(\mathbf{x}) \cdot \mathbf{V}(\mathbf{x}) = 0$ is satisfied. Here, $\mathbf{n}(\mathbf{x})$ is vector field of unit normals to the boundary. Since for the spheres \mathbb{S}_k^2 we have $\mathbf{n}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$ the boundary conditions have the equivalent form

$$\mathbf{x} \cdot \mathbf{V}_n(\mathbf{x}) = 0 \tag{1.10}$$

for $\mathbf{x} \in \mathbb{S}^2_k$, k=1,2. The same solutions with magnetic field $\mathbf{B}(\mathbf{x})$ instead of velocity $\mathbf{V}(\mathbf{x})$ describe the force-free magnetic fields inside a spherical shell (1.9). All constructed exact solutions define by formulas (1.6) and (1.7) exact flows of viscous magnetic fluid inside the spherical shell S_{R,R_1} (1.9) with the nonpenetration conditions (1.10).

The main problem consists of finding the eigenvalues λ_n of the operator curl and its eigenfields $\mathbf{V}_n(\mathbf{x})$ that satisfy Eq. (1.4) inside the spherical shell and the nonpenetration condition (1.10) at its two boundaries. Cantarella, DeTurck, Gluck, and Teytel studied in paper [5] vector fields V that "satisfy the equation $\nabla \times \mathbf{V} = \lambda \mathbf{V}$, where λ is the eigenvalue of curl having smallest nonzero absolute value among such fields. It is shown that on the ball the energy minimizers are axially symmetric spheromak fields found by Woltjer and Chandrasekhar-Kendall, and on spherical shells they are spheromak-like fields" [5], p. 2766. The authors used the spherical coordinate r, θ, φ in \mathbb{R}^3 and the theory of Bessel functions, and the Legendre and Gegenbauer functions applied earlier by Woltjer [6] and by Chandrasekhar and Kendall [7, 8]. The methods of papers [6-8] were used also in the works [1, 9-12] where the eigenvalue problem (1.4), (1.10) was studied in spherical coordinates in the ball using the Bessell functions.

Cantarella, DeTurck, Gluck, and Teytel presented in Table 1 of paper [5] results of their calculations (using Bessel's functions) of the minimal eigenvalues $\lambda_1^{(1)}$ for different values of R_1 and R_2 .

In this paper, we calculate for the first time in the literature the products of the correspondent numbers $\lambda_1^{(1)}$ and R_1-R_2 from Table 1 of [5]. The results of our calculations are presented in the right column of Figure 1.

Reading the right column of Figure 1 down from the top to the bottom we see that numbers $x_1 = \lambda_1^{(1)}(R_1 - R_2)$ monotonously decrease and tend to the number of π = 3.1415926535 Therefore our calculations prove that the dimensionless eigenvalues $x_1 = \lambda_1^{(1)}(R_1 - R_2)$ converge

to the number of π when $(R_1 - R_2) \to 0$ (hence the dimensionless parameter $\mu = R_1 R_2 / (R_1 - R_2) \to \infty$). As a consequence of this we get the asymptotics

$$\lambda_1^{(1)} \approx \frac{\pi}{R_1 - R_2}, \quad \text{as } \mu \to \infty.$$
 (1.11)

This result is the first appearance of the scale invariance of the spectral problem (1.4), (1.10). In Section 3, we prove asymptotics (1.11) more rigorously.

Note that for $R_1 = 10$ and $R_2 = 9.996665555$, the corresponding value of parameter μ is $\mu = 8996999.5$. For $R_1 = 100$ and $R_2 = 99.99996667$ (in the bottom line of Figure 1), the value of μ is enormous: $\mu = 9001797 \times 10^6$.

Due to the scale invariance (1.5) of the Beltrami Eq. (1.4) the eigenvalue problem (1.4), (1.10) is invariant with respect to the scale transformations

$$R_1 \to cR_1, \quad R_2 \to cR_2, \quad \lambda \to \frac{1}{c}\lambda,$$
 (1.12)

where c > 0 is an arbitrary parameter. We define the dimensionless parameters

$$x = \lambda (R_1 - R_2) > 0, \quad \mu = \frac{R_1 R_2}{(R_1 - R_2)^2} > 0,$$
 (1.13)

that are invariant under the group of scaling transformations (1.12). We will call parameter x the dimensionless eigenvalue.

We construct in this paper two infinite series of eigenvalues λ_n and $\tilde{\lambda}_m$ and the corresponding eigenfields $\mathbf{V}_n(\mathbf{x},\mathbf{A})$ and $\tilde{\mathbf{V}}_m(\mathbf{x},\mathbf{A},\mathbf{B})$ which depend on arbitrary constant vectors $\mathbf{A},\mathbf{B} \in \mathbb{R}^3$. Each eigenvalue λ_n has multiplicity 3 and each eigenvalue $\tilde{\lambda}_m$ has multiplicity 5. We study the eigenvalue problem (1.4), (1.10) in an arbitrary shell $S_{R_2R_1}$ (1.9) and use only Cartesian coordinates x^1, x^2, x^3 and elementary functions of them. In Section 3, we derive the first series of eigenvalues $\lambda_n = x_n/(R_1 - R_2)$ where x_n are roots of equation

$$\tan(x) = \frac{x}{1 + \mu x^2}.$$
 (1.14)

In Section 3, we show that any root x_n of Eq. (1.14) defines the eigenvalue λ_n :

$$\lambda_n = \frac{\sqrt{4\mu + 1} + 1}{2R_1} x_n \tag{1.15}$$

of the spectral problem (1.4) and (1.10). Here, radius R_1 is arbitrary due to the scale invariance (1.12). Radius R_2 is connected with R_1 by relation

$$R_2 = \frac{\sqrt{4\mu + 1} - 1}{\sqrt{4\mu + 1} + 1} R_1. \tag{1.16}$$

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	R_2	R_1	$\lambda_1^{(1)}$	$x_1 = \lambda_1^{(1)}(R_1 - R_2)$
	0	1	4.493409	4.493409
	0.144273032	1.001	4.568933565	3.914329
	0.2470322424	1.005	4.799466836	3.637841
	0.3117589910	1.01	5.033680133	3.514722
	0.54018397940	1.05	6.423856305	3.274985
	0.6917396417	1.1	7.858576168	3.208345
	1.334200824	1.5	18.97447753	3.145953
	1.912931183	2	36.08896092	3.142224
	4.986630952	5	234.9901755	3.141595
	9.996665555 	10	942.1635951	3.141593
	99.99996667	100	94257.20533	3.141593

Figure 1: Calculation of dimensionless eigenvalues x_1 .

In Section 4, we derive the second series of eigenvalues $\tilde{\lambda}_m = \tilde{x}_m/(R_1-R_2)$ where \tilde{x}_m are the roots of equation

$$\tan(x) = \frac{x + \mu x^3/3}{1 + (\mu - 1)x^2/3 + \mu^2 x^4/9}.$$
 (1.17)

We prove that any root \tilde{x}_m of Eq. (1.17) defines the eigenvalue $\tilde{\lambda}_m$:

$$\tilde{\lambda}_m = \frac{\sqrt{4\mu + 1} + 1}{2R_1} \tilde{x}_m,\tag{1.18}$$

where radius R_1 is arbitrary in accordance with the scale invariance (1.12) and radius R_2 is defined by Eq. (1.16).

Remark 1. Cantarella, DeTurck, Gluck, and Teytel wrote on p. 2767 of [5]:

"Theorem B. For the spherical shell $B^3(a,b)$ of inner radius a and outer radius b, the eigenvalue of curl having least absolute value is $\lambda_1^{(1)}$, where $\lambda_1^{(1)}$ is the smallest of the infinite sequence of positive numbers x_k that satisfy

$$J_{3/2}(ax)Y_{3/2}(bx) - J_{3/2}(bx)Y_{3/2}(ax) = 0.$$
"

Thus the eigenvalues $\lambda_1^{(1)}$ are defined in paper [5] by this equation in terms of Bessel's functions. The scale invariance (1.12) and the dimensionless Eqs. (1.14) and (1.17) were not derived in paper [5].

