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A basic definition of spin in the new matrix dynamics

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Abstract: We have recently proposed a new matrix dynamics at the Planck scale, building on the theory of trace dynamics. This is a Lagrangian dynamics in which the matrix degrees of freedom are made from Grassmann numbers, and the Lagrangian is trace of a matrix polynomial. Matrices made from even grade elements of the Grassmann algebra are called bosonic, and those made from odd grade elements are called fermionic. In the present article, we provide a basic definition of spin angular momentum in this matrix dynamics, and introduce a bosonic(fermionic) configuration variable conjugate to the spin of a boson(fermion). We then show that at energies below Planck scale, where the matrix dynamics reduces to quantum theory, fermions have half-integer spin (in multiples of Planck's constant), and bosons have integral spin. We also show that this definition of spin agrees with the conventional understanding of spin in relativistic quantum mechanics. Consequently, we obtain an elementary proof for the spin-statistics connection.

Keywords: noncommutative geometry; spin angular momentum; spin-statistics connection; trace dynamics.

1 Introduction

We have recently proposed a new matrix dynamics at the Planck scale [1, 2], building on Adler's theory of trace dynamics [3–5] and by using constructs from Connes' noncommutative geometry programme [6, 7] to incorporate gravity and curvature into trace dynamics. One starts by assuming the existence of a Riemannian differentiable manifold along with the standard Dirac operator $D_B \equiv i\gamma^\mu \nabla_\mu$. Matter is described by relativistic point particles. Einstein field equations are not assumed. As is known from earlier work, the information about metric and

curvature can also be captured by the Dirac operator and its eigenvalues [8, 9].

Given this classical background, the transition to the matrix dynamics is made as follows. The idea is to describe fundamental degrees of freedom by matrices, instead of by real numbers. The motivation is to achieve a formulation of quantum field theory which does not refer to classical time. Doing so also allows one to construct the new dynamics at the Planck scale, from which quantum field theory and classical general relativity are emergent at lower energies and at length and time scales much larger than Planck length and Planck time. Given a matter Lagrangian on a space-time background, all configuration variables and their corresponding velocities are replaced by matrices, and the trace of the resulting matrix polynomial defines the new Lagrangian. Integral of this Lagrangian over time defines the action, whose variation gives the matrix-valued Lagrange equations of motion. These equations define the Lagrangian dynamics, for which an equivalent Hamiltonian dynamics can also be constructed following standard techniques [3].

The next step is to raise space–time points also to the status of matrices, and employ the Dirac operator to describe distance and curvature on the resulting noncommutative geometry. One no longer makes a distinction between the matrix describing a relativistic particle and the matrix describing the space–time geometry it produces. Together, they are described by a Grassman-valued matrix q , which can always be written as a sum: $q \equiv q_B + q_F$, where the bosonic matrix q_B is made of even-grade elements of the Grassmann algebra, and the fermionic matrix q_F is made of odd-grade elements of the Grassmann algebra. The matrix q_F describes the matter part and q_B describes the contribution of q_F to space–time geometry and gravity. We call this entity an ‘atom’ of space–time–matter, or an aikyon. It evolves in Hilbert space, with evolution described by a time-parameter τ intrinsic to a noncommutative geometry [6, 10], and labelled by us as Connes time. The matrix dynamics Lagrangian and action for the aikyon q are given by [1]

$$\frac{S}{C_0} = \frac{1}{2} \int \frac{d\tau}{\tau_P} \text{Tr} \left[\frac{L_p^2}{L^2 c^2} \left(\dot{q}_B + \beta_1 \frac{L_p^2}{L^2} \dot{q}_F \right) \left(\dot{q}_B + \beta_2 \frac{L_p^2}{L^2} \dot{q}_F \right) \right] \quad (1)$$

where β_1 and β_2 are constant self-adjoint fermionic

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matrices. These matrices make the Lagrangian bosonic. The only two fundamental constants are Planck length and Planck time—these scale the length scale L of the aikyon, and the Connes time, respectively. C_0 is a constant with dimensions of action, which will be identified with Planck's constant in the emergent theory. The Lagrangian and action are not restricted to be self-adjoint. A dot denotes derivative with respect to Connes time ($\text{Dot} = \delta/c\delta\tau$). By varying this action w.r.t. q_B and q_F one gets a pair of coupled equations of motion, which can be solved to find the evolution of q_B and q_F . The respective momenta p_B and p_F are constants of motion, and the expression for p_B can be written as an eigenvalue equation for the modified Dirac operator $D \equiv D_B + D_F$:

$$[D_B + D_F]\psi = \frac{1}{L} \left(1 + i \frac{L_p^2}{L^2} \right) \psi \quad (2)$$

where

$$D_B \equiv \frac{1}{Lc} \frac{dq_B}{d\tau}; \quad D_F \equiv \frac{L_p^2}{L^2} \frac{\beta_1 + \beta_2}{2Lc} \frac{dq_F}{d\tau} \quad (3)$$

D_B is defined such that in the commutative c-number limit where space-time emerges, it becomes the standard Dirac operator on a Riemannian manifold. If there are many aikyons in the theory, their total action is the sum of their individual actions. There is no space-time in this dynamics at the Planck scale; only a Hilbert space, from which space-time is emergent.

