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The Rabi problem with elliptical polarization

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Abstract: We consider the solution of the equation of motion of a classical/quantum spin subject to a monochromatical, elliptically polarized external field. The classical Rabi problem can be reduced to third-order differential equations with polynomial coefficients and hence solved in terms of power series in close analogy to the confluent Heun equation occurring for linear polarization. Application of Floquet theory yields physically interesting quantities like the quasienergy as a function of the problem's parameters and expressions for the Bloch–Siegert shift of resonance frequencies. Various limit cases are thoroughly investigated.

Keywords: elliptical polarization; quasienergy; Rabi problem; resonance frequencies.

1 Introduction

In recent years, theoretical and experimental evidence has shown that periodic driving can be a key element for engineering exotic quantum mechanical states of matter, such as time crystals and superconductors at room temperature [1–3]. The renewed interest in Floquet engineering, i.e., the control of quantum systems by periodic driving, is due to (a) the rapid development of laser and ultrashort spectroscopy techniques [4], (b) the discovery and understanding of various "quantum materials" that exhibit interesting exotic properties [5, 6], and (c) the interaction with other emerging fields of physics such as programmable matter [7] and periodic thermodynamics [8–22].

One of the simplest system to study periodic driving is a two level system (TLS) interacting with a classical periodic radiation field. The special case of a constant magnetic field in, say, x-direction plus a circularly polarized field in the y–z-plane was already solved more than eight decades ago by Rabi [23] and can be found in many textbooks. This case is referred to as Rabi problem with circular polarization (RPC) in the following. Shortly thereafter, Bloch and Siegert [24] considered the analogous problem of a linearly polarized magnetic field orthogonal to the direction of the constant field henceforth called, Rabi problem with linear polarization (RPL) and proposed the so-called rotating wave approximation. They also investigated the shift of the resonant frequencies due to the approximation error of the rotating wave approximation, since then called the "Bloch–Siegert shift."

In the following decades, one noticed [25, 26] that the underlying mathematical problem leads to the Floquet Theory [27], which deals with linear differential matrix equations with periodic coefficients [28, 29]. Accordingly, analytical approximations for solutions were worked out, which formed the basis for subsequent research. In particular, the groundbreaking work of Shirley [26] has received widespread attention and many citations. Among the numerous applications of the theory of periodically driven TLS are nuclear magnetic resonance [30], ac-driven quantum dots [31], Josephson qubit circuits [32], and coherent destruction of tunneling [33]. On a theoretical level, the methods for solving the RPL and related problems have been gradually refined and include power series approximations for Bloch–Siegert shifts [34, 35], perturbation theory and/or various boundary cases [36–41] and the hybridized rotating wave approximation [42]. Also, the inverse method yields analytical solutions for certain periodically driven TLS [43–46].

In the meantime also the RPL has been analytically solved [47, 48]. This solution is based on a transformation of the Schrödinger equation into a confluent Heun differential equation. A similar approach was previously applied to the TLS subject to a magnetic pulse [49, 50] and has been extended to other cases of physical interest [51, 52]. In the special case of the RPL, the analytical solution has been further elaborated to include time evolution over a full period and explicit expressions for the quasienergy [53].

In this paper, we will extend these results to the Rabi problem with elliptic polarization (RPE) that is also of experimental interest, see [54, 55]. Here, we will approach the Floquet problem of the TLS via its well-known classical limit, see, e.g., [56]. It has been shown that, for the particular problem of a TLS with periodic driving, the classical limit is already equivalent to the quantum problem [57, 58]. More precisely, to each periodic solution of the classical equation of motion, there exists a Floquet solution of the original Schrödinger equation that can be explicitly calculated via integrations. Especially, the quasienergy is essentially given

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by the action integral over one period of the classical solution. This is reminiscent of the semiclassical Floquet theory developed in a study by Breuer et al. [59].

The motion of a classical spin vector $\mathbf{S}(\tau)$ in a monochromatical magnetic field with elliptic polarization and an orthogonal constant component can be analyzed by following an approach analogous to that leading to the confluent Heun equation in a study by Ma and Li [47] and Xie and Hai [48]. We differentiate the first-order equation of motion twice and eliminate two components of the spin vector. The resulting third-order differential equation for the remaining component $x(\tau)$ can be transformed into a differential equation with polynomial coefficients by the change from the dimensionless time variable τ to $u = \sin^2\frac{\tau}{2}$. The latter differential equation is solved by a power series in u such that its coefficients satisfy a six terms recurrence relation. The second component $y(\tau)$ can be treated in the same way, whereas the third component $z(\tau)$ is obtained in a different way. As in the RPL case, the transformation from τ to u is confined to the half period $0 < \tau < \pi$ and, and moreover, the resulting power series diverges for $u = 1$ corresponding to $\tau = \pi$. Hence, it is necessary to reduce the full-time evolution of the classical spin vector to the first quarter period. This is done analogously to the procedure in a study by Schmidt [53] utilizing the discrete symmetries of the polarization ellipse.

The structure of the paper is the following. In Section 2, we present the scenario of the classical Rabi problem with elliptic polarization and its connection to the underlying Schrödinger equation. The abovementioned reduction of the time evolution to the first quarter period is made in Section 3. Already in the following Section 4, before solving the equation of motion, it can be shown that the fully periodic monodromic matrix depends only on two parameters r and α, which determine the quasienergy and the initial value of the periodic solution $\mathbf{S}(\tau)$, respectively. The Fourier series of this solution necessarily have the structure of an even/odd cos-series for $x(\tau)/y(\tau)$ and an odd sin-series for $z(\tau)$. The third-order differential equations for $X(u(\tau)) = x(\tau)$ and $Y(u(\tau)) = y(\tau)$ and their power series solutions are derived in Section 5. First consequences of this solution for the Fourier series coefficients and the parameters r and α are considered in Section 6. In order to check our results obtained so far, we consider, in Section 7, an example of the time evolution with simple values of the parameters of the polarization ellipse and two different initial values. On the one hand, we calculate the time evolution by using 10 terms of the abovementioned power series solutions for the first quarter period and extend the result to the full period. One the other hand, we numerically calculate the time evolution and find satisfactory agreement between both methods.

The quasienergy is discussed in more details in Section 8 with the emphasis on curves in parameter space where it vanishes. The resonance frequencies $\omega_{res}^{(n)}$ can be expressed in terms of power series in the variables F and G denoting the semiaxes of the polarization ellipse and compared with known results for the limit cases of linear and circular polarization, see Section 9. The next Section 10 is devoted to the discussion of further limit cases along the lines of [57]. In the adiabatic limit of vanishing driving frequency $\omega \to 0$, the spin vector follows the direction of the magnetic field, see Section 10.1. The corresponding quasienergy can be expressed through a complete elliptic integral of the second kind. The next two order corrections proportional to ω^1 and ω^2 can be obtained recursively and yield a kind of asymptotic envelope of a certain branch of the quasienergy as a function of ω. In the next limit case of $F,\ G \to 0$ in Section 10.2, the solution S(t) and the quasienergy can be written in the form of a so-called Fourier–Taylor series. This series is also of interest for the limit case of vanishing energy level splitting $\omega_0 \to 0$ in Section 10.3, where it replaces the exact solution of the RPL for $\omega_0 = 0$, and allows analytical approximations for the further limit cases $F \to 0$ and $F \to G$. An application concerning the work performed on a TLS by an elliptically polarized field is given in Section 11. We close with a summary and outlook in Section 12.

2 The classical Rabi problem: general definitions and results

We consider the Schrödinger equation

$$\mathrm{i}\hbar \frac{d}{dt}\Psi(t) = H(t)\,\Psi(t)\,, \tag{1}$$

of a spin with quantum number $s = 1/2$, $\Psi(t) = \begin{pmatrix} \Psi_1(t) \\ \Psi_2(t) \end{pmatrix}$ and a time-dependent, periodic Hamiltonian

$$H(t) = \frac{\hbar}{2}\left(\omega_0\,\sigma_1 + G\cos(\omega t)\,\sigma_2 + F\,\sin(\omega t)\,\sigma_3\right), \tag{2}$$

where the σ_i, $i = 1, 2, 3$, are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Hence $H(t)$ can be understood as a Zeeman term w. r. t. a (dimensionless) magnetic field

$$\mathbf{H}(t) = \begin{pmatrix} \omega_0 \\ G\cos\,\omega t \\ F\sin\,\omega t \end{pmatrix}. \tag{4}$$

Alternatively, $\frac{\hbar}{2}\omega_0\,\sigma_1$ can be understood as the zero field Hamiltonian of a TLS and (4) without the constant component as a monochromatic, elliptically polarized magnetic field.

Setting $\hbar = 1$ and passing to a dimensionless time variable $\tau = \omega t$ we may rewrite (1) in the form

$$\mathrm{i}\,\frac{d}{d\tau}\begin{pmatrix}\psi_1(\tau)\\ \psi_2(\tau)\end{pmatrix} = \frac{1}{2}\begin{pmatrix} f\,\sin\,\tau & \nu - g\,\cos\,\tau\\ \nu + g\,\cos\,\tau & -f\,\sin\,\tau\end{pmatrix}\begin{pmatrix}\psi_1(\tau)\\ \psi_2(\tau)\end{pmatrix}, \tag{5}$$

where $G = g\omega$, $F = f\omega$ and $\omega_0 = \nu\omega$. The dimensionless period is always $T\omega = 2\pi$. Sometimes, we will denote the derivative w. r. t. τ by an overdot $\frac{d}{d\tau} = \dot{}$.

Let

$$P(\tau) = \left|\begin{pmatrix}\psi_1(\tau)\\ \psi_2(\tau)\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\psi_1(\tau)\\ \psi_2(\tau)\end{pmatrix}\right| \tag{6}$$

denote the one-dimensional time-dependent projector onto a solution of (5) and

$$P(\tau) = \frac{1}{2}\left(1 + x(\tau)\ \sigma_1 + y(\tau)\ \sigma_2 + z(\tau)\ \sigma_3\right) \tag{7}$$

its expansion w. r. t. the basis $(1, \sigma_1, \sigma_2, \sigma_3)$ of Hermitean 2×2-matrices. It follows that the vector $\mathbf{S}(\tau) = (x(\tau), y(\tau), z(\tau))^\top$ satisfies the classical equation of motion

$$\frac{d}{d\tau}\mathbf{S}(\tau) = \mathbf{h}(\tau) \times \mathbf{S}(\tau), \tag{8}$$

and hence $\mathbf{S}(\tau)$ can be viewed as a classical spin vector (not necessarily normalized). Moreover,

$$\mathbf{h}(\tau) = \begin{pmatrix} h_1\\ h_2\\ h_3\end{pmatrix} = \begin{pmatrix}\nu\\ g\,\cos\,\tau\\ f\,\sin\,\tau\end{pmatrix} \tag{9}$$

denotes the dimensionless magnetic field vector (4) written as a function of τ.

Conversely, to each solution of (8) one obtains the corresponding solution of (5) up to a time-dependent phase that can be obtained by an integration, see [57] for the details.

The coefficients of the Taylor series w. r. t. τ of $x(\tau)$, $y(\tau)$ and $z(\tau)$ can be recursively determined by using (8) and the initial values $x(0)$, $y(0)$ and $z(0)$. Note that h_1 and h_2 are even functions of τ and that h_3 is an odd one. Hence, there exist special solutions of (8) such that $x(\tau)$ and $y(\tau)$ are even functions of τ and $z(\tau)$ is an odd one, symbolically:

$$\mathbf{S}(\tau) = \begin{pmatrix}\text{even}\\ \text{even}\\ \text{odd}\end{pmatrix}. \tag{10}$$

In fact, this is consistent with (8) and (9) since

$$\frac{d}{d\tau}\mathbf{S}(\tau) = \begin{pmatrix}\text{odd}\\ \text{odd}\\ \text{even}\end{pmatrix}, \tag{11}$$

and

$$\mathbf{h} \times \mathbf{S} = \begin{pmatrix}\text{even}\\ \text{even}\\ \text{odd}\end{pmatrix} \times \begin{pmatrix}\text{even}\\ \text{even}\\ \text{odd}\end{pmatrix} = \begin{pmatrix}\text{odd}\\ \text{odd}\\ \text{even}\end{pmatrix}, \tag{12}$$

and can be proven by induction over the degree of the Taylor series coefficients of $\mathbf{S}(\tau)$ using the necessary initial condition $z(0) = 0$.

Analogously, there exist solutions $\mathbf{S}(\tau)$ of type

$$\mathbf{S}(\tau) = \begin{pmatrix}\text{odd}\\ \text{odd}\\ \text{even}\end{pmatrix}. \tag{13}$$

satisfying $x(0) = y(0) = 0$. We will state these results in the following form:

Proposition 1: *1. The solution* $\mathbf{S}(\tau)$ *of* (8) *is of type* (10) *iff* $z(0) = 0$.
1. Analogously, the solution $\mathbf{S}(\tau)$ *of* (8) *is of type* (13) *iff* $x(0) = y(0) = 0$.

For general initial conditions, the solution $\mathbf{S}(\tau)$ of (8) will be of mixed type.