Let us prove that for any given radii R_1 and R_2 no one eigenvalue λ_n coincides with an eigenvalue $\tilde{\lambda}_m$. Indeed, let λ be the common value of $\lambda_n = \tilde{\lambda}_m$. Then the dimensionless eigenvalue $x = \lambda (R_1 - R_2)$ satisfies both Eqs (1.14) and (1.17). Subtracting Eqs (1.14) and (1.17) and reducing to the common denominator we arrive at equation

$$2\mu^2 x^2 + 9\mu + 3 = 0 \tag{1.19}$$

that has no real solutions. Hence there are no real numbers x satisfying to both Eqs (1.14) and (1.17). Therefore for any R_1 and R_2 the eigenvalues λ_n and $\tilde{\lambda}_m$ are not equal for any

In Section 3, we prove that all eigenvalues λ_k for $R_2 \rightarrow R_1$ have the following asymptotics

$$\lambda_k pprox rac{k\pi}{R_1 - R_2}, \quad \text{as } \mu o \infty.$$

This asymptotics for k = 1 reduces to the special asymptotics (1.11). Equation (1.14) with arbitrary values of parameter $\mu > 0$ (that means for arbitrary R_1 and R_2) yields the following asymptotics for eigenvalues λ_n :

$$\lambda_n pprox rac{n\pi}{R_1 - R_2}$$
, as $n o \infty$.

In Section 4, we prove the following asymptotics for the eigenvalues $\tilde{\lambda}_m$:

$$\tilde{\lambda}_k \approx \frac{k\pi}{R_1 - R_2} \quad \text{as } \mu \to \infty,$$

$$\tilde{\lambda}_m \approx \frac{m\pi}{R_1 - R_2} \quad \text{as } m \to \infty.$$

In Section 3, we present the three-dimensional (3D) space of eigenfields $V_n(x, A)$ in explicit form that yields that any linear combination of the axisymmetric eigenfields $V_n(x, A)$ with arbitrary axes of symmetry is again axisymmetric.

We present in Section 5 in an explicit form the linear five-dimensional (5D) space of eigenfields $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ and prove in Section 6 that the eigenfields $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ with noncollinear constant vectors A, B are not axisymmetric. The corresponding dynamical systems $d\mathbf{x}/dt = \tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ do not have first integrals. A recent numerical investigation [4] proves that some of these systems possess chaotic streamlines and therefore are not integrable.

The special eigenfields $\kappa \tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{A})$ with $\mathbf{B} = \kappa \mathbf{A}$ are axisymmetric. The general linear combinations of the axisymmetric eigenfields $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}_k, \mathbf{A}_k)$ are not axisymmetric. In Section 7, we demonstrate that topologies of the axisymmetric fluid flows $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{A})$ and $\mathbf{V}_n(\mathbf{x}, \mathbf{A})$ are completely different. All eigenfields $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ are presented in explicit form in terms of elementary functions of the Cartesian coordinates x^1, x^2, x^3 .

2 Functions $H_n(\lambda R)$ and their properties

Using the method of Chandrasekhar and Kendall [8], we study vector fields

$$\mathbf{V}(\mathbf{x}) = \operatorname{curl} \mathbf{U}(\mathbf{x}) + \frac{1}{\lambda} \operatorname{curl} \operatorname{curl} \mathbf{U}(\mathbf{x}),$$

where $\mathbf{U}(\mathbf{x})$ satisfies the vector Helmholtz equation

$$\Delta \mathbf{U}(\mathbf{x}) = -\lambda^2 \mathbf{U}(\mathbf{x}). \tag{2.1}$$

In view of the identity $\operatorname{curl} \mathbf{U}(\mathbf{x}) = \operatorname{grad}(\operatorname{div} \mathbf{U}(\mathbf{x}))$ $-\Delta \mathbf{U}(\mathbf{x})$ and Eq. (2.1), we get

$$\mathbf{V}(\mathbf{x}) = \operatorname{curl} \mathbf{U}(\mathbf{x}) + \lambda \mathbf{U}(\mathbf{x}) + \frac{1}{\lambda} \operatorname{grad}(\operatorname{div} \mathbf{U}(\mathbf{x})). \tag{2.2}$$

Applying to V(x) the operator curl and using the identity $\operatorname{curl}(\operatorname{grad} F(\mathbf{x})) = 0$ we find that vector field $\mathbf{V}(\mathbf{x})$ (2.2) satisfies the Beltrami Eq. (1.4).

We consider vector field $\mathbf{U}(\mathbf{x})$ (2.1) of the form $\mathbf{U}(\mathbf{x}) = H_1(\lambda R)\mathbf{A}$, where **A** is an arbitrary vector and function $H_1(\lambda R)$ satisfies the spherically symmetric Helmholtz equation

$$\Delta u = \frac{d^2 u(R)}{dR^2} + \frac{2}{R} \frac{du(R)}{dR} = -\lambda^2 u(R),$$
 (2.3)

variable *R* is the spherical radius $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$. All solutions to the Helmholtz Eq. (2.3) have the form

$$H_1(\lambda R) = H_1(v) = a \frac{\sin(v)}{v} + b \frac{\cos(v)}{v},$$

 $H_1(v) = \frac{1}{v} \frac{dH_0(v)}{dv},$ (2.4)

where $v = \lambda R$, and a, b are arbitrary constants. Here, function $H_0(v)$ is

$$H_0(\lambda R) = H_0(v) = -a \cos v + b \sin v.$$
 (2.5)

¹ Applications of the exact solution $(\sin R)/R$ to the atomic bomb physics were presented by Heisenberg in his lecture [15], see pp. 195-196.

In this paper, we will use elementary functions

$$H_2(v) = \frac{1}{v} \frac{dH_1(v)}{dv} = \frac{a}{v^2} \left[\cos v - \frac{\sin v}{v} \right]$$
$$-\frac{b}{v^2} \left[\sin v + \frac{\cos v}{v} \right], \qquad (2.6)$$

$$H_3(v) = \frac{1}{v} \frac{dH_2(v)}{dv} = \frac{a}{v^4} \left[(3 - v^2) \frac{\sin v}{v} - 3 \cos v \right] + \frac{b}{v^4} \left[3 \sin v + (3 - v^2) \frac{\cos v}{v} \right], \tag{2.7}$$

$$\begin{split} H_4(v) &= \frac{1}{v} \frac{dH_3(v)}{dv} \\ &= \frac{a}{v^6} \left[(6v^2 - 15) \frac{\sin v}{v} - (v^2 - 15) \cos v \right] \\ &+ \frac{b}{v^6} \left[(v^2 - 15) \sin v + (6v^2 - 15) \frac{\cos v}{v} \right]. \end{split}$$

Let us define by induction the functions

$$H_{n+1}(v) = \frac{1}{v} \frac{dH_n(v)}{dv}.$$
 (2.8)

Lemma 1. Functions $H_0(v), H_1(v), \dots, H_n(v), \dots$ with arbitrary constants a and b satisfy an infinite series of identities

$$H_n(v) + (2n+1)H_{n+1}(v) + v^2H_{n+2}(v) = 0.$$
 (2.9)

Proof. It is easy to verify using formulas (2.4), (2.5), and (2.6) the identity (2.9) for n = 0:

$$H_0(v) + H_1(v) + v^2 H_2(v) = 0.$$

Assume that identity (2.9) is true for an integer $n \ge 0$ and apply to it the operator $v^{-1}d/dv$. Then using the definitions (2.8) we get

$$H_{n+1}(v) + (2n+1)H_{n+2}(v) + 2H_{n+2}(v) + v^2H_{n+3}(v) = 0,$$

that is the identity (2.9) for n + 1. Hence identities (2.9) are proven by induction for all $n \ge 0$.

We apply identity (2.9) for n = 1 and n = 2 in Sections 3, 4, 5, and 7 of this paper.

3 The first series of eigenvalues λ_n and axisymmetric eigenfields $V_n(x, A)$

Substituting formula $\mathbf{U}(\mathbf{x}) = H_1(v)\mathbf{A}$ into Eq. (2.2) we derive

$$\mathbf{V}(\mathbf{x}, \mathbf{A}) = \operatorname{grad} H_1(v) \times \mathbf{A} + \lambda H_1(v) \mathbf{A}$$

$$+ \frac{1}{\lambda} \operatorname{grad} \left[\operatorname{grad} H_1(v) \cdot \mathbf{A} \right]$$

$$= \lambda^2 \frac{1}{v} \frac{dH_1(v)}{dv} \mathbf{x} \times \mathbf{A} + \lambda \left[H_1(v) + \frac{1}{v} \frac{dH_1(v)}{dv} \right] \mathbf{A}$$

$$+ \lambda^3 \frac{1}{v} \frac{d}{dv} \left(\frac{1}{v} \frac{dH_1(v)}{dv} \right) (\mathbf{x} \cdot \mathbf{A}) \mathbf{x}.$$
(3.1)

Remark 2. Vector field (3.1) evidently is invariant with respect to rotations around the axis having direction A. Vector fields (3.1) for any vector **A** and for $\lambda = 1$ and $H_1(v) = (\sin R)/R$ (that corresponds to b = 0) are equivalent to the well-known in plasma physics spheromak exact solution [1, 7, 13, 14].