We next asked what the coarse-grained matrix dynamics looks like at energies much lower than Planck scale; equivalently, at times scales much larger than Planck time. This question can be answered by employing the techniques of statistical thermodynamics, as set up in the theory of trace dynamics. The low energy dynamics falls in two classes. If not too many aikyons are entangled with each other, the anti-self-adjoint component of the net Hamiltonian is negligible, and the emergent dynamics is quantum dynamics without a classical space-time. The canonical variables obey quantum commutation relations, and the Heisenberg equations of motion, for which there is also an equivalent Schrödinger picture. Evolution is still in Connes time, and there is no background space-time, yet.

The other limiting class is when sufficiently many aikyons get entangled; then the anti-self-adjoint part of the net Hamiltonian becomes significant. This causes rapid spontaneous localisation, loss of quantum superposition and the emergence of classicality. The classical space-time manifold emerges, and its points are defined by the position eigenvalues to which the fermions localise. The metric and classical curvature are recovered by localisation of the

Dirac operators of the aikyons to their specific eigenvalues. The net action for the aikyons described above reduces to the action for classical general relativity. In this way Einstein field equations are recovered, with relativistic point particles as sources. Given this space-time background, the above-mentioned quantum dynamics of uncollapsed aikyons can be described as quantum field theory on a background space-time. This background is generated by the classicalised matter degrees of freedom.

Subsequently, we have generalised this action to include Yang–Mills gauge fields [2], and the new action is given by $S/C_0 = \int \tau \mathcal{L} / \tau_P$ where

$$\mathcal{L} = \text{Tr} \left[\frac{L_p^2}{L^4} \left\{ i\alpha \left(q_B + \frac{L_p^2}{L^2} \beta_1 q_F \right) + L \left(\dot{q}_B + \frac{L_p^2}{L^2} \beta_1 \dot{q}_F \right) \right\} \right. \\ \left. \left\{ i\alpha \left(q_B + \frac{L_p^2}{L^2} \beta_2 q_F \right) + L \left(\dot{q}_B + \frac{L_p^2}{L^2} \beta_2 \dot{q}_F \right) \right\} \right] \quad (4)$$

This Lagrangian for an aikyon should be compared with the earlier one (1) which had only gravity and Dirac fermions as unified components of the aikyon. This new Lagrangian here also includes gauge-fields and their currents, through q_B and q_F , assumed self-adjoint. α is the Yang–Mills coupling constant, assumed to be a real number. Gravitation, and Yang–Mills fields, and their corresponding sources, are unified here as the ‘position’ q and ‘velocity’ $dq/d\tau$ of the aikyon. With position being the Yang–Mills part, and velocity being the gravitation part.

By defining new dynamical variables $\dot{Q}_B$ and $\dot{Q}_F$ as

$$\dot{Q}_B = \frac{1}{L} (i\alpha q_B + L\dot{q}_B); \quad \dot{Q}_F = \frac{1}{L} (i\alpha q_F + L\dot{q}_F) \quad (5)$$

this Lagrangian can be brought to the elegant and revealing form.

$$\mathcal{L} = \text{Tr} \left[\frac{L_p^2}{L^2} \left(\dot{Q}_B + \frac{L_p^2}{L^2} \beta_1 \dot{Q}_F \right) \left(\dot{Q}_B + \frac{L_p^2}{L^2} \beta_2 \dot{Q}_F \right) \right] \quad (6)$$

We used this form to express a unification for gravity and gauge fields in our recent work [2] in terms of these new complex variables. These variables imbibe the position and velocity of the aikyon as their real and imaginary parts. We note that $q_B, q_F, \dot{q}_B$ and $\dot{q}_F$ are all assumed to be self-adjoint in the present paper. This new construction amounts to re-expressing the aikyon q by the variable $\tilde{Q} \equiv \tilde{Q}_B + \tilde{Q}_F$ where the bosonic $\tilde{Q}_B$ and the fermionic $\tilde{Q}_F$ are further expressed in terms of their self-adjoint and anti-self-adjoint parts as in Eq. (5) above. The self-adjoint part is velocity, which encodes gravity, and the anti-self-adjoint is position, which encodes Yang–Mills gauge fields. This natural split of a Grassmann matrix into its four parts [bosonic self-adjoint,

bosonic anti-self-adjoint, fermionic self-adjoint, fermionic anti-self-adjoint] captures gravity and Yang–Mills fields, as well as their sources. There is only one term in the Lagrangian of an aikyon, which when opened up using this split, gives rise to 16 different terms. The classical limit is Einstein gravity coupled to Yang–Mills fields and matter sources [2].