Next, let $\mathbf{S}^{(i)}(\tau), i = 1, 2, 3$, denote the three solutions of (8) with initial conditions $\mathbf{S}^{(i)}_j(\tau_0) = \delta_{ij}$ and $R(\tau, \tau_0)$ be the 3×3-matrix with columns $\mathbf{S}^{(i)}(\tau)$. Since the $\mathbf{S}^{(i)}(\tau)$ are mutually orthogonal and right-handed for $\tau = \tau_0$ this holds for all $\tau \in \mathbb{R}$ and hence $R(\tau, \tau_0) \in SO(3)$. It satisfies the differential equation

$$\frac{d}{d\tau}R(\tau, \tau_0) = H(\tau)\ R(\tau, \tau_0), \tag{14}$$

with initial condition

$$R(\tau_0, \tau_0) = 1. \tag{15}$$

Here, $H(\tau) \in so(3)$ is the real antisymmetric 3×3-matrix corresponding to $\mathbf{h}(\tau)$, i.e.,

$$H(\tau) = \begin{pmatrix} 0 & -f\,\sin\,\tau & g\,\cos\,\tau\\ f\,\sin\,\tau & 0 & -\nu\\ -g\,\cos\,\tau & \nu & 0\end{pmatrix}. \tag{16}$$

The differential Equation (14) with initial condition (15) has a unique solution $R(\tau, \tau_0)$ for all $\tau, \tau_0 \in \mathbb{R}$, see, e.g., theorem 3.9 in [29]. Obviously, this implies the composition law

$$R(\tau_2, \tau_0) = R(\tau_2, \tau_1)\ R(\tau_1, \tau_0), \tag{17}$$

and hence

$$R(\tau_2,\tau_1)^{-1} = R(\tau_1,\tau_2)\,, \tag{18}$$

for all $\tau_0, \tau_1, \tau_2 \in \mathbb{R}$.

Usually, we will set $\tau_0 = 0$. The matrix $H(\tau)$ is obviously 2π-periodic. Hence, we may apply Floquet theory to the classical equation of motion (8). The monodromy matrix $R(2\pi, 0)$ has the eigenvalues $\{1, \exp(\pm \mathrm{i}\,\rho\}\}$ which leads to the corresponding classical quasienergy (or Floquet exponent) of the form

$$\epsilon^{(cl)} = 0, \pm\frac{\rho}{2\pi}\,, \tag{19}$$

uniquely defined up to integer multiples (note that effectively $\omega = 1$ in our approach).

The connection to the quasienergy $\epsilon^{(qu)}$ of the underlying spin $s = \frac{1}{2}$ Schrödinger Equation (5) can be given in two ways. Either we may utilize the fact that the classical Rabi problem can be understood as the "lift" of the spin $s = \frac{1}{2}$ problem to spin $s = 1$. Then Equation (38) of [57] implies

$$\epsilon^{(cl)} = 2m\,\epsilon^{(qu)}, \quad \text{where } m = -1, 0, 1. \tag{20}$$

Taking into account the mentioned ambiguity of $\epsilon^{(cl)}$, this means that we have two possibilities: Either $\epsilon^{(qu)} = \pm\frac{1}{2}\epsilon^{(cl)}$ or $\epsilon^{(qu)} = \frac{1}{2}(1 \pm \epsilon^{(cl)})$. Since we have, modulo integers, only two values for $\epsilon^{(qu)}$ these two possibilities are generally exclusive. One way to decide between the two possibilities would be to utilize the well-known quasienergies for the RPC, that agree with the case $\epsilon^{(qu)} = \frac{1}{2}(1 \pm \epsilon^{(cl)})$, and to argue with continuity.

Another way to obtain $\epsilon^{(qu)}$ would be to follow the prescription given in the study by Schmidt [57] and write it as follows:

$$\epsilon^{(qu)} = \overline{\frac{1}{2}\left(h_1 + \frac{h_2 y + h_3 z}{1+z}\right)}, \tag{21}$$

where the overline indicates the time average over one period of a 2π-periodic solution $\mathbf{S}(\tau)$ of (8). An equivalent expression, that is manifestly invariant under rotations, is given by

$$\epsilon^{(cl)} = \overline{\mathbf{h}\cdot\mathbf{S} - \frac{\mathbf{S}\cdot\left(\dot{S}\times\ddot{\mathbf{S}}\right)}{\dot{S}\cdot\dot{S}}}, \tag{22}$$

see Eq. (46) in [58]. Periodic solutions of (8) can be found by using the initial value $\mathbf{S}(0) = \mathbf{r}$, where $\mathbf{r}$ is the normalized eigenvector of $R(2\pi, 0)$ corresponding to the eigenvalue 1, see also in the study by Schmidt et al. [58].

Of course, both ways, (20) and (21), to obtain $\epsilon^{(qu)}$ agree within the usual ambiguity modulo integers. This will be explicitly checked in Section 8 for the case of circular polarization.

3 Reduction to the first quarter period

Due to the discrete symmetries of the polarization ellipse, it is possible to reduce the time evolution of the classical spin to the first quarter period $\tau \in \left[0, \frac{\pi}{2}\right]$. This is similar to the corresponding considerations in [53]. Let $T^{(i)}$, $i = 1, 2, 3$ denote the involutory diagonal 3×3-matrices with entries $T^{(i)}_{jk} = (-1)^{\delta_{ij}}\,\delta_{jk}$, for example,

$$T^{(1)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{23}$$

and $T^{(ij)} \equiv T^{(i)}\,T^{(j)}$, for example,

$$T^{(13)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{24}$$

First, we will formulate a proposition that allows us to reduce the time evolution for the classical spin from the full period to the first half period $\tau \in [0, \pi]$.

Proposition 2:

$$R(\pi+\tau, 0) = T^{(1)} R(\tau, 0) T^{(1)} R(\pi, 0) \tag{25}$$

for all $\tau \in \mathbb{R}$.

Proof: Let $\tilde{R}(\tau) \equiv T^{(1)} R(\pi+\tau, \pi) T^{(1)}$ such that $\tilde{R}(0) = T^{(1)} R(\pi, \pi) T^{(1)} = \mathbb{1}$. It satisfies the differential equation

$$\frac{d}{d\tau}\tilde{R}(\tau) = T^{(1)}\left(\frac{d}{d\tau} R(\pi+\tau, \pi)\right) T^{(1)} \tag{26}$$

$$\overset{(14)}{=} T^{(1)}\left(H(\pi+\tau)\; R(\pi+\tau, \pi)\right) T^{(1)} \tag{27}$$

$$= \left(T^{(1)} H(\pi+\tau) T^{(1)}\right)\left(T^{(1)} R(\pi+\tau, \pi)\, T^{(1)}\right) \tag{28}$$

$$= H(\tau)\left(T^{(1)} R(\pi+\tau, \pi) T^{(1)}\right) \tag{29}$$

$$= H(\tau)\; \tilde{R}(\tau). \tag{30}$$

In (29) we have used that $\sin(\pi+\tau) = -\sin\tau$, $\cos(\pi+\tau) = -\cos\tau$ and hence

$$T^{(1)}\, H(\pi+\tau)\, T^{(1)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & f\sin\tau & -g\cos\tau \\ -f\sin\tau & 0 & -\nu \\ g\cos\tau & \nu & 0 \end{pmatrix} \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{31}$$

$$= \begin{pmatrix} 0 & -f\sin\tau & g\cos\tau \\ f\sin\tau & 0 & -\nu \\ -g\cos\tau & \nu & 0 \end{pmatrix} = H(\tau). \quad (32)$$

It follows that $\tilde{R}(\tau)$ satisfies the same differential equation and initial condition as $R(\tau, 0)$ and hence $T^{(1)} R(\pi+\tau,\pi) T^{(1)} = \tilde{R}(\tau) = R(\tau, 0)$. Consequently,

$$R(\pi+\tau,0) \overset{(17)}{=} R(\pi+\tau,\pi)R(\pi,0) = T^{(1)}R(\tau,0)T^{(1)}R(\pi,0), \quad (33)$$

which completes the proof of the proposition.

Setting $\tau = \pi$ in (25) gives

$$R(2\pi,0) = T^{(1)}R(\pi,0)T^{(1)}R(\pi,0) = \left(T^{(1)}R(\pi,0)\right)^2. \quad (34)$$

Next, we show how to further reduce the time evolution to the first quarter period $\tau \in \left[0, \frac{\pi}{2}\right]$.

Proposition 3:

$$R(\pi-\tau,0) = T^{(13)}R(\tau,0)\,T^{(13)}R(\pi,0) \quad (35)$$

for all $\tau \geq 0$.

Proof: The proof is similar to that of proposition 2 except that an additional time reflection is involved. Let $\tilde{R}(\tau) \equiv T^{(13)}R(\pi-\tau,\pi)\,T^{(13)}$ such that $\tilde{R}(0) = T^{(13)}R(\pi,\pi)\,T^{(13)} = 1$. It satisfies the differential equation

$$\frac{d}{d\tau}\tilde{R}(\tau) = T^{(13)}\left(\frac{d}{d\tau}R(\pi-\tau,\pi)\right)T^{(13)} \quad (36)$$

$$\overset{(14)}{=} T^{(13)}\left(-H(\pi-\tau)R(\pi-\tau,\pi)\right)T^{(13)} \quad (37)$$

$$= \left(T^{(13)}\left(-H(\pi-\tau)\,T^{(13)}\right)\right)\left(T^{(13)}R(\pi-\tau,\pi)T^{(13)}\right) \quad (38)$$

$$= H(\tau)\left(T^{(13)}R(\pi-\tau,\pi)T^{(13)}\right) \quad (39)$$

$$= H(\tau)\tilde{R}(\tau). \quad (40)$$

In (39), we have used that $\sin(\pi-\tau) = \sin\tau$, $\cos(\pi-\tau) = -\cos\tau$ and hence

$$T^{(13)}\,(-H(\pi-\tau))\,T^{(13)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & f\sin\tau & g\cos\tau \\ -f\sin\tau & 0 & \nu \\ -g\cos\tau & -\nu & 0 \end{pmatrix} \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (41)$$

$$= \begin{pmatrix} 0 & -f\sin\tau & g\cos\tau \\ f\sin\tau & 0 & -\nu \\ -g\cos\tau & \nu & 0 \end{pmatrix} = H(\tau). \quad (42)$$

It follows that $\tilde{R}(\tau)$ satisfies the same differential equation and initial condition as $R(\tau, 0)$ and hence

$$T^{(13)}R(\pi-\tau,\pi)T^{(13)} = \tilde{R}(\tau) = R(\tau,0). \quad (43)$$

Consequently,

$$R(\pi-\tau,0) \overset{(17)}{=} R(\pi-\tau,\pi)R(\pi,0) = T^{(13)}R(\tau,0)T^{(13)}R(\pi,0), \quad (44)$$

which completes the proof of the proposition.

Setting $\tau = \pi$ in (43) implies

$$T^{(13)}R(0,\pi)T^{(13)} = R(\pi,0), \quad (45)$$

and hence

$$R(0,\pi) \overset{(18)}{=} R(\pi,0)^\top = T^{(13)}R(\pi,0)T^{(13)}. \quad (46)$$

Moreover, if we set $\tau = \frac{\pi}{2}$ in (35), we obtain

$$R\left(\frac{\pi}{2},0\right) = T^{(13)}R\left(\frac{\pi}{2},0\right)T^{(13)}R(\pi,0), \quad (47)$$

and hence, solving for $R(\pi, 0)$,

$$R(\pi,0) = T^{(13)}R\left(\frac{\pi}{2},0\right)^\top T^{(13)}R\left(\frac{\pi}{2},0\right). \quad (48)$$

Thus (35) can be re-written as

$$R(\pi-\tau,0) = T^{(13)}R(\tau,0)R\left(\frac{\pi}{2},0\right)^\top T^{(13)}R\left(\frac{\pi}{2},0\right), \quad (49)$$

and hence the evolution data for $\tau \in \left[\frac{\pi}{2}, \pi\right]$ can be completely written in terms of those for $\tau \in \left[0, \frac{\pi}{2}\right]$. Together with (25), this shows that the complete time evolution can be reduced to that in the first quarter period.

4 Fourier series and quasienergy: preliminary results

First we will re-derive (46) under more general assumptions.

Proposition 4: *Let $R \in SO(3)$ and $T \in O(3)$ be such that $T^2 = 1$ and hence $T^\top = T$. Define $\tilde{R} \in SO(3)$ by*

$$\tilde{R} \equiv T R^\top T R, \quad (50)$$

then

$$\tilde{R}^\top = T\tilde{R}T \quad (51)$$

holds.

Proof: $\tilde{R}^\top = (TR^\top TR)^\top = R^\top TRT = T^2R^\top TRT = T\,(TR^\top TR)\,T = T\tilde{R}T$.

Let us specialize to the case $T = T^{(13)}$, then (51) is equivalent to the following three equations:

$$\tilde{R}_{12} = -\tilde{R}_{21}, \quad \tilde{R}_{13} = \tilde{R}_{31}, \quad \tilde{R}_{23} = -\tilde{R}_{32}. \tag{52}$$

A general rotational matrix $\tilde{R} \in SO(3)$ can be determined by three real parameters; by the three Equation (52) the number of parameters can be reduced to two:

Proposition 5: *Every rotational matrix $\tilde{R} \in SO(3)$ satisfying (52) will be of the form*

$$\tilde{R} = \begin{pmatrix} r^2 + (1-r^2)\cos(2\alpha) & (1-r^2)\sin(2\alpha) & 2r\sqrt{1-r^2}\sin(\alpha) \\ -(1-r^2)\sin(2\alpha) & (1-r^2)\cos(2\alpha) - r^2 & 2r\sqrt{1-r^2}\cos(\alpha) \\ 2r\sqrt{1-r^2}\sin(\alpha) & -2r\sqrt{1-r^2}\cos(\alpha) & 1-2r^2 \end{pmatrix}, \tag{53}$$

where $r \in [0, 1]$ and $\alpha \in [0, 2\pi]$.