Using formulas (2.6) and (2.7) we represent vector field V(x, A) (3.1) in the form

$$\mathbf{V}(\mathbf{x}, \mathbf{A}) = \lambda^2 H_2(v) \mathbf{x} \times \mathbf{A} + \lambda \left[H_1(v) + H_2(v) \right] \mathbf{A}$$
$$+ \lambda^3 H_2(v) (\mathbf{x} \cdot \mathbf{A}) \mathbf{x}.$$
(3.2)

Hence we get

$$\mathbf{x} \cdot \mathbf{V}(\mathbf{x}, \mathbf{A})$$

$$= \lambda \left[H_1(v) + H_2(v) \right] (\mathbf{x} \cdot \mathbf{A}) + \lambda^3 H_3(v) (\mathbf{x} \cdot \mathbf{x}) (\mathbf{x} \cdot \mathbf{A})$$

$$= \lambda \left[H_1(v) + H_2(v) + v^2 H_3(v) \right] (\mathbf{x} \cdot \mathbf{A}), \tag{3.3}$$

where we put $\lambda^2(\mathbf{x} \cdot \mathbf{x}) = \lambda^2 R^2 = v^2$. Identity (2.9) for n = 1has the form

$$H_1(v) + 3H_2(v) + v^2H_3(v) = 0.$$
 (3.4)

In view of identity (3.4), Eq. (3.3) becomes

$$\mathbf{x} \cdot \mathbf{V}(\mathbf{x}, \mathbf{A}) = -2\lambda H_2(v)(\mathbf{x} \cdot \mathbf{A}). \tag{3.5}$$

Therefore the two nonpenetration boundary conditions (1.10) are satisfied for the fluid flow V(x, A) (3.2) if and only if

$$H_2(v_1) = 0$$
, $H_2(v_2) = 0$, $v_1 = \lambda R_1$, $v_2 = \lambda R_2$. (3.6)

Multiplying Eq. (2.6) with $v^2/(a\cos v)$ we find

$$\frac{v^2}{a\cos v}H_2(v) = 1 - \frac{\tan v}{v} - \frac{b}{a}\left(\frac{1}{v} + \tan v\right).$$

Hence equation $H_2(v) = 0$ takes the form

$$\frac{b}{a} = \frac{v - \tan v}{1 + v \tan v}.\tag{3.7}$$

Therefore Eqs (3.6) and (3.7) yield

$$\frac{b}{a} = \frac{v_1 - \tan v_1}{1 + v_1 \tan v_1} = \frac{v_2 - \tan v_2}{1 + v_2 \tan v_2}.$$
 (3.8)

The second equality implies

$$v_1 + v_1 v_2 \tan v_2 - \tan v_1 - v_2 \tan v_1 \tan v_2$$

= $v_2 + v_1 v_2 \tan v_1 - \tan v_2 - v_1 \tan v_1 \tan v_2$.

Collecting the similar terms we get

$$(v_1 - v_2)(1 + \tan v_1 \tan v_2)$$

= $(1 + v_1v_2)(\tan v_1 - \tan v_2)$.

Hence we find

$$\frac{\tan v_1 - \tan v_2}{1 + \tan v_1 \tan v_2} = \frac{v_1 - v_2}{1 + v_1 v_2}.$$
 (3.9)

Substituting $v_1 = \lambda R_1$ and $v_2 = \lambda R_2$ into Eq. (3.9) and using the trigonometric identity

$$\frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \tan(\alpha - \beta) \tag{3.10}$$

we get from Eq. (3.9) the equation for the eigenvalues λ :

$$\tan\left[\lambda(R_1 - R_2)\right] = \frac{\lambda(R_1 - R_2)}{1 + \lambda^2 R_1 R_2}.$$
 (3.11)

Substituting $v_1 = \lambda R_1$ and $v_2 = \lambda R_2$ into Eq. (3.8) we get

$$\frac{b}{a} = \frac{\lambda R_1 - \tan(\lambda R_1)}{1 + \lambda R_1 \tan(\lambda R_1)} = \frac{\lambda R_2 - \tan(\lambda R_2)}{1 + \lambda R_2 \tan(\lambda R_2)}.$$
 (3.12)

Hence function $H_1(\lambda R)$ (2.4) has the form

$$H_1(\lambda R) = a \left(\frac{\sin(\lambda R)}{\lambda R} + \frac{\lambda R_1 - \tan(\lambda R_1)}{1 + \lambda R_1 \tan(\lambda R_1)} \frac{\cos(\lambda R)}{\lambda R} \right),$$
(3.13)

where *a* is an arbitrary parameter.

Let us express the dimensionless quantities λR_1 and λR_2 in terms of variables (1.13). We have

$$\lambda R_1 = \alpha_1 x, \quad \alpha_1 = \frac{R_1}{R_1 - R_2},$$

$$\lambda R_2 = \alpha_2 x, \quad \alpha_2 = \frac{R_2}{R_1 - R_2}.$$
(3.14)

Evidently $\alpha_1 - \alpha_2 = 1$ and $\alpha_1 \alpha_2 = \mu$. Hence α_1 satisfies equation $\alpha_1^2 - \alpha_1 - \mu = 0$. Therefore by Vieta formula we

$$\alpha_1 = \frac{1}{2} \left(\sqrt{4\mu + 1} + 1 \right), \quad \alpha_2 = \frac{1}{2} \left(\sqrt{4\mu + 1} - 1 \right).$$
 (3.15)

Equations (3.14) and (3.15) yield

$$\lambda = \frac{\alpha_1 x}{R_1} = \frac{\sqrt{4\mu + 1} + 1}{2R_1} x. \tag{3.16}$$

Another consequence of Eqs (3.14) and (3.15) is

$$\lambda R_1 = \frac{x}{2} \left(\sqrt{4\mu + 1} + 1 \right), \quad \lambda R_2 = \frac{x}{2} \left(\sqrt{4\mu + 1} - 1 \right).$$
 (3.17)

Substituting formulas (3.17) into Eq. (3.13) we get

$$H_1(\lambda R) = a \left[\frac{\sin(\lambda R)}{\lambda R} + \frac{\alpha_1 x - \tan(\alpha_1 x)}{1 + \alpha_1 x \tan(\alpha_1 x)} \frac{\cos(\lambda R)}{\lambda R} \right].$$
(3.18)

Equation (3.11) evidently is invariant with respect to the group of scale transformations (1.12). In the dimensionless variables $x = \lambda (R_1 - R_2)$ and $\mu = R_1 R_2 / (R_1 - R_2)^2$ (1.13) Eq. (3.11) takes the form

$$\tan(x) = \frac{x}{1 + \mu x^2}.$$
 (3.19)

Any root x_n of Eq. (3.19) defines the eigenvalue λ_n according to formula (3.16):

$$\lambda_n = \frac{\sqrt{4\mu + 1} + 1}{2R_1} x_n. \tag{3.20}$$

Here, radius R_1 is arbitrary in accordance with the scale invariance (1.12). Equation (3.17) implies that radius R_2 is connected with R_1 by relation

$$R_2 = \frac{\sqrt{4\mu + 1} - 1}{\sqrt{4\mu + 1} + 1} R_1. \tag{3.21}$$

Formula (3.21) agrees with the definition (1.13) of parameter $\mu = R_1 R_2 / (R_1 - R_2)^2$.