Our new Planck scale matrix dynamics has been used to make several predictions. We derived the Bekenstein–Hawking black hole entropy from the microstates of its constituent aikyons [11]. We have predicted the Karolyhazy uncertainty relation as a consequence of our theory [12]. We have used the theory to propose that dark energy is a large-scale quantum gravitational phenomenon [13]. We have explained the remarkable fact that the Kerr–Newman black hole has the same gyromagnetic ratio as a Dirac fermion, both being twice the classical value [2].

In the present article we provide a basic definition of spin angular momentum in this matrix dynamics, and introduce a bosonic(fermionic) configuration variable conjugate to the spin of a boson(fermion). We then show that at energies below Planck scale, where the matrix dynamics reduces to quantum theory, fermions have half-integer spin (in multiples of Planck’s constant), and bosons have integral spin. We also show that this definition of spin coincides with the conventional understanding of spin in relativistic quantum mechanics. Consequently, we obtain an elementary proof for the spin-statistics connection. Essentially, we reverse the arguments of the traditional proof of spin-statistics connection in relativistic quantum field theory [14]. Instead of showing that integer-spin particles obey Bose–Einstein statistics, we show that particles obeying Bose–Einstein statistics have integer spin. Similarly, we show that particles obeying Fermi–Dirac statistics have half-integer spin.

2 A definition for spin in the new matrix dynamics

Our starting point is the Lagrangian for an aikyon, as given in Eqn. (62) of [2], and mentioned above in (6), which we reproduce here again:

$$\mathcal{L} = \text{Tr} \left[\frac{L_p^2}{L^2} \left(\dot{Q}_B + \frac{L_p^2}{L^2} \beta_1 \dot{Q}_F \right) \left(\dot{Q}_B + \frac{L_p^2}{L^2} \beta_2 \dot{Q}_F \right) \right] \quad (7)$$

We now introduce self-adjoint bosonic operators R_B and θ_B , and self-adjoint fermionic operators R_F and θ_F , as follows:

$$\tilde{Q}_B \equiv R_B \exp i\theta_B ; \quad \tilde{Q}_F \equiv R_F \exp i\eta\theta_F \quad (8)$$

Here, η is a real Grassmann number, introduced to ensure that the fermionic phase is bosonic so that $\tilde{Q}_F$ comes out fermionic, as desired, upon the Taylor expansion of its phase. As is known, this will give $\exp i\eta\theta_F = 1 + i\eta\theta_F$, with the higher terms in the Taylor expansion vanishing because $\eta^2 = 0$. These definitions are equivalent to expressing a Grassmann-valued matrix in terms of its ‘amplitude’ matrix and ‘phase’ matrix, as if to represent the matrix on a complex plane.

We note from the definition of $\tilde{Q}_B$ that it remains unchanged under the shift $\theta_B \rightarrow \theta_B + 2\pi I$. In this sense θ_B acts like an angle variable, and we will require all bosonic physical quantities depending on θ_B to remain unchanged under the shift $\theta_B \rightarrow \theta_B + 2\pi I$. We can also reason why the fermionic $\tilde{Q}_F$ should change sign if the bosonic part of the corresponding aikyon undergoes a shift $\theta_B \rightarrow \theta_B + 2\pi I$. The bosonic $\tilde{Q}_B$ is a matrix made from elements of the even-grade Grassmann algebra, so that in principle we can consider the case that it is made from two fermionic matrices: $\tilde{Q}_B = \tilde{Q}_{F1} \times \tilde{Q}_{F2}$. Hence

$$\tilde{Q}_B \equiv R_B \exp i\theta_B = \tilde{Q}_{F1} \times \tilde{Q}_{F2} = R_{F1} \exp i\eta\theta_{F1} \times R_{F2} \exp i\eta\theta_{F2} \quad (9)$$

The shift $\eta\theta_{F1} \rightarrow \eta\theta_{F1} + \pi$ induces a sign change in $\tilde{Q}_{F1}$, and a simultaneous shift $\eta\theta_{F2} \rightarrow \eta\theta_{F2} + \pi$ induces a sign change in $\tilde{Q}_{F2}$. Together these two sign changes imply that the bosonic $\tilde{Q}_B$ does not change sign, and one can conclude that these two shifts are equivalent to $\theta_B \rightarrow \theta_B + 2\pi I$. Conversely, under $\theta_B \rightarrow \theta_B + 2\pi I$, each of the fermionic parts undergo a change of sign. They do not change under $\theta_B \rightarrow \theta_B + 4\pi I$. This observation is analogous to the fact that a spinor changes sign under a 2π rotation in space. We may think of spinors as eigenstates of fermionic matrices, having odd-grade Grassmann numbers as their components. Vectors are eigenstates of bosonic matrices, and clearly, a product of two spinors is a vector, just as a product of two fermionic matrices is a bosonic matrix.