Proof: Obviously, the third column $\tilde{R}_3$ of $\tilde{R}$ according to (53) is the most general form of a unit vector. The second column $\tilde{R}_2$ must be a unit vector orthogonal to $\tilde{R}_3$ with a given component $\tilde{R}_{3,2} = -\tilde{R}_{2,3} = -2r\sqrt{1-r^2}\cos(\alpha)$. If $r > 0$ there are only two possibilities for $\tilde{R}_2$: the first one is given by (53) and the second one is $\mathbf{b} = (-r^2\sin(2\alpha), -r^2\cos(2\alpha) - r^2 + 1, -2r\sqrt{1-r^2}\cos(\alpha))^\top$. The first column of $\tilde{R}$ is uniquely given by $\tilde{R}_1 = \tilde{R}_2 \times \tilde{R}_3$, but $\tilde{R}_2 = \mathbf{b}$ does not yield a matrix satisfying (52) and hence has to be excluded.

We have still to consider the case $r = 0$ such that $\tilde{R}_3 = (0,0,1)^\top$. Then the representation (53) reduces to

$$\begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) & 0 \\ -\sin(2\alpha) & \cos(2\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{54}$$

which is obviously the most general case satisfying (52) and $\tilde{R}_3 = (0,0,1)^\top$.

Recall that the "half period monodromy matrix" $R(\pi, 0)$ satisfies (46), hence, according to Prop. 4, also (52) and, by virtue of Prop. 5, must be of the form (53). In the case of linear polarization ($g = 0$), this result also follows from the form of the half period monodromy matrix $U(\pi, 0)$ of the corresponding Schrödinger equation, see Equation (30) in [53], where the parameters r and α have the same meaning as in this paper. Using (34), we can immediately derive the form of the full period monodromy matrix

It will be instructive to sketch another derivation of (55). To this end, we state without proof that the monodromy matrix of the Schrödinger Equation (5) will assume the form

$$U = U(2\pi, 0) = \begin{pmatrix} 1-2r^2 & 2ir\sqrt{1-r^2}e^{-i\alpha} \\ 2ir\sqrt{1-r^2}e^{i\alpha} & 1-2r^2 \end{pmatrix}, \tag{56}$$

completely analogous to Eq. (33) of [53]. U has the eigenvalues $\exp(\pm 2i\ \arcsin r)$ with respective eigenvectors $(e^{\pm i\alpha}, 1)^\top$. Then the corresponding monodromy matrix ρ of the classical RPE is given by the equation

$$U\,\sigma_i\,U^* = \sum_{j=1}^{3} \rho_{j,i}\sigma_j, o \tag{57}$$

where the σ_i are the Pauli matrices (3). It is easy to check that the so defined matrix ρ coincides with $R(2\pi, 0)$ given by (55).

Like $R(\pi, 0)$ also $R(2\pi, 0)$ depends only on two parameters α and r and satisfies a similar equation that characterizes the corresponding two-dimensional submanifold of $SO(3)$, to wit,

$$R(2\pi, 0)^\top = T^{(3)} R(2\pi, 0) T^{(3)}. \tag{58}$$

This equation can be proven either directly by checking (55) or by applying (34) and (46).

According to the general theory [57], the eigenvalues of $R(2\pi, 0)$ that are generally of the form $(1,\ \exp(\pm i\rho))$ yield the quasienergies $\epsilon_\pm^{(qu)}$ of the underlying Schrödinger equation for spin $s = \frac{1}{2}$ via

$$\exp \pm i\rho = \exp\left(4\pi i\,\epsilon_\pm^{(qu)}\right). \tag{59}$$

As in [53] it follows that

$$\begin{aligned} \epsilon_\pm^{(qu)} &= \pm\frac{1}{4\pi}\arg\left(1 + 8r^4 - 8r^2 + 4ir(1-2r^2)\sqrt{1-r^2}\right) \\ &= \pm\frac{1}{\pi}\arcsin r. \end{aligned} \tag{60}$$

The eigenvector $\mathbf{r}$ corresponding to the real eigenvalue 1 of $R(2\pi, 0)$ is

$$\mathbf{r} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}. \tag{61}$$

Choosing $\mathbf{r}$ as the initial value $\mathbf{r} = \mathbf{S}(0)$ for the time evolution (8) yields a 2π-periodic solution. Any other unit vector in the plane P orthogonal to $\mathbf{r}$ will, in general, not return to its initial value after the time $\tau = 2\pi$ but will be

$$R(2\pi, 0) = \begin{pmatrix} (1-2r^2)^2 + 4r^2(1-r^2)\cos(2\alpha) & 4r^2(1-r^2)\sin(2\alpha) & 4r\sqrt{1-r^2}(2r^2-1)\sin(\alpha) \\ 4r^2(1-r^2)\sin(2\alpha) & 4(r^2-1)r^2\cos(2\alpha) + (1-2r^2)^2 & 4r(1-2r^2)\sqrt{1-r^2}\cos(\alpha) \\ 4r(1-2r^2)\sqrt{1-r^2}\sin(\alpha) & 4r\sqrt{1-r^2}(2r^2-1)\cos(\alpha) & 8r^4 - 8r^2 + 1 \end{pmatrix}. \tag{55}$$

rotated in the plane P by the angle $2\pi\epsilon^{(cl)}$. This endows the parameters r and α occurring in (53) and (55) with a geometrical and dynamical meaning.

Another remarkable result follows from $R(\pi, 0)$ being of the form (53):

$$R(\pi,0)\begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix}=\begin{pmatrix}\cos\alpha\\ -\sin\alpha\\ 0\end{pmatrix}, \tag{62}$$

which means that for the initial value $\mathbf{S}(0)=\mathbf{r}$ the half period time evolution is equivalent to a reflection at the x–z-plane. This has further consequences for the Fourier series of the 2π-periodic functions $x(\tau)$, $y(\tau)$, and $z(\tau)$ with initial values $x(0)=\cos\alpha, y(0)=\sin\alpha$, and $z(0)=0$. Since $x(\tau)$ and $y(\tau)$ will be even functions of τ and $z(\tau)$ will be an odd one, see Proposition 1, we can write their Fourier series in the form

$$x(\tau)=\sum_{\mu=0}^{\infty}x_\mu\,\cos(\mu\tau), \tag{63}$$

$$y(\tau)=\sum_{\mu=0}^{\infty}y_\mu\,\cos(\mu\tau), \tag{64}$$

$$x(\tau)=\sum_{\mu=1}^{\infty}z_\mu\,\sin(\mu\tau). \tag{65}$$

Now consider the sequence of linear mappings

$$\begin{aligned}\mathbf{r}=\begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix}&\overset{R(\pi,0)}{\to}\begin{pmatrix}\cos\alpha\\ -\sin\alpha\\ 0\end{pmatrix}\overset{T^{(1)}}{\to}\begin{pmatrix}-\cos\alpha\\ -\sin\alpha\\ 0\end{pmatrix}\\ &\overset{R(\tau,0)}{\to}\begin{pmatrix}-x(\tau)\\ -y(\tau)\\ -z(\tau)\end{pmatrix}\overset{T^{(1)}}{\to}\begin{pmatrix}x(\tau)\\ -y(\tau)\\ -z(\tau)\end{pmatrix}\\ &\overset{(29)}{=}R(\pi+\tau,0)\mathbf{r}=\begin{pmatrix}x(\pi+\tau)\\ y(\pi+\tau)\\ z(\pi+\tau)\end{pmatrix}.\end{aligned} \tag{66}$$

From this we conclude

$$\begin{aligned}x(\pi+\tau)&=\sum_{\mu=0}^{\infty}x_\mu\,\cos(\mu(\pi+\tau))\\ &=\sum_{\mu\text{ even}}x_\mu\,\cos(\mu\tau)-\sum_{\mu\text{ odd}}x_\mu\cos(\mu\tau)=x(\tau)\\ &=\sum_{\mu\text{ even}}x_\mu\cos(\mu\tau)+\sum_{\mu\text{ odd}}x_\mu\,\cos(\mu\tau).\end{aligned} \tag{67}$$

Hence, the odd terms of the cos-series must vanish and $x(\tau)$ is an even cos-series. Similarly, we conclude from (66) that $y(\tau)$ is an odd cos-series and $z(\tau)$ an odd sin-series. Summarizing, we have proven the following

Proposition 6: *The components of the 2π-periodic solution $\mathbf{S}(\tau)$ of (8) with initial values $\mathbf{S}(0)=\mathbf{r}$ according to (61) have the Fourier series*

$$x(\tau)=\sum_{\mu\text{ even}}x_\mu\cos(\mu\tau), \tag{68}$$

$$y(\tau)=\sum_{\mu\text{ odd}}y_\mu\cos(\mu\tau), \tag{69}$$

$$z(\tau)=\sum_{\mu\text{ odd}}z_\mu\sin(\mu\tau). \tag{70}$$

In particular, the time averages of $y(\tau)$ and $z(\tau)$ over one period vanish.

5 Third order differential equations for single spin components

We consider again (8) and its higher derivatives that read

$$\frac{d}{d\tau}\mathbf{S}=\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix}=\begin{pmatrix}gz\,\cos\tau-fy\,\sin\tau\\ fx\,\sin\tau-vz\\ vy-gx\,\cos\tau\end{pmatrix}, \tag{71}$$

$$\frac{d^2}{d\tau^2}\mathbf{S}=\begin{pmatrix}\ddot x\\ \ddot y\\ \ddot z\end{pmatrix}=\begin{pmatrix}-\sin\tau\,(f^2x\,\sin\tau+z\,(g-fv))+y\,\cos\tau\,(gv-f)-g^2x\,\cos^2\tau\\ \cos\tau\,(x\,(f+gv)+f\,g\,z\,\sin\tau)-y\,(f^2\sin^2\tau+v^2)\\ \sin\tau\,(x\,(fv+g)+f\,g\,y\,\cos\tau)-z\,(g^2\cos^2\tau+v^2)\end{pmatrix}, \tag{72}$$

$$\frac{d^3}{d\tau^3}\mathbf{S}=\begin{pmatrix}\dddot x\\ \dddot y\\ \dddot z\end{pmatrix}=:x\,\mathbf{S}_1^{(3)}+y\,\mathbf{S}_2^{(3)}+z\,\mathbf{S}_3^{(3)}, \tag{73}$$

with

$$\mathbf{S}_1^{(3)}=\begin{pmatrix}-3(f^2-g^2)\sin\tau\,\cos\tau\\ -\sin\tau\,(f^3\,\sin^2\tau+fg^2\cos^2\tau+fv^2+f+gv)\\ \cos\tau\,(f^2g\,\sin^2\tau+fv+g^3\cos^2\tau+gv^2+g)\end{pmatrix}, \tag{74}$$

$$\mathbf{S}_2^{(3)}=\begin{pmatrix}\sin\tau\,(f^3\sin^2\tau+f\,g^2\cos^2\tau+fv^2+f-2gv)\\ -3f^2\sin\tau\cos\tau\\ -f\,\sin^2\tau\,(fv+2g)-g\,\cos^2\tau\,(gv-f)-v^3\end{pmatrix}, \tag{75}$$

and

$$\mathbf{S}_3^{(3)}=\begin{pmatrix}-\cos\tau\,(f^2g\,\sin^2\tau-2fv+g^3\,\cos^2\tau+gv^2+g)\\ f\,\sin^2\tau\,(fv-g)+g\,\cos^2\tau\,(2f+gv)+v^3\\ 3g^2\,\sin\tau\,\cos\tau\end{pmatrix}. \tag{76}$$

It is obvious that $\dot x$ and $\ddot x$ depend linearly on y and z and that this dependence can be inverted to express y and z in terms of x, $\dot x$ and $\ddot x$. Inserting this result into $\dddot x$ yields a third-order linear differential equation for $x(\tau)$, where the coefficients are trigonometric functions of τ.

Similarly, we can obtain third-order differential equations for $y(\tau)$ and $z(\tau)$. For the preparation of the next step, we make the restriction to solutions of (71) such that $x(\tau)$ and $y(\tau)$ are even functions of τ, whereas $z(\tau)$ is an odd one, according to Prop. 1. In this way, we could obtain two solutions $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ with different initial conditions for $x(\tau)$ and $y(\tau)$ and the initial condition $z(0) = 0$, the latter being a consequence of the restriction to odd functions $z(\tau)$. The third solution $\mathbf{S}^{(3)}$ with $x(\tau)$ and $y(\tau)$ odd and $z(\tau)$ even is then uniquely determined by $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$. For example, if $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are chosen to be orthogonal for $\tau = 0$ then they will be orthogonal for all τ and $\mathbf{S}^{(3)}$ is just the vector product of $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$.