Since function tan(x) is π -periodic and $tan(x) \to \pm \infty$ as $x \to (n \pm 1/2)\pi$ we see that Eq. (3.19) for any constant $\mu > 0$ has infinitely many roots x_n that satisfy the inequalities $n\pi < x_n < n + 1/2$) π . The maximal value of function $f(x) = x/(1 + \mu x^2)$ is $1/(2\sqrt{\mu})$ that is attained at $x_* =$ $1/\sqrt{\mu}$. Therefore at $\mu \gg 1$ function $f(x) \le 1/(2\sqrt{\mu}) \ll 1$ for all x. Hence solutions x_k of Eq. (3.19) are close to the roots of function tan(x) that means to $k\pi$. Hence we get the asymptotics at $\mu \gg 1$ or at $R_2 \rightarrow R_1$:

$$\lambda_k \approx \frac{k\pi}{R_1 - R_2}, \quad \text{as } R_2 \to R_1.$$
 (3.22)

Since the right hand side of Eq. (3.19) tends to zero as $x \to \infty$ we get that Eq. (3.19) with arbitrary value of parameter $\mu > 0$ yields the following asymptotics for eigenvalues λ_n as $n \to \infty$:

$$\lambda_n \approx \frac{n\pi}{R_1 - R_2}. (3.23)$$

Example 1. Let $R_1/R_2 = 10$. Then parameter $\mu = 10/81$ = 0.1234568. For any root x_k to Eq. (3.19) the corresponding eigenvalue $\lambda_k = x_k/(R_1 - R_2) = x_k/(9R_2)$. The first five roots of Eq. (3.19) are:

$$x_1 = 4.070059$$
, $x_2 = 7.061906$, $x_3 = 10.065190$, $x_4 = 13.099750$, $x_5 = 16.160470$. (3.24)

Example 2. Let $R_1/R_2 = 2$. Then parameter $\mu = 2$ and any root x_k to Eq. (3.19) defines an eigenvalue $\lambda_k = x_k/(R_1$ $-R_2$) = x_k/R_2 . The first five roots to Eq. (3.19) are:

$$x_1 = 3.286005$$
, $x_2 = 6.360678$, $x_3 = 9.477196$, $x_4 = 12.605920$, $x_5 = 15.739670$. (3.25)

Comparing these sequences with the sequence of numbers $z_k = k\pi$:

$$z_1 = 3.141593$$
, $z_2 = 6.283185$, $z_3 = 9.424778$, $z_4 = 12.56637$, $z_5 = 15.70796$, (3.26)

we observe that the roots x_k (3.25) corresponding to $\mu_2 = 2$ tend to the sequence $z_k = k\pi$ faster than the roots (3.24) corresponding to $\mu_1 = 0.1234568$.

Let x_n be a root of Eq. (3.19). Substituting $x = x_n$ into formula (3.18) we obtain function

$$H_{1,n}(\lambda_n R) = a \left[\frac{\sin(\lambda_n R)}{\lambda_n R} + \frac{\alpha_1 x_n - \tan(\alpha_1 x_n)}{1 + \alpha_1 x_n \tan(\alpha_1 x_n)} \frac{\cos(\lambda_n R)}{\lambda_n R} \right], \quad (3.27)$$

corresponding to the eigenvalue $\lambda_n = x_n/(R_1 - R_2)$. Vector field $\mathbf{V}_n(\mathbf{x}, \mathbf{A})$ (3.1) and (3.2) corresponding to function $H_{1n}(\lambda_n R)$ (3.27) satisfies the Beltrami Eq. (1.4) with $\lambda = \lambda_n$ and the two nonpenetration boundary conditions (1.10). Therefore it is one of the corresponding eigenfields.

Remark 3. Any linear combination of eigenfields $\mathbf{V}_n(\mathbf{x}, \mathbf{A}_k)$ (3.2) depending on arbitrary vectors \mathbf{A}_k has the form

$$\sum_{k=1}^{N} c_k \mathbf{V}_n(\mathbf{x}, \mathbf{A}_k) = \mathbf{V}_n \left(\mathbf{x}, \sum_{k=1}^{N} c_k \mathbf{A}_k \right), \tag{3.28}$$

due to the linearity of formula (3.2) with respect to vector **A.** Hence an arbitrary linear combination (3.28) is another Beltrami vector field (3.2). Therefore for any eigenvalue λ_n the linear space of the corresponding eigenfields (3.2) has dimension 3. Hence each eigenvalue $\lambda_n = x_n/(R_1 - R_2)$ where x_n is a root of Eq. (3.19) has multiplicity 3.

In view of formula (3.2), the eigenfield (3.28) is invariant under rotations around vector $\sum_{k=1}^{N} c_k \mathbf{A}_k$. Formula (3.28) provides an explanation why any linear combination of the axisymmetric eigenfields $\mathbf{V}_n(\mathbf{x}, \mathbf{A}_k)$ is also axisymmetric. For the second series of eigenfields in Section 4, this is not so.

4 The second series of eigenvalues $\tilde{\lambda}_m$ and eigenfields $\tilde{V}_m(x,A,B)$

For any λ vector field **V**(**x**, **A**) (3.1), (3.2) satisfies Beltrami Eq. (1.4). Hence its derivative $\nabla_{\mathbf{R}}\mathbf{V}(\mathbf{x},\mathbf{A})$ in direction of any vector **B** also satisfies Eq. (1.4). By definition we have

$$\nabla_{\mathbf{B}} \mathbf{V}(\mathbf{x}, \mathbf{A}) = \sum_{i=1}^{3} \frac{\partial \mathbf{V}(\mathbf{x}, \mathbf{A})}{\partial x_{i}} B_{i}. \tag{4.1}$$

For functions $H_{\nu}(v)$ we find using the equality $v = \lambda R$ and definition (2.8):

$$\nabla_{\mathbf{B}} H_k(v) = \sum_{i=1}^3 \frac{dH_k(v)}{dv} \frac{\partial v}{\partial x_i} B_i = \frac{dH_k(v)}{dv} \frac{\lambda}{R} (\mathbf{x} \cdot \mathbf{B})$$

$$= \frac{\lambda^2}{v} \frac{dH_k(v)}{dv} (\mathbf{x} \cdot \mathbf{B}) = \lambda^2 H_{k+1}(v) (\mathbf{x} \cdot \mathbf{B}).$$
(4.2)

Applying formulas (4.1) and (4.2) to vector field (3.2) we find

$$\nabla_{\mathbf{B}}\mathbf{V}(\mathbf{x}, \mathbf{A}) = \lambda^{2}H_{2}(v)\mathbf{B} \times \mathbf{A} + \lambda^{4}H_{3}(v)(\mathbf{x} \cdot \mathbf{B})\mathbf{x} \times \mathbf{A}$$

$$+ \lambda^{3} \left[H_{2}(v) + H_{3}(v)\right](\mathbf{x} \cdot \mathbf{B})\mathbf{A}$$

$$+ \lambda^{5}H_{4}(v)(\mathbf{x} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{B})\mathbf{x}$$

$$+ \lambda^{3}H_{3}(v)\left[(\mathbf{A} \cdot \mathbf{B})\mathbf{x} + (\mathbf{x} \cdot \mathbf{A})\mathbf{B}\right]. \tag{4.3}$$

Permuting here vectors **A** and **B** we get vector field

$$\nabla_{\mathbf{A}}\mathbf{V}(\mathbf{x}, \mathbf{B}) = \lambda^{2}H_{2}(v)\mathbf{A} \times \mathbf{B} + \lambda^{4}H_{3}(v)(\mathbf{x} \cdot \mathbf{A})\mathbf{x} \times \mathbf{B}$$

$$+ \lambda^{3} \left[H_{2}(v) + H_{3}(v)\right](\mathbf{x} \cdot \mathbf{A})\mathbf{B}$$

$$+ \lambda^{5}H_{4}(v)(\mathbf{x} \cdot \mathbf{B})(\mathbf{x} \cdot \mathbf{A})\mathbf{x}$$

$$+ \lambda^{3}H_{3}(v)\left[(\mathbf{B} \cdot \mathbf{A})\mathbf{x} + (\mathbf{x} \cdot \mathbf{B})\mathbf{A}\right]. \tag{4.4}$$