Each of these four newly introduced self-adjoint operators are functions of Connes time, and are the four configuration variables which define the aikyon. By substituting these definitions of $\tilde{Q}_B$ and $\tilde{Q}_F$ in the above Lagrangian (7), we can write the Lagrangian in terms of time derivatives of these four configuration variables. We will do that in the next section. For now, it suffices to note that the canonical linear momenta p_{BR} and p_{FR} are defined as usual, as derivatives of the Lagrangian with respect to the corresponding velocities:

$$p_{BR} = \frac{\delta \mathcal{L}}{\delta \dot{R}_B} ; \quad p_{FR} = \frac{\delta \mathcal{L}}{\delta \dot{R}_F} \quad (10)$$

The novel part is the following. We define bosonic and fermionic spin angular momenta as follows:

$$p_{B\theta} = \frac{\delta \mathcal{L}}{\delta \dot{\theta}_B} ; \quad p_{F\theta} = \frac{\delta \mathcal{L}}{\delta \dot{\theta}_F} \quad (11)$$

[A word about dimensions. The Lagrangian and action as introduced here are dimensionless, and hence so is the linear momentum. However, when care is taken of the τ_p present in the action integral, and the C_0 on the left hand side of the action integral brought to the right, linear momentum acquires familiar correct dimensions. The same reasoning applies for the dimensions of angular momentum]. The following proof for spin quantisation is independent of the specific form of the Lagrangian for the matrix dynamics. All that is required is that the configuration variables have a self-adjoint part as well as an anti-self-adjoint part. As is known from Adler's theory of trace dynamics, and is true also for the present matrix dynamics, there is a conserved charge known as the Adler–Millard charge [15]. This charge results from the invariance of the trace Lagrangian under global unitary transformations of the degrees of freedom. The charge has dimensions of action and is denoted by the symbol $\tilde{C}$:

$$\tilde{C} = \sum_{r \in B} [q_r, p_r] - \sum_{r \in F} \{q_r, p_r\} \quad (12)$$

which is the sum over the shown commutators for bosonic degrees of freedom, minus the sum over the shown anti-commutators for fermionic degrees of freedom. If there are many aikyons in the system, the conserved charge is the sum over all aikyons, of their individual contributions. For the present set of momenta, the Adler–Millard charge is

$$\tilde{C} = [R_B, p_{BR}] + [\theta_B, p_{B\theta}] - \{R_F, p_{FR}\} - \{\theta_F, p_{F\theta}\} \quad (13)$$

As we know from trace dynamics and our own earlier work, if we observe this matrix dynamics at energy scales much lower than Planck scale, the emergent dynamics is quantum theory. This is shown by coarse-graining the matrix dynamics over times much larger than Planck times, and using the techniques of statistical thermodynamics to find out the coarse-grained dynamics [There is an additional requirement that any anti-self-adjoint component in the momenta and in the Hamiltonian must be negligible for the emergence of quantum theory]. In particular, the Adler–Millard charge gets equipartitioned over all the degrees of freedom, and the constant value of the equipartitioned charge per degree of freedom is identified with Planck's constant $\hbar$. This implies, from the structure of the

charge $\tilde{C}$ above, that the self-adjoint part of statistically averaged canonical variables (identified with the dynamical variables of quantum field theory) obey the canonical commutation relations of quantum theory:

$$\begin{aligned} [R_B, p_{BR}] &= i\hbar ; & [\theta_B, p_{B\theta}] &= i\hbar ; & \{R_F, p_{FR}\} &= i\hbar ; \\ \{\theta_F, p_{F\theta}\} &= i\hbar \end{aligned} \quad (14)$$

It is understood in these commutators that only the self-adjoint component of the momenta is present, and this component has been averaged over the canonical ensemble of the microstates allowed at statistical equilibrium. From here, it is possible to deduce the quantization of spin angular momentum. From the second commutation relation, between θ_B and $p_{B\theta}$, we deduce that this spin angular momentum is a displacement operator, whose eigenvalues are quantized:

$$\begin{aligned} p_{\theta B} &= -i\hbar \frac{\delta}{\delta \theta_B} ; & -i\hbar \frac{\delta}{\delta \theta_B} \psi &= \lambda \psi ; & \psi &\sim \exp \left[i \frac{\lambda}{\hbar} \theta_B \right] \Rightarrow \lambda \\ &= n\hbar \end{aligned} \quad (15)$$

where n is an integer. Moreover, since θ_B is an even grade Grassmann matrix, two such matrices commute, leaving the state of a multiparticle bosonic system unchanged upon interchange of two identical bosons. The state is hence symmetric, and the system obeys Bose–Einstein statistics.

The situation regarding fermions is more subtle. Because the fermionic spin $p_{F\theta}$ satisfies an *anti-commutation* relation with the dynamical variable θ_F , one can construct a displacement operator for it using Berezin calculus. [By itself, θ_F does not permit any angle interpretation for itself. However we can infer fermion spin quantisation indirectly. Consider a bosonic degree of freedom B made from a product of two identical fermions F_1 and F_2 , i.e., $B = F_1 F_2$. Since B has integral spin and since spin is additive, and since we cannot discriminate between the contribution of spin from F_1 and spin from F_2 , we can conclude the following. If the boson has spin $\hbar$, the fermions each have spin $\hbar/2$. And because fermions are made from odd-grade Grassmann numbers which anti-commute, a state for a system of identical fermions is anti-symmetric under exchange of particles, implying that the statistics is Fermi–Dirac].