Following the study by Xie and Hai [48], we will consider a transformation $\tau \mapsto u$ of the independent variable such that the coefficients of the transformed differential equations become rational functions of u. This transformation will be chosen as

$$u(\tau) = \sin^2\frac{\tau}{2} = \frac{1}{2}(1 - \cos\tau), \tag{77}$$

the same as in the study by Xie and Hai [48], and maps the half period $\tau \in [0, \pi]o$ bijectively onto $u \in [0, 1]$. Since (77) defines an even function of τ the corresponding transformation is only appropriate for the even functions $x(\tau)$ and $y(\tau)$. Their transforms will be denoted by $X(u)$ and $Y(u)$ such that

$$X(u(\tau)) = x(\tau), \quad \text{and } Y(u(\tau)) = y(\tau) \text{for } \tau \in [0, \pi]. \tag{78}$$

The remaining function $z(\tau)$ has to be calculated differently, e.g., by using that the length of $\mathbf{S}(\tau)$ is conserved under time evolution according to (8). This gives the result

$$z(\tau) = \pm\sqrt{x(0)^2 + y(0)^2 - x(\tau)^2 - y(\tau)^2}, \tag{79}$$

where $z(0) = 0$ has been used, and the sign has to be chosen in such a way that $z(\tau)$ remains a smooth function in the neighborhood of its zeros. An alternative procedure would be possible if $x(\tau)$ and $y(\tau)$ can be written as Fourier series (maybe only locally valid for $\tau \in [0, \pi/2]$). Then $z(\tau)$ could be obtained by a direct integration of $\dot{z}(\tau) = v y(\tau) - g x(\tau)\cos(\tau)$. This last procedure will be applied in Section 7.

We come back to the differential equation for $X(u)$ and write it with polynomial coefficients $p_n(u)$ in the form

$$0 = \sum_{n=0}^{3} p_n(u) X^{(n)}(u). \tag{80}$$

The coefficients are the following ones:

$$p_3(u) = u(1-u)\left(4f^2 v(u-1)u + fg - g^2 v(1-2u)^2\right) \tag{81}$$

$$p_2(u) = -\frac{1}{2}(2u-1)\left(4f^2 v(u-1)u + 3fg + g^2 v(-4(u-1)u - 3)\right) \tag{82}$$

$$\begin{aligned} p_1(u) = &-16f^4 v(u-1)^2 u^2 - 4f^3 g(u-1)u \\ &+4f^2 v(u-1)u\left(2g^2(1-2u)^2 + v^2\right) \\ &+fg^3(1-2u)^2 + 3fgv^2 - g^2 v\left(g^2(1-2u)^4 + v^2(1-2u)^2 + 2\right) \end{aligned} \tag{83}$$

$$\begin{aligned} p_0(u) = &-2(2u-1)(f-g)(f+g)\left(4f^2 v(u-1)u + 3fg\right. \\ &\left.- g^2 v(1-2u)^2\right). \end{aligned} \tag{84}$$

The singular points of the differential equation are the zeros of $p_3(u)$. Except the points $u = 0$ and $u = 1$ that occur also for the confluent Heun equation, see [48] and [53], we have an additional pair of singular points, real or complex ones, depending on the parameters f, g and v. The obvious ansatz to obtain a physically relevant solution of (80) is a power series

$$X(u) = \sum_{n=0}^{\infty} \xi_n u^n \tag{85}$$

at the singular point given by $u = 0$. We have not investigated its radius of convergence, but it is clear that the series diverges at least for the second singular point $u = 1$, which has been our motivation to restrict the application of (85) to $|u| \le \frac{1}{2}$ corresponding to the first quarter period $\tau \in [0, \pi/2]$. In contrast to a study by Xie and Hai [48], we need only one real solution and can neglect further solutions of the fundamental system. However, due to the degree three of the differential equation and the additional singular points we need a six-term recurrence relation for the coefficients of the power series.

We will not give the details of the recurrence relation but rather sketch how to obtain it by means of computer-algebraic aids. We take a finite part $\sum_{n=m-3}^{m+2} \xi_n u^n$ of the power series and insert it into the differential Equation (80). The result is expanded into a u-polynomial and the coefficient of u^m is set to 0. It has been checked that only the above considered finite part of the power series influences this coefficient. Thus, we obtain a six-term recurrence relation of the form

$$\xi_{m+2} = \sum_{i=m-3}^{m+1} a_i \xi_i, \tag{86}$$

where the a_i have been determined as rational functions of f, g, v, but they are too complicated to be presented here.

The next problem is that we need the first five coefficients of $X(u) = \sum_n \xi_n u^n$ to get the next coefficients using the recursion relation. Since the original Equation (8) is of the first order, we have only two undetermined initial values $x(0)$ and $y(0)$, taking into account that $z(0) = 0$. To solve this problem, we have compared the first terms of the τ-power series of $x(\tau)$ and $X(u(\tau))$, using the differential Equation (8), and thereby determined $\xi_0, \ldots, \xi_4$ as functions of $x(0)$ and $y(0)$. This also compensates the enlargement of the solution space by passing from a first-order differential equation to a third-order one. To give an impression of the kind of results, we display the first three coefficients:

$$\xi_0 = x(0) \tag{87}$$

$$\xi_1 = -2\left(y(0)(f - gv) + g^2 x(0)\right) \tag{88}$$

$$\begin{aligned}\xi_2 = \frac{1}{3}&\left(2x(0)\left(-3f^2 - 2fgv + g^2\left(g^2 + v^2 + 3\right)\right)\right.\\ &\left.+ 2y(0)\left(f\left(g^2 + 3v^2\right) - gv\left(g^2 + v^2 + 2\right)\right)\right).\end{aligned} \tag{89}$$

Obviously, ξ_n is a linear function of $x(0)$ and $y(0)$ that can be written as

$$\xi_n = \xi_n^{(x)} x(0) + \xi_n^{(y)} y(0). \tag{90}$$

After these preparations it is, in principle, possible to calculate any finite number of power series coefficients ξ_n as a function of the physical parameters f, g and v and the initial values $x(0)$ and $y(0)$, although the expressions become more and more intricate, and finally to obtain a truncated approximation of $X(u(\tau))$. For a comparison to a numerical solution of (8) see Section 8.

Analogous considerations apply for the case of the solution $y(\tau) = Y(u(\tau))$. This time we obtain a differential equation of the form

$$0 = \sum_{n=0}^{3} q_n(u) Y^{(n)}(u), \tag{91}$$

where

$$\begin{aligned}q_3(u) = &(u-1)\ u(2u-1)^3\left(f^2 g + fv + gv^2\right)\\ &\times\left(4f^2 g(u-1)u - fv - gv^2\right)\end{aligned} \tag{92}$$

$$\begin{aligned}q_2(u) = &\frac{1}{2}(1-2u)^2\left(f^2 g + fv + gv^2\right)\left(4f^2 g(u-1)u\right.\\ &\left.+ fv(-8(u-1)u-3) + gv^2(-8(u-1)u-3)\right)\end{aligned} \tag{93}$$

$$\begin{aligned}q_1(u) = &\left(1-2u^2\right)(2u-1)\left(f^2 g + fv + gv^2\right)\Big(16f^4 g(u-1)^2 u^2\\ &-4f^3 v(u-1)u - 4f^2 g(u-1)u\left(g^2(1-2u)^2 + 2v^2\right)\\ &+fv\left(3g^2(1-2u)^2 + v^2\right) + g^3 v^2(1-2u)^2 + gv^4\Big)\end{aligned} \tag{94}$$

$$\begin{aligned}q_0(u) = &\,2(1-2u)^2\left(f^2 g + fv + gv^2\right)\left(4f^4 g(u-1)u\right.\\ &+ f^3 v(-8(u-1)u-3)\\ &\left.+ f^2 gv^2(4(u-1)u-1) - fv^3 - gv^4\right).\end{aligned} \tag{95}$$

The zeros of $q_3(u)$ yield five singular points. The power series solution ansatz

$$Y(u) = \sum_{n=0}^{\infty} \eta_n u^n \tag{96}$$

leads to a 9-term recursion relation and the first eight coefficients are again determined by calculating the corresponding t-power series coefficients. We show the first three ones.

$$\eta_0 = y(0) \tag{97}$$

$$\eta_1 = 2x(0)(f + gv) - 2v^2 y(0) \tag{98}$$

$$\begin{aligned}\eta_2 = \frac{1}{3}&\left(2y(0)\left(-3f^2 + 2fgv + v^2\left(g^2 + v^2 - 1\right)\right)\right.\\ &\left.- 2x(0)\left(f\left(3g^2 + v^2\right) + gv\left(g^2 + v^2\right)\right)\right).\end{aligned} \tag{99}$$

Analogously to (90), η_n is a linear function of $x(0)$ and $y(0)$ that can be written as

$$\eta_n = \eta_n^{(x)} x(0) + \eta_n^{(y)} y(0). \tag{100}$$

The further details are too intricate to be displayed here, but, in principle, the procedure is completely analogous to the power series solution of the confluent Heun equation investigated in [53].

6 Fourier series and quasienergy: results based on the power series solutions

It is clear that $u^n = \sin^{2n}\frac{\tau}{2} = \left(\frac{1}{2}(1 - \cos\tau)\right)^n$ is a finite Fourier series including only cos-terms. It explicitly reads

$$\sin^{2n}\frac{\tau}{2} = \frac{(2n-1)!!}{2^n n!} + \sum_{\mu=1}^{n} \frac{(2n-1)!!(1-\mu+n)_\mu(-1)^\mu}{2^{n-1}(\mu+n)!}\cos(\mu\tau), \tag{101}$$

where $(a)_\mu = a(a+1)\ldots(a+\mu-1)$ denotes the Pochhammer symbol. Inserting (101) into the power series (85) and (96) for $x(\tau)$ and $y(\tau)$ yields Fourier series representations valid within the convergence radius of the power series. This does *not* mean that $x(\tau)$ and $y(\tau)$ are generally periodic functions but only that they locally, within the respective domains of convergence, coincide with periodic functions. We may explicitly write down the corresponding Fourier coefficients of

$$x(\tau) = \sum_{\mu=0}^{\infty} x_\mu \cos(\mu\tau), \tag{102}$$

$$y(\tau) = \sum_{\mu=0}^{\infty} y_\mu \cos(\mu\tau), \tag{103}$$

to wit,

$$x_\mu = \begin{cases} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} \xi_n & : \mu = 0, \\ \sum_{n=\mu}^{\infty} \frac{(2n-1)!!(1-\mu+n)_\mu (-1)^\mu}{2^{n-1}(\mu+n)!} \xi_n & : \mu > 0, \end{cases} \tag{104}$$

$$y_\mu = \begin{cases} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} \eta_n & : \mu = 0, \\ \sum_{n=\mu}^{\infty} \frac{(2n-1)!!(1-\mu+n)_\mu (-1)^\mu}{2^{n-1}(\mu+n)!} \eta_n & : \mu > 0. \end{cases} \tag{105}$$

Recall that the ξ_n and η_n are the coefficients of the power series (85) and (96) to be determined by means of recurrence relations.

The case of $z(\tau)$ is a bit more complicated. Using the above local Fourier series representation of $x(\tau)$ and $y(\tau)$, we may directly solve the differential equation

$$\dot{z}(\tau) = \nu\, y(\tau) - g \cos\tau\, x(\tau), \tag{106}$$

since the r. h. s. of (106) is again a cos-series. In general, there will be a nonvanishing constant term z_0 at the r. h. s. of (106) that generates a corresponding part $z_0\,\tau$ of $z(\tau)$ taking into account that $z(0) = 0$.

The complete result is the following:

$$z(\tau) = z_0\,\tau + \sum_{\mu=1}^{\infty} z_\mu \sin(\mu\tau), \tag{107}$$

$$z_\mu = \begin{cases} \nu y_0 - \frac{g}{2} x_1 & : \mu = 0, \\ \nu y_1 - g x_0 - \frac{g}{2} x_2 & : \mu = 1, \\ \frac{1}{\mu}\left(\nu y_\mu - \frac{g}{2}(x_{\mu-1} + x_{\mu+1})\right) & : \mu > 1. \end{cases} \tag{108}$$

The expressions (104) and (105) for the Fourier coefficients still depend, via ξ_n and η_n, on the initial conditions $x(0)$ and $y(0)$. In the special case of $x(0) = \cos\alpha, y(0) = \sin\alpha$ according to (61) the solutions $x(\tau)$ and $y(\tau)$ will be 2π-periodic functions and hence, according to proposition 6, can be written as even resp. odd cos-series valid for all $\tau \in \mathbb{R}$. In particular,

$$\begin{aligned} y_0 = 0 &= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} \eta_n \\ &= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} \left(\eta_n^{(x)} x(0) + \eta_n^{(y)} y(0)\right) \\ &= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} \left(\eta_n^{(x)} \cos\alpha + \eta_n^{(y)} \sin\alpha\right). \end{aligned} \tag{109}$$

This equation can be solved for the auxiliary parameter α:

$$\alpha = -\arctan\left(\frac{\sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} \eta_n^{(x)}}{\sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} \eta_n^{(y)}}\right), \tag{110}$$

if the numerator and denominator of this fraction do not vanish simultaneously. This solution is only determined modulo π in accordance with the fact that $x(0) = -\cos\alpha$, $y(0) = -\sin\alpha$ also gives a periodic solution.