Adding formulas (4.3) and (4.4), we get Beltrami field that is symmetric with respect to vectors **A** and **B**:

$$\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B}) = \frac{1}{2\lambda^{3}} \left[\nabla_{\mathbf{A}} \mathbf{V}(\mathbf{x}, \mathbf{B}) + \nabla_{\mathbf{B}} \mathbf{V}(\mathbf{x}, \mathbf{A}) \right]$$

$$= \frac{1}{2} \lambda H_{3}(v) \left[(\mathbf{x} \cdot \mathbf{B}) \mathbf{x} \times \mathbf{A} + (\mathbf{x} \cdot \mathbf{A}) \mathbf{x} \times \mathbf{B} \right]$$

$$+ \frac{1}{2} \left[H_{2}(v) + H_{3}(v) \right] \left[(\mathbf{x} \cdot \mathbf{A}) \mathbf{B} + (\mathbf{x} \cdot \mathbf{B}) \mathbf{A} \right]$$

$$+ \lambda^{2} H_{4}(v) (\mathbf{x} \cdot \mathbf{A}) (\mathbf{x} \cdot \mathbf{B}) \mathbf{x}$$

$$+ \frac{1}{2} H_{3}(v) \left[2(\mathbf{A} \cdot \mathbf{B}) \mathbf{x} + (\mathbf{x} \cdot \mathbf{B}) \mathbf{A} + (\mathbf{x} \cdot \mathbf{A}) \mathbf{B} \right].$$

$$(4.5)$$

Formula (4.5) and $\lambda^2(\mathbf{x} \cdot \mathbf{x}) = \lambda^2 R^2 = v^2$ yield

$$\mathbf{x} \cdot \tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B})$$

$$= [H_2(v) + H_3(v)] (\mathbf{x} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{B}) + v^2 H_4(v)(\mathbf{x} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{B})$$

$$+ (\mathbf{x} \cdot \mathbf{x})H_3(v)(\mathbf{A} \cdot \mathbf{B}) + H_3(v)(\mathbf{x} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{B})$$

$$= H_3(v)(\mathbf{A} \cdot \mathbf{B})(\mathbf{x} \cdot \mathbf{x})$$

$$+ [H_3(v) + 2H_3(v) + v^2H_b(v)] (\mathbf{x} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{B}). \tag{4.6}$$

Identity (2.9) for n = 2 takes the form

$$H_2(v) + 5H_3(v) + v^2H_4(v) = 0.$$
 (4.7)

Equation (4.6) in view of identity (4.7) becomes

$$\mathbf{x} \cdot \tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B}) = H_3(v) \left[(\mathbf{A} \cdot \mathbf{B})(\mathbf{x} \cdot \mathbf{x}) - 3(\mathbf{x} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{B}) \right].$$
(4.8)

Remark 4. Equation (4.8) implies that on each sphere \mathbb{S}^2_* satisfying equation $R = R_*$, where $H_3(v_*) = H_3(\lambda R_*) = 0$ we have $\mathbf{x} \cdot \tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B}) = 0$. That means vector field $\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B})$ is tangent to the spheres \mathbb{S}^2_* : $R = R_*$. Hence the spheres \mathbb{S}^2_* are invariant submanifolds for the flows $\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B})$ (4.5) with arbitrary vectors \mathbf{A} and \mathbf{B} . Therefore the two nonpenetration boundary conditions (1.10) are satisfied for the fluid flows $\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B})$ (4.5) if and only if

$$H_3(v_1) = 0$$
, $H_3(v_2) = 0$, $v_1 = \lambda R_1$, $v_2 = \lambda R_2$. (4.9)

Multiplying function $H_3(v)$ (2.7) with $v^4/(a\cos v)$ we get

$$\frac{v^4}{a\cos v}H_3(v)$$
= $(3 - v^2)\frac{\tan v}{v} - 3 + \frac{b}{a}\left(3\tan v + \frac{3 - v^2}{v}\right)$.

Hence equation $H_3(v) = 0$ is equivalent to

$$\frac{b}{a} = \frac{3v - (3 - v^2)\tan v}{3v\tan v + 3 - v^2}.$$
 (4.10)

Since b/a is const, the two nonpenetration conditions (4.9) and Eq. (4.10) imply the equality

$$\frac{b}{a} = \frac{3v_1 - (3 - v_1^2)\tan v_1}{3v_1 \tan v_1 + 3 - v_1^2} = \frac{3v_2 - (3 - v_2^2)\tan v_2}{3v_2 \tan v_2 + 3 - v_2^2}, (4.11)$$

where $v_1 = \lambda R_1$ and $v_2 = \lambda R_2$. Equation (4.11) leads to

$$\begin{split} 9v_1v_2 & \tan v_2 + 3(3 - v_2^2)v_1 - 3v_2(3 - v_1^2)\tan v_1 \tan v_2 \\ & - (3 - v_1^2)(3 - v_2^2)\tan v_1 \\ &= 9v_1v_2 \tan v_1 + 3(3 - v_1^2)v_2 - 3v_1(3 - v_2^2)\tan v_1 \tan v_2 \\ & - (3 - v_1^2)(3 - v_2^2)\tan v_2. \end{split}$$

Collecting here similar terms we find

$$3(v_1 - v_2)(3 + v_1v_2) [1 + \tan v_1 \tan v_2]$$

= $[9v_1v_2 + (3 - v_1^2)(3 - v_2^2)] (\tan v_1 - \tan v_2).$

This equation evidently implies

$$\frac{\tan v_1 - \tan v_2}{1 + \tan v_1 \tan v_2} = \frac{3(v_1 - v_2)(3 + v_1 v_2)}{9v_1 v_2 + (3 - v_1^2)(3 - v_2^2)}$$

Applying here the trigonometric identity (3.10) and substituting $v_1 = \lambda R_1$ and $v_2 = \lambda R_2$ we get the equation for the eigenvalues λ :

$$\tan \left[\lambda (R_1 - R_2) \right]$$

$$= \frac{3\lambda (R_1 - R_2)(3 + \lambda^2 R_1 R_2)}{9\lambda^2 R_1 R_2 + 9 - 3\lambda^2 (R_1^2 + R_2^2) + \lambda^4 R_1^2 R_2^2}.$$
 (4.12)

Equation (4.12) evidently is invariant with respect to the scale transformations (1.12). Using formulas (3.17) it is easy to verify that in the dimensionless variables x, μ (1.13) Eq. (4.12) takes the form

$$\tan(x) = \frac{x + \mu x^3/3}{1 + (\mu - 1)x^2/3 + \mu^2 x^4/9}.$$
 (4.13)

Due to Eq. (3.16), any root \tilde{x}_m of Eq. (4.13) defines the eigenvalue $\tilde{\lambda}_m$:

$$\tilde{\lambda}_m = \frac{\alpha_1 \tilde{x}_m}{R_1} = \frac{\sqrt{4 \mu + 1} + 1}{2R_1} \tilde{x}_m. \tag{4.14}$$

Here, radius R_1 is arbitrary in agreement with the scale invariance (1.12). For any fixed μ (1.13) radius R_2 is connected with R_1 by Eq. (1.16).

Example 3. Let $R_1/R_2 = (3 + \sqrt{5})/2 = 2.618034$. Then $\mu = 1$ and any root \tilde{x}_m to Eq. (4.13) defines an eigenvalue $\tilde{\lambda}_m = \tilde{x}_m/(R_1 - R_2) = 2\tilde{x}_m/[(1 + \sqrt{5})R_2] = \tilde{x}_m/(1.618034R_2)$. The first five roots to Eq. (4.13) with $\mu = 1$ are:

$$\tilde{x}_1 = 3.871221, \quad \tilde{x}_2 = 6.725464, \quad \tilde{x}_3 = 9.732386,$$

$$\tilde{x}_4 = 12.800570, \quad \tilde{x}_5 = 15.896620. \tag{4.15}$$

It is evident that sequence (4.15) is an approximation of sequence $z_k = k\pi$ (3.26).

Since $\tan(x)$ is periodic with period π , $\tan(k\pi) = 0$ and $\tan(x) \to \pm \infty$ as $x \to (k \pm 1/2)\pi$ we get that Eq. (4.13) has infinitely many roots. The function $f_2(x)$ in the right hand side of Eq. (4.13) tends to zero when either $\mu \to \infty$ (that means $R_2 \to R_1$) or when $x \to \infty$. Hence we get the asymptotics

$$\tilde{\lambda}_k \approx \frac{k\pi}{R_1 - R_2} \quad \text{as } R_2 \to R_1,$$

$$\tilde{\lambda}_m \approx \frac{m\pi}{R_1 - R_2} \quad \text{as } m \to \infty.$$
(4.16)

Let $\tilde{\lambda}_m$ be a root of Eq. (4.12). Then from Eq. (4.14), we get $\tilde{\lambda}_m R_1 = \alpha_1 \tilde{x}_m$ where \tilde{x}_m satisfies Eq. (4.13) Substituting v_1 $= \tilde{\lambda}_m R_1 = \alpha_1 \tilde{x}_m$ into the first Eq. (4.11) we find the constant

$$\frac{b}{a} = \frac{3\alpha_1 \tilde{x}_m - (3 - (\alpha_1 \tilde{x}_m)^2) \tan(\alpha_1 \tilde{x}_m)}{3\alpha_1 \tilde{x}_m \tan(\alpha_1 \tilde{x}_m) + 3 - (\alpha_1 \tilde{x}_m)^2}.$$
 (4.17)

Therefore the corresponding function $H_1(\lambda R)$ (2.4) takes the form

$$H_{1,m}(v) = a \left[\frac{\sin(v)}{v} + \left(\frac{3\alpha_1 \tilde{x}_m - (3 - (\alpha_1 \tilde{x}_m)^2)}{\tan(\alpha_1 \tilde{x}_m)} \frac{\cos(v)}{v} + 3 - (\alpha_1 \tilde{x}_m)^2 \right) \right],$$

$$(4.18)$$

where a is an arbitrary constant. Vector fields $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ (4.5) with $\lambda = \tilde{\lambda}_m$ and constant b/a (4.17) and corresponding to the function $H_{1,m}(v)$ (4.18) satisfy the Beltrami Eq. (1.4) with $\lambda = \tilde{\lambda}_m$ in the spherical shell $S_{R_2R_1}$ (1.9) and the two nonpenetration boundary conditions (1.10) on the spheres $R = R_1$ and $R = R_2$.