The fermionic Berezin displacement operator corresponding to θ_F is $i\hbar \delta / \delta \theta_F$ and its eigenvalue will be a Grassmann number, not a c-number. However, we can construct the bosonic operator $i\hbar \delta / \delta \theta'_F$ where $\theta'_F = \eta \theta_F$ and this has a c-number eigenvalue λ and an eigenstate

proportional to $\exp[-i\lambda\eta\theta_F/\hbar]$. Since the phase must change sign under a shift of 2π in $\eta\theta_F$, it follows that the fermion has a spin $\lambda = \pm\hbar/2$.

The novelty of the present proposal is the introduction of the fermionic configuration variable θ_F . There is no analogue for it in quantum mechanics. That is because one develops quantum mechanics by quantising classical dynamical theories. In so doing, we never arrive at this dynamical variable θ_F , which indeed comes down to us from the Planck scale matrix dynamics. Moreover, there is no space-time, yet, in our analysis. This is another piece of evidence to suggest that quantum mechanics is a low energy limit of a [more complete] underlying dynamics: a dynamics in which classical space-time is absent. It also seems to be the case that this proof of the spin-statistics connection does not manifestly require a space-time symmetry such as Lorentz invariance.

Next, we show, using our specific Lagrangian, that the spin angular momentum introduced here agrees with the conventional understanding of spin in relativistic quantum mechanics.

3 Relating the spin in matrix dynamics to the spin in quantum mechanics

We work out the expressions for the four momenta by first substituting the forms (8) into the Lagrangian (6). The velocities are given by (in the small angle approximation)

$$\begin{aligned}\dot{\vec{Q}}_B &= \dot{R}_B \exp i\theta_B + R_B \exp i\theta_B \times i\dot{\theta}_B; \\ \dot{\vec{Q}}_F &= \dot{R}_F \exp i\eta\theta_F + R_F \exp i\eta\theta_F \times i\eta\dot{\theta}_F\end{aligned}\quad (16)$$

These are substituted in the Lagrangian, and they yield the following expressions for the four momenta. We first open the brackets in the expression for the Lagrangian, and write it as a sum of four terms: $\mathcal{L} = T_1 + T_2 + T_3 + T_4$:

$$\begin{aligned}T_1 &= Tr \frac{L_P^2}{L^2} (\dot{\vec{Q}}_B^2) = Tr \frac{L_P^2}{L^2} (\dot{R}_B \exp[i\theta_B] \dot{R}_B \exp[i\theta_B] \\ &\quad + \dot{R}_B \exp[i\theta_B] R_B \exp[i\theta_B] i\dot{\theta}_B \\ &\quad + R_B \exp[i\theta_B] i\dot{\theta}_B \dot{R}_B \exp[i\theta_B] \\ &\quad + R_B \exp[i\theta_B] i\dot{\theta}_B R_B \exp[i\theta_B] i\dot{\theta}_B)\end{aligned}\quad (17)$$

$$\begin{aligned}T_2 &= Tr \frac{L_P^4}{L^4} [\dot{\vec{Q}}_B \beta_2 \dot{\vec{Q}}_F] = Tr \frac{L_P^4}{L^4} (\dot{R}_B \exp[i\theta_B] \beta_2 \dot{R}_F \exp[i\theta_F] \\ &\quad + \dot{R}_B \exp[i\theta_B] \beta_2 R_F \exp[i\theta_F] i\eta\dot{\theta}_F \\ &\quad + R_B \exp[i\theta_B] i\dot{\theta}_B \beta_2 \dot{R}_F \exp[i\theta_F] \\ &\quad + R_B \exp[i\theta_B] i\dot{\theta}_B \beta_2 R_F \exp[i\theta_F] i\eta\dot{\theta}_F)\end{aligned}\quad (18)$$

$$\begin{aligned}T_3 &= Tr \frac{L_P^4}{L^4} [\beta_1 \dot{\vec{Q}}_F \dot{\vec{Q}}_B] = Tr \frac{L_P^4}{L^4} (\beta_1 \dot{R}_F \exp[i\theta_F] \dot{R}_B \exp[i\theta_B] \\ &\quad + \beta_1 \dot{R}_F \exp[i\theta_F] R_B \exp[i\theta_B] i\dot{\theta}_B \\ &\quad + \beta_1 R_F \exp[i\theta_F] i\eta\dot{\theta}_F \dot{R}_B \exp[i\theta_B] \\ &\quad + \beta_1 R_F \exp[i\theta_F] i\eta\dot{\theta}_F R_B \exp[i\theta_B] i\dot{\theta}_B)\end{aligned}\quad (19)$$