The determination of the second auxiliary parameter r is more involved. We consider the following procedure that does not presuppose the determination of α. First, we calculate the quarter period monodromy matrix $R\left(\frac{\pi}{2}, 0\right)$ by means of the local Fourier series representation considered above. From this, we obtain $R(\pi, 0)$ via (48) and finally r by

$$r = \pm\sqrt{\frac{1 - R(\pi, 0)_{3,3}}{2}}. \tag{111}$$

The latter holds since $R(\pi, 0)$ is of the form (53). It will be instructive to give some more details.

First consider $R\left(\frac{\pi}{2}, 0\right)_{1,1} = x\left(\frac{\pi}{2}\right)$, where $x(\tau)$ has the initial values $x(0) = 1$, $y(0) = z(0) = 0$. It follows that

$$x\left(\frac{\pi}{2}\right) = \sum_{\mu=0}^{\infty} x_\mu \cos\left(\mu\frac{\pi}{2}\right) = \sum_{\mu=0,4,\ldots} x_\mu - \sum_{\mu=2,6,\ldots} x_\mu, \tag{112}$$

because the only nonvanishing terms are $\cos\left(\mu\frac{\pi}{2}\right) = 1$ for μ being an integer multiple of 4 and $\cos\left(\mu\frac{\pi}{2}\right) = -1$ for even μ such that $\mu/2$ is odd. Recall that the Fourier coefficients x_μ have to be determined via (104) where the ξ_n have to be chosen as $\xi_n^{(x)}$ according to (90) and the above initial values.

The procedure for the calculation of $R\left(\frac{\pi}{2}, 0\right)_{2,1} = y\left(\frac{\pi}{2}\right)$ is completely analogous. For $R\left(\frac{\pi}{2}, 0\right)_{3,1} = z\left(\frac{\pi}{2}\right)$ we employ (107) and (108) as well as

$$z\left(\frac{\pi}{2}\right) = z_0\,\frac{\pi}{2} + \sum_{\mu=1,5,\ldots} z_\mu - \sum_{\mu=3,7,\ldots} z_\mu. \tag{113}$$

For the second column of $R\left(\frac{\pi}{2}, 0\right)$ the calculation is again the same as for the first column except that $\xi_n^{(x)}$ has to be replaced by $\xi_n^{(y)}$ and $\eta_n^{(x)}$ by $\eta_n^{(y)}$. As mentioned before, the third column of $R\left(\frac{\pi}{2},0\right)$ is the vector product of the first and the second ones. Since we only need a particular matrix element of the half period monodromy, namely $R(\pi,0)_{3,3}$, it suffices to use the following equation resulting from (48):

$$\begin{aligned} R(\pi,0)_{3,3} &= R\left(\frac{\pi}{2},0\right)_{1,3}^2 - R\left(\frac{\pi}{2},0\right)_{2,3}^2 + R\left(\frac{\pi}{2},0\right)_{3,3}^2 \\ &= 1 - 2R\left(\frac{\pi}{2},0\right)_{2,3}^2, \end{aligned} \tag{114}$$

and hence

$$\begin{aligned} r &= \left| R\left(\frac{\pi}{2},0\right)_{2,3} \right| \\ &= \left| R\left(\frac{\pi}{2},0\right)_{3,1} R\left(\frac{\pi}{2},0\right)_{1,2} - R\left(\frac{\pi}{2},0\right)_{1,1} R\left(\frac{\pi}{2},0\right)_{3,2} \right|, \end{aligned} \tag{115}$$

where the entries from the first and second column of $R\left(\frac{\pi}{2},0\right)$ have been calculated above. This completes the determination of the auxiliary parameter r and the quasienergy via (60).

We have checked the results (110) and (115) by comparison with a numerical solution of the $s=\frac{1}{2}$ Schrödinger Equation (5) for the choice of the parameters $v=1$, $f=1$, and $g=1/2$. For this case, both methods come to the same conclusion

$$\begin{aligned} &\alpha = 1.40464\ldots, \quad r = 0.387328\ldots, \quad \text{and hence} \\ &\epsilon^{(qu)} = 0.126602\ldots. \end{aligned} \tag{116}$$

7 Time evolution: an example

As an example, we consider the time evolution over one period $\tau \in [0,2\pi]$ according to (8). We choose the values of the parameters $f=1, g=1/2$ and $v=1$ and analytically calculate three mutually orthogonal solutions $\mathbf{S}^{(i)}(\tau), i=1,2,3$ for $\tau \in \left[0,\frac{\pi}{2}\right]$ by evaluating the corresponding power series solutions with 10 terms. For the remaining three quarter periods, we adopt the reduction Equations (25) and (35) for $\mathbf{S}^{(1)}$, where $R(\pi,0)$ can be expressed through $R\left(\frac{\pi}{2},0\right)$ via (48). We observe a satisfactory agreement with the numerical solution of (8), see Figure 1.

The alternative choice of the initial conditions as $x(0)=\cos\alpha$ and $y(0)=\sin\alpha$, whereas $z(0)=0$ remains unchanged, leads to 2π-periodic solutions, see Figure 2.

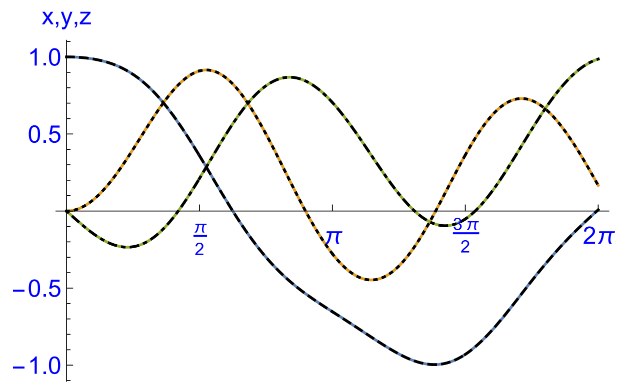

Figure 1: The three components of the classical spin vector as functions of dimensionless time τ over one period according to the equation of motion (8). We have chosen the parameters $f=1, g=1/2$ and $v=1$ and the initial conditions $x(0)=1$, $y(0)=z(0)=0$. The solid curves are numerical results; the dashed curve represents $x(\tau)$ as calculated analytically, likewise $y(\tau)$ (dotted curve) and $z(\tau)$ (dotted-dashed curve).

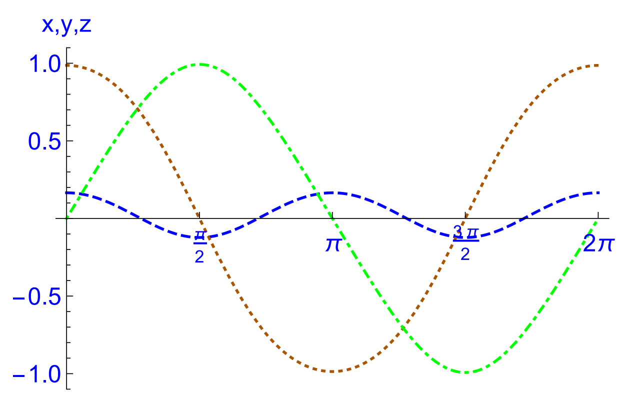

Figure 2: The three components of the classical spin vector as periodic functions of dimensionless time τ over one period according to the equation of motion (8). We have chosen the parameters $f=1, g=1/2$ and $v=1$ and the initial conditions $x(0)=\cos\alpha$, $y(0)=\sin\alpha$ and $z(0)=0$. The dashed curve represents $x(\tau)$ as calculated numerically and analytically, likewise $y(\tau)$ (dotted curve) and $z(\tau)$ (dotted-dashed curve).

This calculation uses the value of the auxiliary parameter α that has been determined according to (110).

The first few terms of the corresponding Fourier series read as follows:

$$\begin{aligned} x(\tau) &= 0.0240019 + 0.144012\cos(2t) \\ &\quad - 0.00263811\cos(4t) + \ldots \end{aligned} \tag{117}$$

$$\begin{aligned} y(\tau) &= 1.01784\cos(t) - 0.0319147\cos(3t) \\ &\quad + 0.000303923\cos(5t) + \ldots \end{aligned} \tag{118}$$

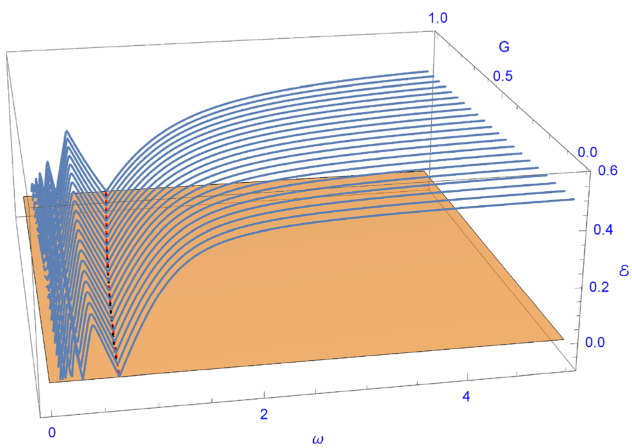

Figure 3: The branch of the quasienergy $\mathcal{E}(\omega_0, F, G, \omega)$ satisfying (121) as a function of ω and G for fixed values of $\omega_0 = 1$ and $F = 1$. G varies from $G = 0$ (linear polarization) to $G = F = 1$ (circular polarization). Along the red, dashed curve the quasienergy vanishes. An analytical approximation of this curve according to (131) is shown as a black, dashed curve.

$$z(\tau) = 0.969835\sin(t) - 0.0224197\sin(3t) + 0.000192725\sin(5t) + \ldots . \tag{119}$$

8 Vanishing of the quasienergy

We will discuss the quasienergy in physical units

$$\mathcal{E}(\omega_0, F, G, \omega) \equiv \hbar\,\omega\,\epsilon^{(qu)}\left(\frac{\omega_0}{\omega}, \frac{F}{\omega}, \frac{G}{\omega}\right) = \hbar\,\omega\,\epsilon^{(qu)}(\nu, f, g), \tag{120}$$

where usually $\hbar$ is set to 1. Analogously to the ambiguity of $\epsilon^{(qu)}$ also $\mathcal{E}$ will be only defined up to integer multiples of $\hbar\omega$. A typical plot of the functions $\omega \mapsto \mathcal{E}(\omega_0, F, G, \omega)$ for the values $\omega_0 = F = 1$ and G varying from $G = 0$ (linear polarization) to $G = F = 1$ (circular polarization) is shown in Figure 3, where the branch and the sign of the quasienergy are chosen according to

$$0 \le \frac{1}{\hbar\omega}\mathcal{E}(1, 1, G, \omega) \le \frac{1}{2}. \tag{121}$$

We notice that these curves qualitatively all look the same. First we note that the family of curves shows the same asymptotic behavior of $\mathcal{E}(\omega_0, F, G, \omega)$ for $\omega \to \infty$. In the case of circular polarization, we have $\mathcal{E}(\omega_0, F, G, \omega) \to \frac{\omega_0}{2}$ for $G = F$, as well as $\mathcal{E}(\omega_0, F, G, \omega) \to \frac{\omega_0}{2}$ for linear polarization, see Eq. (269) in [57]. Further, the quasienergy functions have an infinite number of zeros with a nonvanishing slope, the largest being slightly below $\omega = 1$.

To better understand this behavior in detail, we revisit the RPC. It is well known that in the special case of circular polarization the quasienergy can be analytically determined in a rather simple form. In the context of the present discussion, we note that the fundamental matrix solution of (14) with initial condition (15) assumes the form $R(\tau, 0) = (R_1, R_2, R_3)$ with the three columns reading

$$R_1 = \begin{pmatrix} f^2\cos(\tau\Omega) + (\nu-1)^2 \\ f\left(2(\nu-1)\cos(\tau)\sin^2\left(\frac{\tau\Omega}{2}\right) + \Omega\sin(\tau)\sin(\tau\Omega)\right) \\ f\left(2(\nu-1)\sin(\tau)\sin^2\left(\frac{\tau\Omega}{2}\right) - \Omega\cos(\tau)\sin(\tau\Omega)\right) \end{pmatrix}, \tag{122}$$

$$R_2 = \begin{pmatrix} -f(\nu-1)(\cos(\tau\Omega)-1) \\ \cos(\tau)\left(f^2 + (\nu-1)^2\cos(\tau\Omega)\right) - (\nu-1)\Omega\sin(\tau)\sin(\tau\Omega) \\ \sin(\tau)\left(f^2 + (\nu-1)^2\cos(\tau\Omega)\right) + (\nu-1)\Omega\cos(\tau)\sin(\tau\Omega) \end{pmatrix}, \tag{123}$$

$$R_3 = \begin{pmatrix} f\,\Omega\sin(\tau\Omega) \\ -\Omega\left((\nu-1)\cos(\tau)\sin(\tau\Omega) + \Omega\sin(\tau)\cos(\tau\Omega)\right) \\ \Omega\left(\Omega\cos(\tau)\cos(\tau\Omega) - (\nu-1)\sin(\tau)\sin(\tau\Omega)\right) \end{pmatrix}, \tag{124}$$

where we have used the abbreviation

$$\Omega \equiv \sqrt{f^2 + (1-\nu)^2}, \tag{125}$$

known as the "Rabi frequency". The corresponding monodromy matrix $R(2\pi, 0)$ reads:

$$R(2\pi, 0) = \begin{pmatrix} \frac{f^2\cos(2\pi\Omega) + (\nu-1)^2}{\Omega^2} & \frac{2f(\nu-1)\sin^2(\pi\Omega)}{\Omega^2} & \frac{f\sin(2\pi\Omega)}{\Omega} \\ \frac{2f(\nu-1)\sin^2(\pi\Omega)}{\Omega^2} & \frac{f^2 + (\nu-1)^2\cos(2\pi\Omega)}{\Omega^2} & -\frac{(\nu-1)\sin(2\pi\Omega)}{\Omega} \\ -\frac{f\sin(2\pi\Omega)}{\Omega} & \frac{(\nu-1)\sin(2\pi\Omega)}{\Omega} & \cos(2\pi\Omega) \end{pmatrix}. \tag{126}$$