Vector fields $\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B})$ (4.5) are bilinear symmetric functions of vectors A and B because of equality $\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{B}) = \tilde{\mathbf{V}}(\mathbf{x}, \mathbf{B}, \mathbf{A})$. Expanding vectors **A** and **B** in an orthonormal basis $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ we find that any eigenfield $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ is a linear combination of six eigenfields $\tilde{\mathbf{V}}_m(\mathbf{x},\hat{\mathbf{e}}_i,\hat{\mathbf{e}}_i) = \tilde{\mathbf{V}}_m(\mathbf{x},\hat{\mathbf{e}}_i,\hat{\mathbf{e}}_i)$. However, in view of identity (5.1) proved in Section 5 below only five of these eigenfields are linearly independent. Therefore for any eigenvalue $\tilde{\lambda}_m$ the space of eigenfields $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ has dimension 5. Hence the multiplicity of each eigenvalue $\tilde{\lambda}_m$ is equal to 5.

5 An identity for axisymmetric eigenfields

Vector field $\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{A})$ (4.5) with $\mathbf{B} = \mathbf{A}$ has the form

$$\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{A}) = \lambda H_3(v) \left[(\mathbf{x} \cdot \mathbf{A}) \mathbf{x} \times \mathbf{A} \right] + \lambda^2 H_4(v) (\mathbf{x} \cdot \mathbf{A})^2 \mathbf{x}$$

$$+ \left[H_2(v) + H_3(v) \right] (\mathbf{x} \cdot \mathbf{A}) \mathbf{A} + H_3(v) \left[(\mathbf{A} \cdot \mathbf{A}) \mathbf{x} + (\mathbf{x} \cdot \mathbf{A}) \mathbf{A} \right] .$$
(5.1)

Vector field $\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}, \mathbf{A})$ (5.1) evidently is invariant under rotations around the axis generated by vector A. The corresponding fluid flow is integrable and its streamlines belong to certain axisymmetric tori \mathbb{T}^2 , see Section 7.

Lemma 2. Let vectors A_1 , A_2 , and A_3 are mutually orthogo*nal:* $(\mathbf{A}_i \cdot \mathbf{A}_i) = 0$ and $i \neq j$. Then the identity holds:

$$\frac{1}{|\mathbf{A}_1|^2} \tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}_1, \mathbf{A}_1) + \frac{1}{|\mathbf{A}_2|^2} \tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}_2, \mathbf{A}_2)$$
$$+ \frac{1}{|\mathbf{A}_3|^2} \tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}_3, \mathbf{A}_3) = 0. \tag{5.2}$$

Proof. Vector fields $\hat{\mathbf{e}}_k = \mathbf{A}_k/|\mathbf{A}_k|$ have unit lengths and are mutually orthogonal. Evidently we have $\tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}_k, \mathbf{A}_k)/|\mathbf{A}_k|^2 = \tilde{\mathbf{V}}(\mathbf{x}, \hat{\mathbf{e}}_k, \hat{\mathbf{e}}_k)$. Vector field $\tilde{\mathbf{V}}(\mathbf{x}, \hat{\mathbf{e}}_k, \hat{\mathbf{e}}_k)$ (4.5) has the form

$$\tilde{\mathbf{V}}(\mathbf{x}, \hat{\mathbf{e}}_k, \hat{\mathbf{e}}_k) = \lambda H_3(v) \left[\mathbf{x} \times ((\mathbf{x} \cdot \hat{\mathbf{e}}_k) \hat{\mathbf{e}}_k) \right]
+ \left[H_2(v) + H_3(v) \right] \left[(\mathbf{x} \cdot \hat{\mathbf{e}}_k) \hat{\mathbf{e}}_k \right]
+ \lambda^2 H_4(v) (\mathbf{x} \cdot \hat{\mathbf{e}}_k)^2 \mathbf{x} + H_3(v)
\times \left[(\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_k) \mathbf{x} + (\mathbf{x} \cdot \hat{\mathbf{e}}_k) \hat{\mathbf{e}}_k \right].$$
(5.3)

The following identities evidently hold in the basis $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$:

$$\sum_{k=1}^{3} (\mathbf{x} \cdot \hat{\mathbf{e}}_k) \hat{\mathbf{e}}_k = \mathbf{x}, \quad \sum_{k=1}^{3} (\mathbf{x} \cdot \hat{\mathbf{e}}_k)^2 = |\mathbf{x}|^2 = R^2,$$
$$\sum_{k=1}^{3} (\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_k) = 3.$$

Hence using Eq. (5.3) and equation $\lambda^2 R^2 = v^2$ we find

$$\sum_{k=1}^{3} \tilde{\mathbf{V}}(\mathbf{x}, \hat{\mathbf{e}}_{k}, \hat{\mathbf{e}}_{k})$$

$$= \lambda H_{3}(v)[\mathbf{x} \times \mathbf{x}] + [H_{2}(v) + 5H_{3}(v) + v^{2}H_{4}(v)] \mathbf{x}.$$

This expression vanishes due to the identity $\mathbf{x} \times \mathbf{x} = 0$ and identity (4.7). Hence we get

$$\sum_{k=1}^{3} \frac{1}{|\mathbf{A}_k|^2} \tilde{\mathbf{V}}(\mathbf{x}, \mathbf{A}_k, \mathbf{A}_k) = \sum_{k=1}^{3} \tilde{\mathbf{V}}(\mathbf{x}, \hat{\mathbf{e}}_k, \hat{\mathbf{e}}_k) = 0,$$

that proves the identity (5.2).

Remark 5. An arbitrary linear combination of two axisymmetric eigenfields $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}_1, \mathbf{A}_1)$ and $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}_2, \mathbf{A}_2)$ is not axisymmetric. However, if vectors \mathbf{A}_1 and \mathbf{A}_2 are orthogonal and have equal lengths $A = |\mathbf{A}_1| = |\mathbf{A}_2|$ then using identity (5.2) of Lemma 2 we get

$$\tilde{\mathbf{V}}_{m}(\mathbf{x}, \mathbf{A}_{1}, \mathbf{A}_{1}) + \tilde{\mathbf{V}}_{m}(\mathbf{x}, \mathbf{A}_{2}, \mathbf{A}_{2}) = -\tilde{\mathbf{V}}_{m}(\mathbf{x}, \mathbf{A}_{3}, \mathbf{A}_{3}),$$
 (5.4)

where $A_3 = (A_1 \times A_2)/A$, $|A_3| = A$. Equations (5.4), (5.1) imply that for this case the eigenfield $\tilde{\mathbf{V}}_{m}(\mathbf{x}, \mathbf{A}_{1}, \mathbf{A}_{1})$ $+\tilde{\mathbf{V}}_m(\mathbf{x},\mathbf{A}_2,\mathbf{A}_2)$ is axisymmetric with respect to the axis generated by vector \mathbf{A}_3 that is orthogonal to both vectors \mathbf{A}_1 and \mathbf{A}_2 .