$$\begin{aligned}T_4 &= Tr \frac{L_P^6}{L^6} [\beta_1 \dot{\vec{Q}}_F \beta_2 \dot{\vec{Q}}_F] = (\beta_1 \dot{R}_F \exp[i\theta_F] \beta_2 \dot{R}_F \exp[i\theta_F] \\ &\quad + \beta_1 \dot{R}_F \exp[i\theta_F] \beta_2 R_F \exp[i\theta_F] i\eta\dot{\theta}_F \\ &\quad + \beta_1 R_F \exp[i\theta_F] i\eta\dot{\theta}_F \beta_2 \dot{R}_F \exp[i\theta_F] \\ &\quad + \beta_1 R_F \exp[i\theta_F] i\eta\dot{\theta}_F \beta_2 R_F \exp[i\theta_F] i\eta\dot{\theta}_F)\end{aligned}\quad (20)$$

The momenta can be worked out by taking appropriate trace derivatives of the Lagrangian, using the rules of differentiation from trace dynamics. The varied matrix should be moved to the extreme right by cyclic permutation inside the trace, keeping in mind that exchange of two fermionic matrices results in a change of sign in the overall expression.

The fermionic spin angular momentum is

$$\begin{aligned}p_{F\theta} &= \frac{\delta \mathcal{L}}{\delta \dot{\theta}_F} = \frac{L_P^4}{L^4} \left[-R_B \exp[i\theta_B] i\dot{\theta}_B (\beta_1 + \beta_2) R_F \exp[i\eta\theta_F] \eta \right. \\ &\quad \left. + i(\dot{R}_B \exp[i\theta_B] \beta_1 + \beta_2) R_F \exp[i\eta\theta_F] \eta \right] + \mathcal{O}\left(\frac{L_P^6}{L^6}\right)\end{aligned}\quad (21)$$

The higher order terms do not contribute to the present discussion. As we will see below, this expression has the desired form for matching with the conventional discussion of spin in the Dirac equation. We note that this spin angular momentum is not a conserved quantity; the Lagrangian explicitly depends on θ_F .

The bosonic spin angular momentum is given by

$$\begin{aligned}p_{B\theta} &= \frac{\delta \mathcal{L}}{\delta \dot{\theta}_B} = -2 \frac{L_P^2}{L^2} R_B \exp[i\theta_B] \dot{\theta}_B R_B \exp[i\theta_B] \\ &\quad + 2i \frac{L_P^2}{L^2} \dot{R}_B \exp[i\theta_B] R_B \exp[i\theta_B] \\ &\quad + \frac{L_P^4}{L^4} (\beta_1 + \beta_2) \dot{R}_F \exp[i\eta\theta_F] R_B \exp[i\theta_B] \\ &\quad + i \frac{L_P^4}{L^4} (\beta_1 + \beta_2) R_F \exp[i\eta\theta_F] \eta \dot{\theta}_F R_B \exp[i\theta_B]\end{aligned}\quad (22)$$

The fermionic and bosonic linear momenta are given by

$$\begin{aligned}p_{BR} &= \frac{\delta \mathcal{L}}{\delta \dot{R}_B} = 2 \frac{L_P^2}{L^2} \exp[i\theta_B] \dot{R}_B \exp[i\theta_B] \\ &\quad + 2i \frac{L_P^4}{L^4} \exp[i\theta_B] R_B \exp[i\theta_B] \dot{\theta}_B\end{aligned}$$

$$\begin{aligned}
& + \frac{L_P^4}{L^4} \exp[i\theta_B] (\beta_1 + \beta_2) \dot{R}_F \exp[i\theta_F] \\
& + i \frac{L_P^4}{L^4} \exp[i\theta_B] (\beta_1 + \beta_2) R_F \exp[i\eta\theta_F] \eta \dot{\theta}_F
\end{aligned} \quad (23)$$

$$\begin{aligned}
p_{FR} = \frac{\delta \mathcal{L}}{\delta \dot{R}_F} = \frac{L_P^4}{L^4} & \left[\exp[i\eta\theta_F] \dot{R}_B \exp[i\theta_B] (\beta_1 + \beta_2) \right. \\
& \left. + i \exp[i\eta\theta_F] R_B \exp[i\theta_B] \dot{\theta}_B (\beta_1 + \beta_2) \right] \\
& + \mathcal{O}\left(\frac{L_P^6}{L^6}\right)
\end{aligned} \quad (24)$$

In our earlier work, we constructed the variables $\tilde{Q}_B$ and $\tilde{Q}_F$ as follows:

$$\dot{\tilde{Q}}_B = \frac{1}{L} (i\alpha q_B + L\dot{q}_B); \quad \dot{\tilde{Q}}_F = \frac{1}{L} (i\alpha q_F + L\dot{q}_F) \quad (25)$$