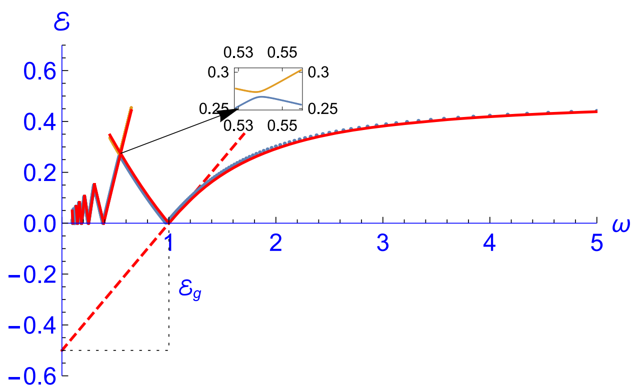

Figure 4: The branch of the quasienergy $\mathcal{E}(\omega_0, F, G, \omega)$ satisfying (121) as a function of ω for fixed $\omega_0 = 1$, $F = 1$ and two values $G = 1$ (red curves) and $G = 0.95$ (blue and orange curves). In the circular case ($G = 1$), we observe level crossing whereas in the case with small eccentricity ($G = 0.95$) this crossing is avoided as demonstrated by the inset. The dashed red line is the tangent of $\mathcal{E}(1, 1, 1, \omega)$ at $\omega = 1$. According to (132), it has the slope $\mathcal{E}_g = \frac{1}{2}$.

Its trace is evaluated as $\operatorname{Tr}(R(2\pi, 0)) = 1 + 2\cos(2\pi\,\Omega)$ and yields the eigenvalues $(1, \exp(\pm 2\pi\,\mathrm{i}\,\Omega))$, corresponding to a classical quasienergy

$$\epsilon^{(cl)} = \Omega. \tag{127}$$

On the other hand, we may apply (21) to the periodic solution

$$\mathbf{S}(\tau) = \frac{1}{\Omega}\begin{pmatrix} \nu - 1 \\ f\cos\tau \\ f\sin\tau \end{pmatrix} \tag{128}$$

with the well-known result

$$\epsilon^{(qu)} = \frac{1 \pm \Omega}{2}, \tag{129}$$

that is compatible with (127) and (20).

For $G = F = \omega_0 = 1$, the quasienergy curve has a zero at $\omega = 1$, i.e., $\mathcal{E}(1, 1, 1, 1) = 0$. For slightly lower values of G, this zero shifts to lower values of ω, see Figures 3 and 4. We will denote by $G = \mathcal{E}_0(\omega\,; F, \omega_0)$ the position of the largest zero of the quasienergy.

The vanishing of the quasienergy is in so far interesting as it means that *all* solutions of (8) will be 2π-periodic, not only the special one with the initial condition $\mathbf{S}(0) = (\cos\alpha, \sin\alpha, 0)^\top$ according to (61). Moreover, $\mathcal{E}(\omega_0, F, G, \omega) = 0$ means degeneracy of the Floquet states for the TLS, which may produce some second-order phase transition in the parameter space, see [21].

The vanishing of the quasienergy implies that the linear term $z_0\tau$ in (107) must vanish and hence

$$0 = z_0 \overset{(108)}{=} \nu y_0 - \frac{g}{2} x_1. \tag{130}$$

In order to check the consistency, we will evaluate the condition (130) by using a truncation of the power series solutions (85) and (96) to the first 10 terms. This yields the exact first five terms of $G = \mathcal{E}_0(\omega;\, 1,\, 1)$ expanded into a power series in terms of $\omega - \omega_0 = \omega - 1$:

$$\begin{aligned}\mathcal{E}_0(\omega\,;\ 1,\ 1) &= 1 + 2(\omega - 1) - \frac{5}{6}(\omega - 1)^2 + \frac{49}{36}(\omega - 1)^3 \\ &\quad - \frac{577}{240}(\omega - 1)^4 + \frac{58357}{12960}(\omega - 1)^5 \\ &\quad + O\big((\omega - 1)^6\big).\end{aligned} \tag{131}$$

The result is shown in Figure 3 as a black dashed curve and fits to the numerically determined red dashed curve of vanishing quasienergy in the domain $0.7 < \omega < 1$.

Further we note that according to [57] the quasienergy $\mathcal{E}$ can be split into a geometrical part $\mathcal{E}_g$ and a dynamical part $\mathcal{E}_d$ such that $\mathcal{E} = \mathcal{E}_g + \mathcal{E}_d$ and the slope relation

$$\frac{\partial\mathcal{E}}{\partial\omega} = \frac{\mathcal{E}_g}{\omega} \tag{132}$$

holds, see Eq. (151) in a study by Schmidt [57]. Recall that $\mathcal{E}_d$ is the time average of the energy, i. e, $\mathcal{E}_d = \frac{1}{2}\overline{\mathbf{h}(t)\cdot\mathbf{S}(t)}$ and $\mathcal{E}_g = \frac{\omega}{4\pi}\,|\mathscr{A}|$, where $|\mathscr{A}|$ denotes the signed area of the Bloch sphere swept by $\mathbf{S}(t)$ over one period. In our case, this implies that for vanishing quasienergy and hence $G = \mathcal{E}_0(\omega)$ we have $\mathcal{E}_g + \mathcal{E}_d = 0$ and the slope of the curve $\omega \mapsto \mathcal{E}(\omega_0, F, G, \omega)$ equals $|\mathcal{E}_g| = |\mathcal{E}_d|$. We have illustrated this relation for the special case of circular polarization in Figure 4 by drawing the tangent (dashed red line) with the slope $\frac{1}{2}$. This corresponds to a periodic solution of (8) tracing a great circle on the Bloch sphere with solid angle $|\mathscr{A}| = 2\pi$.

In general the quasienergy (129) of the RPC has its first zero at $\omega = \omega_1 \equiv \frac{F^2 + \omega_0^2}{2\omega_0}$. Hence, the series (131) will assume its general form as $G = \mathcal{E}_0(\omega\,; F, \omega_0) = F + \sum_{n=1}^{\infty} g_n\,(\omega - \omega_1)^n$. It begins with

$$\begin{aligned} G &= \mathcal{E}_0(\omega) \\ &= F \\ &\quad - \frac{6\left(-60F^4 + 50F^4\omega_0 - 100F^2\omega_0^2 + 71F^2\omega_0^3 - 25\omega_0^4\right)\left(\omega - \frac{F^2 + \omega_0^2}{2\omega_0}\right)}{F\left(30F^4 + 20F^2\omega_0 + 67F^2\omega_0^2 + 10\omega_0^3 + 65\omega_0^4\right)} \\ &\quad + \ldots, \end{aligned} \tag{133}$$

but the next terms are too intricate to be shown here.

We will consider the case of vanishing quasienergy along the curve $G = \mathcal{E}_0(\omega)$ in more details. As already mentioned, in this case all solutions of (8) will be 2π-periodic and hence the local Fourier series representations

(102), (103) and (107) can be extended to all times τ. It turns out that the slope relation (132) cannot be satisfied by *all* periodic solutions of (8) but only by a particular one that is the limit of the (up to a sign) unique periodic solutions for nonvanishing quasienergy. Instead of again using Eq. (110) to determine this limit, we will proceed in a different way.

Recall that for a general periodic, not necessarily normalized solution $\mathbf{S}(\tau) = (x(\tau), y(\tau), z(\tau))^\top$ of (8) of the form (102), (103) and (106) the functions $x(\tau)$ and $y(\tau)$ are represented by cos-series, whereas $z(\tau)$ will be a sin-series. Consider the decomposition of $x(\tau) = x_e(\tau) + x_o(\tau)$ into an even cos-series and an odd one and analogously for $y(\tau) = y_e(\tau) + y_o(\tau)$ and $z(\tau) = z_e(\tau) + z_o(\tau)$. Consequently, the time derivative $\dot{x}$ can be uniquely split into an even sin-series and an odd one:

$$\begin{aligned}\dot{x} &= \dot{x}_e + \dot{x}_o \\ &= (g\cos(\tau) z_o - f\sin(\tau) y_o) + (g\cos(\tau) z_e - f\sin(\tau) y_e).\end{aligned} \tag{134}$$

Analogous decompositions for $\dot{y}$ and $\dot{z}$ lead to a decomposition of $\mathbf{S}(\tau)$ into two separate solutions

$$\mathbf{S}(\tau) = \mathbf{S}^{(1)}(\tau) + \mathbf{S}^{(2)}(\tau) \equiv \begin{pmatrix} x_e(\tau) \\ y_o(\tau) \\ z_o(\tau) \end{pmatrix} + \begin{pmatrix} x_o\tau) \\ y_e(\tau) \\ z_e(\tau) \end{pmatrix}. \tag{135}$$

It is clear that $\mathbf{S}^{(1)}(\tau)$ equals the limit of periodic solutions for nonvanishing quasienergies since these periodic solutions have the same even/odd character as $\mathbf{S}^{(1)}(\tau)$, see Proposition 6. Moreover, the two solutions (135) are orthogonal for all τ: Their scalar product is constant in time, on the other hand an odd cos-series and has thus a vanishing time average. Further, it follows that $\mathbf{S}^{(2)}(\tau)$ belongs to $\mathcal{E}_d = 0$ since $\mathbf{h}\cdot\mathbf{S}^{(2)}$ will be an odd COS-series and thus has a vanishing time average. In contrast, $\mathbf{h}\cdot\mathbf{S}^{(1)}$ will be an even cos-series which is compatible with $|\mathcal{E}_d| > 0$ and a positive slope of the quasienergy curves at $G = \mathcal{E}_0(\omega)$, see Figure 3.

For the sake of completeness, we note that the third solution $\mathbf{S}^{(3)} = \mathbf{S}^{(1)}\times\mathbf{S}^{(2)}$ will be of the following type: $x^{(3)}(\tau)$ is an odd sin-series, $y^{(3)}(\tau)$ is an even sin-series, and $z^{(3)}(\tau)$ is an even cos-series. Hence also for this solution, the time average of $\mathbf{h}\cdot\mathbf{S}^{(3)}$ will be an odd sin-series and hence $\mathcal{E}$ vanishes. An example is shown in Figure 5.

In the case of the periodic solutions $\mathbf{S}^{(1)}(\tau)$ or $\mathbf{S}^{(2)}(\tau)$, the time average of $\mathbf{h}\cdot\mathbf{S}$ can be expressed in terms of the first Fourier coefficients:

$$\epsilon_d^{(qu)} = \frac{1}{2}\overline{\mathbf{h}\cdot\mathbf{S}} = \frac{1}{2}\left(\nu x_0 + \frac{g}{2}y_1 + \frac{f}{2}z_1\right), \tag{136}$$

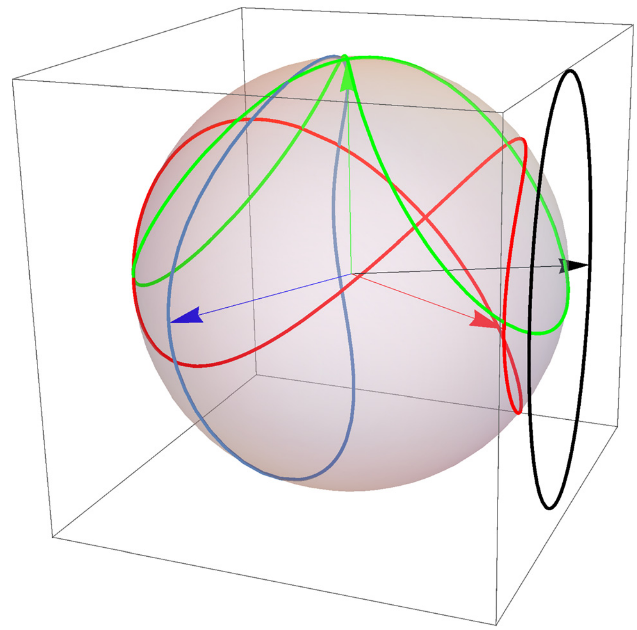

Figure 5: Three periodic solutions $\mathbf{S}^{(i)}(\tau)$, $i = 1, 2, 3$ (blue, red and green curves) of (8) for the parameters $\omega_0 = F = 1$, $G = \frac{1}{2}$ and $\omega = \omega_1$ such that the quasienergy vanishes. The magnetic field vector (black arrow) moves on the black ellipse; the initial vectors $\mathbf{S}^{(i)}(0)$, $i = 1, 2, 3$ are shown as colored arrows. The time average of the energy vanishes for $\mathbf{S}^{(2)}$ and $\mathbf{S}^{(3)}$.