6 Eigenfields $\tilde{V}_m(x, A, B)$ with any noncollinear vectors A, B are not axisymmetric

Fluid streamlines defined by an eigenfield $\tilde{\mathbf{V}}_{m}(\mathbf{x}, \mathbf{A}, \mathbf{B})$ are trajectories of the dynamical system

$$\frac{d\mathbf{x}(t)}{dt} = \tilde{\mathbf{V}}_m(\mathbf{x}(t), \mathbf{A}, \mathbf{B}). \tag{6.1}$$

Theorem 1. For any noncollinear vectors A and B vector field $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ and dynamical system (6.1) are not axisvmmetric.

Proof. On the boundaries of the spherical shell (1.9) \mathbb{S}_{ν}^2 : $R = R_k$ and $v = v_k = \lambda R_k$ we have $H_3(v_k) = 0$. Hence from identity (4.7) we get $v_{\nu}^2 H_4(v_k) = -H_2(v_k)$. Note that $H_2(v_k) \neq 0$ because if both $H_3(v_k) = 0$ and $H_2(v_k) = 0$ then from identities (2.9) for n = 1 and n = 0 we get $H_1(v_k) = 0$ and $H_0(v_k) = 0$ that implies $a^2 + b^2 = 0$. Therefore in view of Eq. (4.5) dynamical system (6.1) at $H_3(v_k) = 0$ takes the

$$\frac{d\mathbf{x}}{dt} = \frac{1}{2}H_2(v_k) \left[(\mathbf{x} \cdot \mathbf{A})\mathbf{B} + (\mathbf{x} \cdot \mathbf{B})\mathbf{A} - \frac{2}{R_k^2}(\mathbf{x} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{B})\mathbf{x} \right].$$
(6.2)

System (6.2) evidently has critical points

$$\mathbf{x}_{k\pm} = \pm \frac{R_k}{|\mathbf{A} \times \mathbf{B}|} \mathbf{A} \times \mathbf{B}. \tag{6.3}$$

Note that $\mathbf{A} \times \mathbf{B} \neq 0$ because vectors **A** and **B** are noncollinear. The critical points (6.3) are intersections of the sphere \mathbb{S}_k^2 with the axis

$$L: \mathbf{x}(t) = r(t)(\mathbf{A} \times \mathbf{B}). \tag{6.4}$$

Substituting $\mathbf{x} = r(\mathbf{A} \times \mathbf{B})$ into formula (4.5) and using $((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A}) = 0$ and $((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B}) = 0$ we get

$$\tilde{\mathbf{V}}(r(\mathbf{A} \times \mathbf{B}), \mathbf{A}, \mathbf{B}) = H_3(v)(\mathbf{A} \cdot \mathbf{B})r(\mathbf{A} \times \mathbf{B}).$$

Therefore dynamical system (6.1) on the axis L (6.4) reduces to equation

$$\frac{dr(t)}{dt} = (\mathbf{A} \cdot \mathbf{B})H_3(v(t))r(t), \tag{6.5}$$

where $v(t) = \lambda |r(t)||\mathbf{A} \times \mathbf{B}|$. Equation (6.5) yields that the axis *L* is an invariant submanifold of system (6.1).

Equation (6.5) shows that if $(\mathbf{A} \cdot \mathbf{B}) = 0$ then the axis Lconsists of critical points of the system (6.1). Equation (6.5) defines nontrivial dynamics on the line *L* if $(\mathbf{A} \cdot \mathbf{B}) \neq 0$. In both cases, if system (6.1) is axisymmetric then its axis of symmetry can be only the axis L and then system (6.1) has no isolated critical points outside of the axis L.

However, let us show that system (6.2) has another critical points which belong to the intersection of the sphere \mathbb{S}^2_L with the plane P orthogonal to the axis L and have the form

$$\tilde{\mathbf{x}} = \alpha \mathbf{A} + \beta \mathbf{B}. \tag{6.6}$$

Equation (6.2) at a critical point $\tilde{\mathbf{x}}$ yields

$$(\tilde{\mathbf{x}} \cdot \mathbf{A})\mathbf{B} + (\tilde{\mathbf{x}} \cdot \mathbf{B})\mathbf{A} = \frac{2}{R_{\nu}^2}(\tilde{\mathbf{x}} \cdot \mathbf{A})(\tilde{\mathbf{x}} \cdot \mathbf{B}) [\alpha \mathbf{A} + \beta \mathbf{B}].$$
 (6.7)

Since vectors A and B are linearly independent, we get from Eq. (6.7) two equations

$$2\alpha(\tilde{\mathbf{x}} \cdot \mathbf{A}) = R_{\nu}^{2}, \quad 2\beta(\tilde{\mathbf{x}} \cdot \mathbf{B}) = R_{\nu}^{2}. \tag{6.8}$$

Using expression (6.6) we find

$$\alpha(\tilde{\mathbf{x}} \cdot \mathbf{A}) = \alpha^2 |\mathbf{A}|^2 + \alpha \beta (\mathbf{A} \cdot \mathbf{B}),$$

$$\beta(\tilde{\mathbf{x}} \cdot \mathbf{B}) = \beta^2 |\mathbf{B}|^2 + \alpha \beta (\mathbf{A} \cdot \mathbf{B}),$$
 (6.9)

$$R_{\nu}^{2} = (\tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}}) = \alpha^{2} |\mathbf{A}|^{2} + 2\alpha \beta (\mathbf{A} \cdot \mathbf{B}) + \beta^{2} |\mathbf{B}|^{2}.$$
 (6.10)

Substituting formulas (6.9) and (6.10) we find that each of the two Eq. (6.8) is equivalent to equation

$$\alpha^2 |\mathbf{A}|^2 = \beta^2 |\mathbf{B}|^2$$
.

Hence we get $\beta = \pm \alpha |\mathbf{A}|/|\mathbf{B}|$. Substituting this into Eq. (6.10) we find

$$R_k^2 = \frac{2\alpha^2 |\mathbf{A}|}{|\mathbf{B}|} \left\{ |\mathbf{A}||\mathbf{B}| \pm (\mathbf{A} \cdot \mathbf{B}) \right\}.$$

The bracket here is nonzero due to the Cauchy inequality and the noncollinearity of vectors A and B. Therefore we derive

$$\alpha = \pm \sqrt{\frac{|\mathbf{B}|}{2|\mathbf{A}|}} \frac{R_k}{\sqrt{|\mathbf{A}||\mathbf{B}| \pm (\mathbf{A} \cdot \mathbf{B})}},$$
$$\beta = \pm \sqrt{\frac{|\mathbf{A}|}{2|\mathbf{B}|}} \frac{R_k}{\sqrt{|\mathbf{A}||\mathbf{B}| + (\mathbf{A} \cdot \mathbf{B})}}.$$

Hence we find the critical points (6.6):

$$\tilde{\mathbf{x}} = \frac{R_k}{\sqrt{|\mathbf{A}||\mathbf{B}| + (\mathbf{A} \cdot \mathbf{B})}} \left[\pm \sqrt{\frac{|\mathbf{B}|}{2|\mathbf{A}|}} \mathbf{A} \pm \sqrt{\frac{|\mathbf{A}|}{2|\mathbf{B}|}} \mathbf{B} \right]. \quad (6.11)$$

If system (6.1) were axisymmetric then the orbit \tilde{S}^1 of any critical point $\tilde{\mathbf{x}}$ under the rotations around the axis of symmetry L would be a continuous set of critical points. Each of the four critical points $\tilde{\mathbf{x}}$ (6.11) belong to the plane P orthogonal to the axis of symmetry L. Hence their onedimensional orbits $\tilde{S}^1(\tilde{\mathbf{x}})$ consisting of critical points also would belong to the plane P. However, we have demonstrated above that the critical set of system (6.1) in the plane P is not continuous but consists only of the four isolated critical points $\tilde{\mathbf{x}}$ (6.11). The obtained contradiction proves that the dynamical system (6.1) and eigenfield $\tilde{\mathbf{V}}_{m}(\mathbf{x}, \mathbf{A}, \mathbf{B})$ are not axisymmetric if vectors \mathbf{A} and \mathbf{B} are noncollinear.

Remark 6. For any collinear vectors **A** and **B**, the axis L(6.4)degenerates into one point $\mathbf{x} = 0$ and therefore the above proof is not applicable. A recent numerical investigation [4] proves that some systems (6.1) with noncollinear vectors A and B possess chaotic streamlines and hence are not integrable.