Here, the self-adjoint operator q_B corresponds to the Yang–Mills potential in the classical limit, and $\dot{q}_B$ to gravity. Here $\dot{q}_F$ is the matter source for gravity, and q_F is the current that sources the Yang–Mills fields. α is the gauge coupling constant. These operators are related to the standard Dirac operator D_B , and we also defined an operator D_F :

$$D_B \equiv \frac{1}{Lc} \frac{dq_B}{d\tau}; \quad D_F \equiv \frac{L_P^2}{L^2} \frac{\beta_1 + \beta_2}{2Lc} \frac{dq_F}{d\tau} \quad (26)$$

D_B is defined such that in the commutative c-number limit where space-time emerges, it becomes the standard Dirac operator on a Riemannian manifold. D_F is defined such that upon spontaneous localisation, it gives rise to the classical action for a relativistic point particle. The modified Dirac operators which take into account the presence of the Yang–Mills potential q_B and the corresponding current q_F , are given by

$$D_{Bnewi} = \frac{1}{L} \dot{\tilde{Q}}_B \quad \text{and} \quad D_{Fnewi} = \frac{L_P^2}{L^2} \frac{\beta_1 + \beta_2}{2Lc} \dot{\tilde{Q}}_F \quad (27)$$

q_B is related to the gauge-potential by $\alpha q_B/L^2 = \gamma^\mu A_\mu$, and q_F is related to the gauge current.

The constancy of the bosonic momentum corresponding to $\tilde{Q}_B$ implies that we have a constant net Dirac operator which can be expressed as an eigenvalue equation given by:

$$[D_{Bnewi} + D_{Fnewi}] \psi = \lambda \psi \quad (28)$$

where the eigenvalues λ are assumed to be $\mathbb{C}$ -numbers [since the operator is bosonic] and are independent of the Connes' time τ .

We now work out what this Dirac equation looks like in terms of the variables θ_B , θ_F , R_B and R_F introduced in the present paper. This will unearth the presence of spin as defined in the current paper, and show its presence in the conventional Dirac equation. By comparing the real and imaginary parts of Eqs. (16) and (25) we obtain that

$$\dot{q}_B = \dot{R}_B (\cos \theta_B) - R_B (\sin \theta_B) \dot{\theta}_B \quad (29)$$

$$\frac{\alpha}{L} q_B = \dot{R}_B (\sin \theta_B) + R_B (\cos \theta_B) \dot{\theta}_B \quad (30)$$

If we make the assumption that introducing the gauge-potential does not change the background space-time geometry too much, we should have that θ_B is small and that q_B is nearly the same as R_B and $\dot{q}_B$ is nearly the same as $\dot{R}_B$. This trivializes the first of these equations [$q_B = R_B$] whereas the second equation gives that $\alpha q_B/L^2 = R_B \dot{\theta}_B/L$. The left hand side in this equality is of course the same as the contribution of the gauge potential to the Dirac operator D_{Bnewi} . We now show that the right hand side is proportional to the spin angular momentum, as defined above. Let us look at the self-adjoint part of the expression (21) for fermionic spin $p_{F\theta}$. For small θ_B we can approximate it as $-R_B \dot{\theta}_B (\beta_1 + \beta_2) R_F \exp[i\eta\theta_F] \eta$. This means that $p_{F\theta} \sim -(\alpha q_B/L) \tilde{Q}_F \eta$ which represents the coupling of the fermion to the gauge potential. Equivalently, the term $\alpha q_B/L$ represents the correction to the Dirac operator enforced by the spin angular momentum $p_{F\theta}$. We also know that this correction term in the standard Dirac equation allows us to conclude that the electron has a gyromagnetic ratio of 2 and hence has a spin $\hbar/2$. We therefore conclude that our definition of the spin angular momentum $p_{F\theta}$ is consistent with the conventional understanding of spin in relativistic quantum mechanics. Of course this Dirac equation is still not on space-time, and evolution is with respect to Connes time; however the transition from here to the conventional Dirac equation on a space-time is straightforward [after spontaneous localisation of macroscopic systems gives rise to emergent space-time]. The Dirac operator is the standard one, and the gauge potential represents an external potential to which the fermion is coupled.

One subtle point to note is the following. In the definition of $\dot{\tilde{Q}}_B$ and $\dot{\tilde{Q}}_F$, the gravity part $\dot{q}_B$ and the gauge part $\dot{q}_F$ have a relative i factor between them, and the same is true for the pair $(\dot{q}_F, q_F)$. This is different from the standard theory where there is of course no i factor between D_B and the gauge potential αA in the modified Dirac operator $D_B + \alpha A$. We believe our construction to be more fundamental, as it allows us to think of the gauge interaction as the phase part of the complex $\tilde{Q}_B$. Equivalence with the

standard theory can be restored if in the classical (gauge field plus current) Lagrangian we introduce an i factor in the current part of the interaction term $j_\mu A^\mu$ and also an i factor in front of the potential term A^μ in this interaction term, and make an overall change of sign for this term in the Lagrangian. And similarly, an overall change in sign in front of the gauge-field Lagrangian because $i^2 = -1$.