where we have again passed to the dimensionless quasienergy and the Fourier coefficients are given in (104), (105) and (107). The suitable initial conditions $x(0) = \cos\beta$ and $y(0) = \sin\beta$ can be derived from the result $\epsilon_d^{(qu)} = 0$ analogously to (110):

$$\beta = -\arctan\frac{\nu x_0^{(x)} + \frac{g}{2}y_1^{(x)} + \frac{f}{2}z_1^{(x)}}{\nu x_0^{(y)} + \frac{g}{2}y_1^{(y)} + \frac{f}{2}z_1^{(y)}}. \tag{137}$$

Here, the superscript (*x*) or (*y*) refers to the dependence of the Fourier coefficients, via ξ_n and η_n, on the initial conditions $x(0)$ and $y(0)$. Finally, the initial conditions $x(0) = \cos\alpha$ and $y(0) = \sin\alpha$ for the first solution $(\mathbf{S})^{(1)}(\tau)$ are given by

$$\alpha = \beta \pm \frac{\pi}{2}, \tag{138}$$

using the orthogonality of $\mathbf{S}^{(1)}(\tau)$ and $\mathbf{S}^{(2)}(\tau)$. After some calculations, it follows that the dynamical part of the quasienergy of the first solution $\mathbf{S}^{(1)}(\tau)$ assumes the value

$$\begin{aligned}\epsilon_d &= \frac{1}{2}\overline{\mathbf{h}\cdot\mathbf{S}^{(1)}} \\ &= \frac{1}{2}\left(\left(\nu x_0^{(x)} + \frac{g}{2}y_1^{(x)} + \frac{f}{2}z_1^{(x)}\right)\cos\,\alpha \right.\\ &\quad \left. + \left(\nu x_0^{(y)} + \frac{g}{2}y_1^{(y)} + \frac{f}{2}z_1^{(y)}\right)\sin\,\alpha\right)\end{aligned} \tag{139}$$

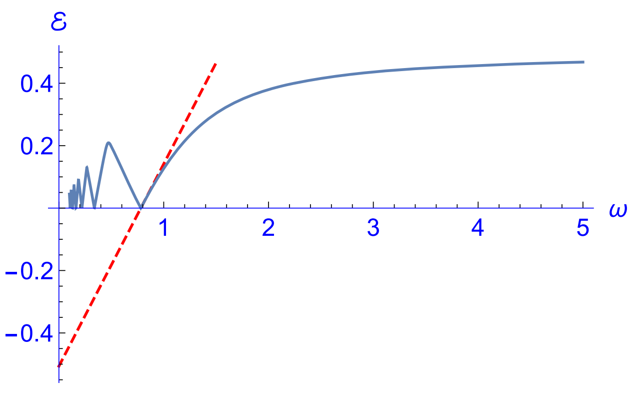

Figure 6: The branch of the quasienergy $\mathcal{E}(\omega_0, F, G, \omega)$ satisfying (121) as a function of ω for fixed values of $\omega_0 = F = 1$ and $G = \frac{1}{2}$. The function has its largest zero at $\omega_1 = 0.781665$ where the tangent (dashed red line) has a slope of 0.64787, see the text after Eq. (140).

$$= \frac{1}{2}\sqrt{\left(\nu\, x_0^{(x)} + \frac{g}{2} y_1^{(x)} + \frac{f}{2} z_1^{(x)}\right)^2 + \left(\nu\, x_0^{(y)} + \frac{g}{2} y_1^{(y)} + \frac{f}{2} z_1^{(y)}\right)^2}\,. \tag{140}$$

As an example we consider the parameters $\omega_0 = F = 1$ and $G = \frac{1}{2}$. The quasienergy curve $\omega \mapsto \mathcal{E}\left(1, 1, \frac{1}{2}, \omega\right)$ has its largest zero at $\omega_1 = 0.781665$. This value has been determined numerically; the analytical approximation (131) yields $\omega_1 = 0.781023$. At this point, the two solutions $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are obtained with initial values $x(0) = \cos\alpha$, $y(0) = \sin\alpha$, and $x(0) = \cos\beta, y(0) = \sin\beta$, respectively, where $\beta = -0.489254$ has been calculated according to (137) and $\alpha = \beta + \frac{\pi}{2}$, see Figure 5. The slope of the tangent of the quasienergy curve at $\omega = \omega_1$ has been determined via (140) and assumes the value $\epsilon_d = 0.64787$, see Figure 6.

9 Resonances

The function $\omega_0 \mapsto \mathcal{E}(\omega_0, F, G, \omega)$, restricted to the domain (121), has an infinite number of maxima, see Figure 7. These satisfy the condition

$$0 = \frac{\partial}{\partial \omega_0} \mathcal{E}(\omega_0, F, G, \omega)\,. \tag{141}$$

defining an infinite number of hypersurfaces in the parameter space $\mathscr{P}$ with points $(\omega_0, F, G, \omega) \in \mathscr{P}$. Solving (141) for ω gives the so-called “resonance frequencies”

$$\omega = \omega_{\text{res}}^{(n)}(\omega_0, F, G), n = 1, 2, 3, \ldots. \tag{142}$$

In the circular case, a smooth representative of the quasienergy $\mathcal{E}_c$ assumes the form

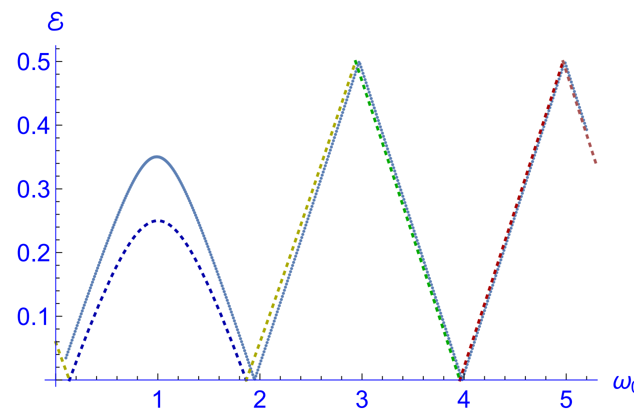

Figure 7: The quasienergy $\mathcal{E}(\omega_0, F, G, \omega)$ as a function of ω_0 for fixed values of $\omega = 1$, $F = 0.5$ and $G = 0.1$ (solid curves) calculated by numerical solutions of the Schrödinger Equation (5). One observes maxima of the quasienergy at $\omega_0 \approx 1, 3, 5, \ldots$. The dotted curves are various branches of the analytical form (143) of the quasienergy for the case of circular polarization, i.e., $G = 0$ and $\omega = 1$, $F = 0.5$.

$$\mathcal{E}_c = \frac{1}{2}(\omega - \Omega) = \frac{1}{2}\left(\omega - \sqrt{F^2 + (\omega - \omega_0)^2}\right), \tag{143}$$

and has a unique maximum at $\omega_0 = \omega$, see Figure 7. This conforms with the intuitive picture that a resonance occurs if the driving frequency ω equals the Larmor frequency ω_0 of the energy level splitting. The other maxima of the quasienergy, restricted to the domain (121), are represented by intersections of suitable branches of the quasienergy of the form $\pm\mathcal{E}_c + n\omega$, $n \in \mathbb{Z}$. For example, the next maximum at $\omega_0 \approx 3\,\omega$ is obtained by the intersection of $-\mathcal{E}_c$ and $\mathcal{E}_c + \omega$ at

$$\omega_0 = \omega + \sqrt{4\,\omega^2 - F^2} = 3\,\omega - \frac{F^2}{4\,\omega} + O(F^4)\,. \tag{144}$$

Note that an arbitrarily small admixture of eccentricity to the polarization leads to an avoided level crossing and a smooth maximum close to the value ω_0 of the intersection, see Figure 7.

According to [26], the time average of the transition probability between different Floquet states assumes its maximum value $\bar{P} = \frac{1}{2}$ at the resonance frequencies, which justifies the denotation. Although Shirley's derivation of the resonance condition refers to the RPL case, see (1) in [26], one can easily check that it also holds in the more general RPE case. Moreover, it has been shown [57] that for $\omega = \omega_{res}^{(n)}$ the classical periodic solution of (8) has a vanishing time-average into the direction of the constant component of the magnetic field. According to our definitions, this means that

$$x_0 = x_0^{(x)} \cos\alpha + x_0^{(y)} \sin\alpha = 0\,, \tag{145}$$

where α is the auxiliary parameter leading to a periodic solution given by (110). Together with

$$y_0 = y_0^{(x)}\cos\alpha + y_0^{(y)}\sin\alpha = 0\,, \tag{146}$$

see (109), this implies that the matrix

$$\Xi \equiv \begin{pmatrix} x_0^{(x)} & x_0^{(y)} \\ y_0^{(x)} & y_0^{(y)} \end{pmatrix} \tag{147}$$

has a nonvanishing null-vector and hence

$$\det\Xi = x_0^{(x)}\,y_0^{(y)} - x_0^{(y)}\,y_0^{(x)} = 0\,. \tag{148}$$

We use truncated versions of (85) and (96) in order to derive the first terms of the power series representations

$$\frac{\omega_{res}^{(n)}}{\omega_0} = \sum_{m,k=0}^{\infty} \Omega_{m,k}^{(n)} \left(\frac{F}{\omega_0}\right)^m \left(\frac{G}{\omega_0}\right)^k, \tag{149}$$

analogously to [57]. We will show a few results. The first resonance $\omega_{\rm res}^{(1)}$ is determined by

We note that $\Omega^{(1)}$ is a symmetric matrix due to the symmetry of the Rabi problem under the exchange $G \leftrightarrow F$. The matrix elements $\Omega_{m,k}^{(1)}$ vanish for odd $m + k$. Further, it is instructive to look at the limit cases of linear or circular polarization. For $G = 0$, the first column of $\Omega^{(1)}$ agrees with the corresponding known results in the case of linear polarization, see Table 1 in the study by Schmidt [57]. For $F = G$, the power series (149) coalesces into a series of a single variable F with coefficients $\tilde{\Omega}_M^{(1)} = \sum_{m=0}^{M} \Omega_{m,M-m}^{(1)}$, $M = 0, 2, 4, \ldots$. On the other hand, the resonance frequency $\omega_{res}^{(1)}$ of the circularly polarized case is known to be $\omega_{res}^{(1)} = \omega_0$. Hence, the antidiagonal sums of $\Omega^{(1)}$-entries $\tilde{\Omega}_M^{(1)}$ must vanish for $M = 2, 4, \ldots$. This can be confirmed for $M = 0, 2, \ldots, 8$ in the above shown part of $\Omega^{(1)}$, see (150).

The second resonance is described by the matrix

$$\Omega^{(1)} = \begin{pmatrix}
1 & 0 & \frac{1}{16} & 0 & \frac{1}{1024} & 0 & -\frac{35}{131072} & 0 & \frac{103}{8388608} & \ldots \\
0 & -\frac{1}{8} & 0 & \frac{3}{256} & 0 & \frac{27}{65536} & 0 & -\frac{69}{262144} & 0 & \ldots \\
\frac{1}{16} & 0 & -\frac{13}{512} & 0 & \frac{611}{131072} & 0 & \frac{433}{2097152} & 0 & \star & \ldots \\
0 & \frac{3}{256} & 0 & -\frac{315}{32768} & 0 & \frac{609}{262144} & 0 & \star & 0 & \\
\frac{1}{1024} & 0 & \frac{611}{131072} & 0 & -\frac{19115}{4194304} & 0 & \star & 0 & \star & \ldots \\
0 & \frac{27}{65536} & 0 & \frac{609}{262144} & 0 & \star & 0 & \star & 0 & \ldots \\
-\frac{35}{131072} & 0 & \frac{433}{2097152} & 0 & \star & 0 & \star & 0 & \star & \ldots \\
0 & -\frac{69}{262144} & 0 & \star & 0 & \star & 0 & \star & 0 & \ldots \\
\frac{103}{8388608} & 0 & \star & 0 & \star & 0 & \star & 0 & \star & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}. \tag{150}$$

$$\Omega^{(2)} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{3}{32} & 0 & -\frac{135}{8192} & 0 & \frac{2133}{1048576} & 0 & \dots \\ 0 & \frac{1}{16} & 0 & -\frac{9}{2048} & 0 & -\frac{3591}{524288} & 0 & \star & \dots \\ \frac{3}{32} & 0 & -\frac{21}{4096} & 0 & \frac{6075}{1048576} & 0 & \star & 0 & \dots \\ 0 & -\frac{9}{2048} & 0 & \frac{4095}{262144} & 0 & \star & 0 & \star & \dots \\ -\frac{135}{8192} & 0 & \frac{6075}{1048576} & 0 & \star & 0 & \star & 0 & \dots \\ 0 & -\frac{3591}{524288} & 0 & \star & 0 & \star & 0 & \star & \dots \\ \frac{2133}{1048576} & 0 & \star & 0 & \star & 0 & \star & 0 & \dots \\ 0 & \star & 0 & \star & 0 & \star & 0 & \star & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{151}$$

Here, analogous remarks apply as in the case of $\Omega^{(1)}$, except that the antidiagonal sums of $\Omega^{(2)}$-entries $\tilde{\Omega}_M^{(2)}$ no longer vanish. They can be determined by the following consideration. In the circular limit, the second resonance is defined by the level crossing

$$\frac{1}{2}\left(-\omega + \Omega\right) = \frac{1}{2}\left(3\,\omega - \Omega\right), \tag{152}$$

where $\Omega \equiv \sqrt{F^2 + (\omega_0 - \omega)^2}$. (Recall that an arbitrary small amount of eccentricity F-G produces an avoided level crossing and hence a smooth maximum of the quasienergy). After some manipulations, the condition (152) can be transformed into

$$\frac{\omega_{res}^{(2)}}{\omega_0} = \frac{1}{3}\left(-1 + \sqrt{3\left(\frac{F}{\omega_0}\right)^2 + 4}\right) \tag{153}$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n-1} (2n-3)!!}{2^{3n-1}\, n!} \left(\frac{F}{\omega_0}\right)^{2n} \tag{154}$$

$$= \frac{1}{3} + \frac{1}{4}\left(\frac{F}{\omega_0}\right)^2 - \frac{3}{64}\left(\frac{F}{\omega_0}\right)^4 + \frac{9}{512}\left(\frac{F}{\omega_0}\right)^6 + O\left(\frac{F}{\omega_0}\right)^8. \tag{155}$$

It can be easily checked that the coefficients of the power series (155) coincide with the antidiagonal sums, i.e., $\tilde{\Omega}_2^{(0)} = \frac{1}{3}$, $\tilde{\Omega}_2^{(2)} = \frac{1}{4}$, $\tilde{\Omega}_4^{(2)} = -\frac{3}{64}$, and $\tilde{\Omega}_6^{(2)} = \frac{9}{512}$.