7 Topology of axisymmetric eigenfields

The axisymmetry and incompressibility of eigenfields $\tilde{\mathbf{V}}_{m}(\mathbf{x}, \mathbf{A}, \mathbf{A})$ (5.1) lead to the existence of a first integral of the corresponding dynamical systems

$$\frac{d\mathbf{x}(t)}{dt} = \tilde{\mathbf{V}}_m(\mathbf{x}(t), \mathbf{A}, \mathbf{A}). \tag{7.1}$$

Dynamical system (7.1) possesses a first integral

$$F_2(\mathbf{x}) = H_3(v)(\mathbf{x} \cdot \mathbf{A})Z(\mathbf{x}), \quad Z(\mathbf{x}) = (\mathbf{A} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{x}) - (\mathbf{x} \cdot \mathbf{A})^2.$$
(7.2)

Indeed, differentiating function $F_2(\mathbf{x}(t))$ along the flow (7.1) we find

$$\frac{dF_2(\mathbf{x}(t))}{dt} = (\mathbf{A} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{A})H_3(v) \left[(\mathbf{A} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{x}) - 3(\mathbf{x} \cdot \mathbf{A})^2 \right] \times \left[H_2(v) + 5H_3(v) + v^2H_4(v) \right]. \tag{7.3}$$

Applying in Eq. (7.3) the identity (4.7) we obtain $dF_2(\mathbf{x}(t))/dt \equiv 0$. Therefore function $F_2(\mathbf{x})$ (7.2) is the first integral of dynamical system (7.1).

The existence of the first integral $F_2(\mathbf{x}) = H_3(v)$ $(\mathbf{x} \cdot \mathbf{A})Z(\mathbf{x})$ implies that all surfaces of its constant levels $F_2(\mathbf{x})$ = const are invariant submanifolds of dynamical system (7.1). Inside the spherical shell $S_{R_2R_1}$ all surfaces of nonzero levels of first integral $F_2(\mathbf{x}) = c$ are compact axisymmetric tori \mathbb{T}_c^2 . Dynamics of the fluid streamlines (7.1) on the tori \mathbb{T}_c^2 is quasi-periodic. The invariant surface of zero level $F_2(\mathbf{x}) = 0$ consists of the plane $(\mathbf{x} \cdot \mathbf{A}) = 0$, the straight line $\mathbf{x} = c\mathbf{A}$, and infinitely many spheres \mathbb{S}_m^2 : $R = R_m$ where R_m are roots of equation $H_3(\lambda R_m) = 0$.

The axisymmetry and incompressibility of the flow with velocity field V(x, A) (3.1) lead to the existence of a first integral for the fluid streamlines defined by equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{V}(\mathbf{x}(t), \mathbf{A}). \tag{7.4}$$

Dynamical system (7.4) has first integral

$$F_1(\mathbf{x}) = H_2(v)Z(\mathbf{x}), \quad Z(\mathbf{x}) = (\mathbf{A} \cdot \mathbf{A})(\mathbf{x} \cdot \mathbf{x}) - (\mathbf{x} \cdot \mathbf{A})^2.$$
 (7.5)

Indeed, differentiating function $F_1(\mathbf{x}(t))$ along the flow (7.4) we get equation

$$\frac{dF_1(\mathbf{x})}{dt} = -2\lambda(\mathbf{x} \cdot \mathbf{A})H_2(v)(\mathbf{A} \cdot \mathbf{A})$$
$$\times \left[H_1(v) + 3H_2(v) + v^2H_3(v)\right], \qquad (7.6)$$

where we used equation $\lambda^2(\mathbf{x} \cdot \mathbf{x}) = v^2$. Applying to the last formula the identity (3.4) we get from Eq. (7.6) $dF_1(\mathbf{x}(t))/dt \equiv 0$. Hence function $F_1(\mathbf{x})$ (7.5) is a first integral of system (7.4).

The existence of the first integral $F_1(\mathbf{x})$ (7.5) yields that all surfaces of its constant level $F_1(\mathbf{x}) = \text{const}$ are invariant submanifolds of the system (7.4). Inside the spherical shell $S_{R,R}$ all nonzero levels of first integral $F_1(\mathbf{x}) = c$ are axisymmetric tori \mathbb{T}^2_c . Dynamics of the fluid streamlines (7.4) on the tori \mathbb{T}_c^2 is quasi-periodic. The surface of zero level $F_1(\mathbf{x}) = H_2(\lambda R)Z(\mathbf{x}) = 0$ consists of the straight line $\mathbf{x} = c\mathbf{A}$ and infinitely many spheres \mathbb{S}^2_k : $R = R_k$ satisfying equation $H_2(\lambda R_k) = 0$.

Remark 7. Topology of the axisymmetric fluid flows $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{A})$ and that of $\mathbf{V}_n(\mathbf{x}, \mathbf{A})$ are completely different. Indeed, invariant tori \mathbb{T}_c^2 for $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{A})$ belong either to the half space $(\mathbf{x} \cdot \mathbf{A}) > 0$ or to the half space $(\mathbf{x} \cdot \mathbf{A}) < 0$ and there is invariant plane $(\mathbf{x} \cdot \mathbf{A}) = 0$. For fluid flow $\mathbf{V}_n(\mathbf{x}, \mathbf{A})$ each invariant torus \mathbb{T}_c^2 belongs to the both sides of any plane $(\mathbf{x} \cdot \mathbf{C}) = 0$ where **C** is an arbitrary constant vector.

8 Summary

We have studied the boundary eigenvalue problem for vector fields V(x) satisfying the Beltrami equation curl V(x) $= \lambda V(\mathbf{x})$ inside a spherical shell $S_{R_2R_1}: R_2 < \sqrt{x_1^2 + x_2^2 + x_3^2}$ $< R_1$ with nonpenetration boundary conditions $\mathbf{x} \cdot \mathbf{V}(\mathbf{x})$ = 0 on the spheres $R = R_2$ and $R = R_1$. We have introduced the scale invariance (1.12) of the eigenvalue problem and have defined the dimensionless parameters $x = \lambda (R_1 - R_2)$

and $\mu = R_1 R_2 / (R_1 - R_2)^2$. We have shown that eigenvalues λ are connected with parameters x, μ by equation

$$\lambda = \frac{\sqrt{4\mu + 1} + 1}{2R_1}x,\tag{8.1}$$

where radius R_1 is arbitrary due to the scale invariance. Radius $R_2 = R_1(\sqrt{4\mu + 1} - 1)/(\sqrt{4\mu + 1} + 1)$. We have derived equations

$$\tan(x) = \frac{x}{1 + \mu x^2},\tag{8.2}$$

$$\tan(x) = \frac{x + \mu x^3 / 3}{1 + (\mu - 1)x^2 / 3 + \mu^2 x^4 / 9}$$
(8.3)

for the dimensionless parameters x, μ . For any $\mu > 0$ the roots x_n of Eq. (8.2) define by formula (8.1) eigenvalues λ_n . Any root \tilde{x}_m of Eq. (8.3) specifies eigenvalue $\tilde{\lambda}_m$ by formula (8.1). Using Eqs. (8.2) and (8.3) we have obtained asymptotics of eigenvalues λ_n , $\tilde{\lambda}_m$ at $R_2 \to R_1$ (that is equivalent to $\mu \to \infty$) and at $n, m \to \infty$ and have proved that the eigenvalues λ_n and $\tilde{\lambda}_m$ are not equal for any n and m.

We have derived in elementary functions the corresponding to λ_n 3D linear space of eigenfields $\mathbf{V}_n(\mathbf{x}, \mathbf{A})$ and the corresponding to $\tilde{\lambda}_m$ 5D linear space of eigenfields $ilde{\mathbf{V}}_m(\mathbf{x},\mathbf{A},\mathbf{B})$ where \mathbf{A} and \mathbf{B} are arbitrary constant vectors in \mathbb{R}^3 . We have proved that eigenfields $\tilde{\mathbf{V}}_m(\mathbf{x}, \mathbf{A}, \mathbf{B})$ with noncollinear vectors A and B are not axisymmetric and eigenfields $\tilde{\mathbf{V}}_{m}(\mathbf{x}, \mathbf{A}, \mathbf{A})$ are axisymmetric. The topologies of fluid flows $V_n(x, A)$, $\tilde{V}_m(x, A, A)$, and $\tilde{V}_m(x, A, B)$ with $\mathbf{A} \times \mathbf{B} \neq 0$ are completely different.

The constructed eigenfields $V_n(x, A)$ and $\tilde{V}_m(x, A, B)$ define the incompressible fluid and plasma equilibria in the shell S_{R-R} , and also time-dependent flows (1.6) and (1.7) of viscous magnetic fluid in the shell.

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