4 Understanding spin

From the Lagrangian above, we can write the first integrals for the equations of motion for $\tilde{Q}_B$ and $\tilde{Q}_F$. The two corresponding canonical momenta are constants of motion, implying that

$$\tilde{p}_B = \frac{\partial \mathcal{L}}{\partial \dot{\tilde{Q}}_B} = \frac{L_p^2}{L^2} \left[2\dot{\tilde{Q}}_B + \frac{L_p^2}{L^2} (\beta_1 + \beta_2) \dot{\tilde{Q}}_F \right] \quad (31)$$

$$\tilde{p}_F = \frac{\partial \mathcal{L}}{\partial \dot{\tilde{Q}}_F} = \frac{L_p^4}{L^4} \left[\dot{\tilde{Q}}_B (\beta_1 + \beta_2) + \frac{L_p^2}{L^2} \beta_1 \dot{\tilde{Q}}_F \beta_2 + \frac{L_p^2}{L^2} \beta_2 \dot{\tilde{Q}}_F \beta_1 \right] \quad (32)$$

The presence of spin in quantum mechanics is also indicated from this expression above for the bosonic momentum, from which the Dirac equation is constructed. It depends not only on the bosonic velocity but also on the fermionic velocity, which is related to the fermionic spin angular momentum.

Here the conjugate momenta, $\tilde{p}_B$ and $\tilde{p}_F$ are constants as the trace Lagrangian is independent of $\tilde{Q}_B$ and $\tilde{Q}_F$, similar to what happened for pure gravity. This implies,

$$2\dot{\tilde{Q}}_B + \frac{L_p^2}{L^2} (\beta_1 + \beta_2) \dot{\tilde{Q}}_F = C_1 \quad (33)$$

$$\dot{\tilde{Q}}_B (\beta_1 + \beta_2) + \frac{L_p^2}{L^2} \beta_1 \dot{\tilde{Q}}_F \beta_2 + \frac{L_p^2}{L^2} \beta_2 \dot{\tilde{Q}}_F \beta_1 = C_2 \quad (34)$$

for some C_1 and C_2 which are constant bosonic and fermionic matrices, respectively. We can deduce from the definition of $\dot{\tilde{Q}}_B$ and $\dot{\tilde{Q}}_F$ in terms of $(q_B, \dot{q}_B, q_F, \dot{q}_F)$ as to how the former set of variables evolve. The latter set evolve as harmonic oscillators [2]:

$$q_B = B_+ e^{i(\alpha\tau/L)} + B_- e^{-i(\alpha\tau/L)} ; \quad q_F = F_+ e^{i(\alpha\tau/L)} + F_- e^{-i(\alpha\tau/L)} \quad (35)$$

This implies that in the complex ‘plane’ formed by $\dot{q}_B$ along the horizontal axis, and $i\dot{q}_B$ along the vertical axis, the complex dynamical variable $\dot{\tilde{Q}}_B$ executes periodic motion with a time-period L/α . The angular momentum associated with this periodic motion is the bosonic spin angular momentum. An analogous interpretation holds for

the fermionic spin, *vis a vis* the motion of $\dot{\tilde{Q}}_F$ in the complex plane formed by $\dot{q}_F$ and $i\dot{q}_F$. In the emergent quantum theory, this spin angular momentum is quantized in units of Planck’s constant, just like angular momentum is, except that, because fermions are described by odd-grade Grassmann matrices, their spin is half-integral.

To put it more physically, spin is the angular momentum associated with the motion of an aikyon in the phase space of matrix dynamics. The motion takes place in the two dimensional ‘plane’ formed by the self-adjoint and anti-self-adjoint parts of the Grassmann matrix which describes an aikyon. The self-adjoint part relates to gravity and the anti-self-adjoint part to Yang–Mills gauge interactions. We can decompose this motion into a sum of linear motion and angular motion. Since in both the linear motion as well as in angular motion, both the self-adjoint and anti-self-adjoint parts vary, each of these motions relate both to gravity and to gauge fields. However, since spin gets switched on only after the imaginary axis of the plane is switched on because of introducing gauge fields, it could be the case that there is an intimate connection between spin and gauge interactions. In particular, since spin relates to torsion in geometry, one should investigate if gauge interactions are manifestations of torsion, and of a complex antisymmetric part to the space–time metric. This kind of a suggested unification of gravity and gauge fields on a complex plane might help get rid of the need for extra hidden space–time dimensions as required in Kaluza–Klein theories. The fact that gauge-interactions are related to the phase which obeys periodic boundary conditions might help understand why the standard model symmetry groups have to do with rotational invariance, whereas gravity, related to the amplitude R_B has to do with diffeomorphisms.

Another way to think of spin is to regard the self-adjoint fermionic position and velocity operators, $\dot{q}_F$ and q_F , as the real and imaginary parts of the complex velocity $\dot{\tilde{Q}}_F$ [which is what they precisely are]. Spin is the momentum associated with change of phase during evolution (in Connes time) in the complex plane defined by position and velocity. Linear momentum is associated with change in amplitude of $\dot{\tilde{Q}}_F$ during evolution.

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