Finally, we consider the third resonance described by

$$\Omega^{(3)} = \begin{pmatrix} \frac{1}{5} & 0 & \frac{5}{96} & 0 & -\frac{2125}{221184} & \dots \\ 0 & \frac{1}{48} & 0 & -\frac{125}{55296} & 0 & \dots \\ \frac{5}{96} & 0 & -\frac{205}{36864} & 0 & \star & \dots \\ 0 & -\frac{125}{55296} & 0 & \star & 0 & \dots \\ -\frac{2125}{221184} & 0 & \star & 0 & \star & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{156}$$

the antidiagonal sums of which are obtained via

$$\frac{\omega_{res}^{(3)}}{\omega_0} = \frac{1}{15}\left(-1 + \sqrt{15\left(\frac{F}{\omega_0}\right)^2 + 16}\right) \tag{157}$$

$$= \frac{1}{5} + \sum_{n=1}^{\infty} \frac{(-15)^{n-1} (-3+2n)!!}{2^{4n-1}\, n!} \left(\frac{F}{\omega_0}\right)^{2n} \tag{158}$$

$$= \frac{1}{5} + \frac{1}{8}\left(\frac{F}{\omega_0}\right)^2 - \frac{15}{512}\left(\frac{F}{\omega_0}\right)^4 + O\left(\frac{F}{\omega_0}\right)^6. \tag{159}$$

The first nontrivial antidiagonal of $\Omega^{(n)}$ can be given in closed form. According to the recurrence relation given in [57], we conjecture that

$$\Omega_{2,0}^{(n)} = \Omega_{0,2}^{(n)} = \frac{2n-1}{16\,(n-1)n}\,, \tag{160}$$

for $n > 1$. Employing the circular limit

$$\frac{\omega_{res}^{(n)}}{\omega_0} = \frac{\sqrt{(2n-3)(2n-1)\left(\frac{F^2}{\omega_0^2}+1\right)+1}-1}{4\,(n-2)n+3}$$
$$= \frac{1}{2n-1} + \frac{1}{4\,(n-1)}\left(\frac{F}{\omega_0}\right)^2 + O\left(\frac{F}{\omega_0}\right)^4 \tag{161}$$

we obtain

$$\Omega_{1,1}^{(n)} = \frac{1}{8\,(n-1)n}\,, \tag{162}$$

for $n > 1$. These results can be checked for $n = 2, 3$ by inspection of (151) and (156).

10 Special limit cases

10.1 Limit case $\omega \to 0$

This limit case ("adiabatic limit") has been already treated in [57] in sufficient generality, such that we only need to recall the essential issues. We adopt a series representation

$$\mathbf{S}(\omega t) = \sum_{n=0}^{\infty} \omega^n\, \mathbf{S}^{(n)}(\omega t) \tag{163}$$

of the periodic solution of (8) and obtain a recursive system of inhomogeneous linear differential equations for the $\mathbf{S_n}$. The starting point is

$$\mathbf{S}^{(0)}(\omega t) = \frac{\mathbf{h}(t)}{\|\mathbf{h}(t)\|}$$
$$= \frac{1}{\sqrt{F^2\ \sin^2(\omega t) + G^2\ \cos^2(\omega t) + \omega_0^2}} \begin{pmatrix} \omega_0 \\ G\cos\ \omega t \\ F\sin\ \omega t \end{pmatrix}, \tag{164}$$

that is, the spin vector follows the direction of the slowly varying magnetic field. The corresponding zeroth term of the series for the quasienergy

$$\mathcal{E} = \frac{1}{2}\left(\mathbf{h}_1 + \frac{\overline{\mathbf{h}_2\mathbf{S}_2 + \mathbf{h}_3\mathbf{S}_3}}{1+\mathbf{S}_1}\right) = \sum_{n=0}^{\infty} \mathcal{E}_n\, \omega^n \tag{165}$$

can be obtained as

$$\mathcal{E}_0 = \frac{1}{2}\left(\mathbf{h}_1 + \frac{\overline{\mathbf{h}_2\mathbf{S}_2^{(0)} + \mathbf{h}_3\mathbf{S}_3^{(0)}}}{1+\mathbf{S}_1^{(0)}}\right) = \frac{1}{2}\overline{\sqrt{\mathbf{h}\cdot\mathbf{h}}} \tag{166}$$

$$= \frac{1}{2}\overline{\sqrt{F^2\ \sin^2(\omega t) + G^2\ \cos^2(\omega\, t) + \omega_0^2}} \tag{167}$$

$$= \frac{\sqrt{G^2+\omega_0^2}}{\pi} E\left(\frac{G^2-F^2}{G^2+\omega_0^2}\right), \tag{168}$$

where $E(\ldots)$ denoted the complete elliptic integral of the second kind. Note that in the adiabatic limit the quasienergy $\mathcal{E}_0$ can be completely reduced to its dynamical part $\mathcal{E}_d$, since the geometrical part $\mathcal{E}_g$ is proportional to ω and only contributes to the next term $\mathcal{E}_1$. For $G = 0$, the formula for $\mathcal{E}_0$ agrees with Eq. (253) in [57]. In the circular case ($_{G} = _{F}$), the series expansion

$$\mathcal{E} = \frac{\omega+\Omega}{2}$$
$$= \frac{1}{2}\sqrt{F^2+\omega_0^2} + \left(\frac{1}{2} - \frac{\omega_0}{2\sqrt{F^2+\omega_0^2}}\right)\omega + \frac{F^2\omega^2}{4\,(F^2+\omega_0^2)^{3/2}}$$
$$+\, O(\omega^3) \tag{169}$$

yields the zeroth order contribution $\lim_{\omega\to 0}\mathcal{E} = \lim_{\omega\to 0}\frac{\omega+\Omega}{2} = \frac{1}{2}\sqrt{F^2+\omega_0^2}$ that also follows from (168) and $E(0) = \frac{\pi}{2}$.

The next term $\mathbf{S}^{(1)}$ of the series (163) is obtained as the solution of

$$\frac{d}{dt}\mathbf{S}^{(0)} = \mathbf{h}\times\mathbf{S}^{(1)}\,, \tag{170}$$

such that $\mathbf{S}^{(0)}\cdot\mathbf{S}^{(1)} = 0$ in order to guarantee normalization in linear ω-order. The result is

$$\omega\,\mathbf{S}^{(1)}(t) = \left(\frac{d}{dt}\mathbf{S}^{(0)}(t)\right)\times\frac{\mathbf{h}(t)}{\|\mathbf{h}(t)\|^2}$$
$$= \frac{2\sqrt{2}\,\omega}{\left((G^2-F^2)\cos(2\omega t) + F^2 + G^2 + 2\omega_0^2\right)^{3/2}}$$
$$\times\begin{pmatrix} -FG \\ F\,\omega_0\cos\ \omega t \\ G\,\omega_0\sin\ \omega t \end{pmatrix}. \tag{171}$$

It leads to a linear contribution to the quasienergy of

$$\omega\,\mathcal{E}_1 = \omega\left(\frac{1}{2} - \frac{F\omega_0\Pi\left(1-\frac{F^2}{G^2}\middle|\frac{G^2-F^2}{G^2+\omega_0^2}\right)}{\pi G\sqrt{G^2+\omega_0^2}}\right), \tag{172}$$

where $\Pi(\ldots|\ldots)$ denotes the complete elliptic integral of the third kind. According to (171), $\mathbf{S_1}\cdot\mathbf{h} = 0$ and hence the dynamical part $\mathcal{E}_{1d}$ of $\mathcal{E}_1$ vanishes. $\mathcal{E}_1$ consists only of the geometrical part that can be identified with the Berry phase [60–62] divided by the period since in the adiabatic limit the solid angles swept by the elliptically polarized magnetic field and by the spin vector are identical. Consequently, $\mathcal{E}_1$ vanishes in the limit of linear polarization. For $G = F$, the limit of (172) and the linear term in the series expansion (169) agree since $\Pi(0, 0) = \frac{\pi}{2}$.

According to [57], the next, quadratic term of (163) is given by

$$\mathbf{S}^{(2)} = \left(\frac{1}{\omega}\frac{d}{dt}\mathbf{S}^{(1)}\right) \times \frac{\mathbf{h}}{\|\mathbf{h}\|^2} - \frac{\mathbf{h}}{2\|\mathbf{h}\|^2}\mathbf{S}^{(1)}\cdot\mathbf{S}^{(1)} \tag{173}$$

$$= \frac{1}{\left((G^2-F^2)\cos(2t\omega)+F^2+G^2+2\omega_0^2\right)^{7/2}} \times \begin{pmatrix} \mathbf{S}^{(2)}_{1,0}+\mathbf{S}^{(2)}_{1,2}\ \cos(2\omega t)+\mathbf{S}^{(2)}_{1,4}\ \cos(4\omega t) \\ \mathbf{S}^{(2)}_{2,1}\ \cos(\omega t)+\mathbf{S}^{(2)}_{2,3}\ \cos(3\omega t) \\ \mathbf{S}^{(2)}_{3,1}\ \sin(\omega t)+\mathbf{S}^{(2)}_{3,3}\ \sin(3\omega t) \end{pmatrix}, \tag{174}$$

where

$$\mathbf{S}^{(2)}_{1,0} = -6\sqrt{2}\omega_0\left(F^4+\omega_0^2(F^2+G^2)+G^4\right), \tag{175}$$

$$\mathbf{S}^{(2)}_{1,2} = 2\sqrt{2}\omega_0\left(F^2-G^2\right)\left(2F^2+2G^2+\omega_0^2\right), \tag{176}$$

$$\mathbf{S}^{(2)}_{1,4} = 2\sqrt{2}\omega_0\left(F^2-G^2\right)^2, \tag{177}$$

$$\mathbf{S}^{(2)}_{2,1} = \sqrt{2}G\left(-6F^4+F^2(2G^2-7\omega_0^2)+11G^2\omega_0^2+8\omega_0^4\right), \tag{178}$$

$$\mathbf{S}^{(2)}_{2,3} = 3\sqrt{2}G(F-G)(F+G)\left(2F^2+\omega_0^2\right), \tag{179}$$

$$\mathbf{S}^{(2)}_{3,1} = \sqrt{2}F\left(F^2(2G^2+11\omega_0^2)-6G^4-7G^2\omega_0^2+8\omega_0^4\right), \tag{180}$$

$$\mathbf{S}^{(2)}_{3,3} = 3\sqrt{2}F\left(F^2-G^2\right)\left(2G^2+\omega_0^2\right). \tag{181}$$

The corresponding quadratic correction to the quasienergy is too complicated to be calculated here. We confine ourselves to determine $\mathcal{E}_2$ for a special set of physical parameters, namely $F=3$, $G=2$, and $\omega_0=1$. The result is

$$\mathcal{E}_2 = 57 - \frac{16147}{200\sqrt{2}} + \frac{1}{\pi}\left(\frac{4777\,\Gamma\left(\frac{1}{4}\right)^2+21036\,\Gamma\left(\frac{3}{4}\right)^2}{240\sqrt{10\pi}} - \frac{171\,\Pi\left(-\frac{5}{4}\middle|-1\right)}{\sqrt{5}}\right) \approx 0.217319\ldots\,. \tag{182}$$

The corresponding adiabatic approximation of the quasienery has been shown in Figure 8 together with the various branches of the form $n\omega \pm \mathcal{E}$. It turns out that the adiabatic approximation is a kind of envelope of a certain family of branches that interpolates between the numerous avoided level crossings of this family. This finding is insofar plausible, since by definition the adiabatic limit of quasi-energy is an analytical function of ω, while the different branches $n\omega \pm \mathcal{E}$ for $\omega \to 0$ get stronger and stronger kinks.

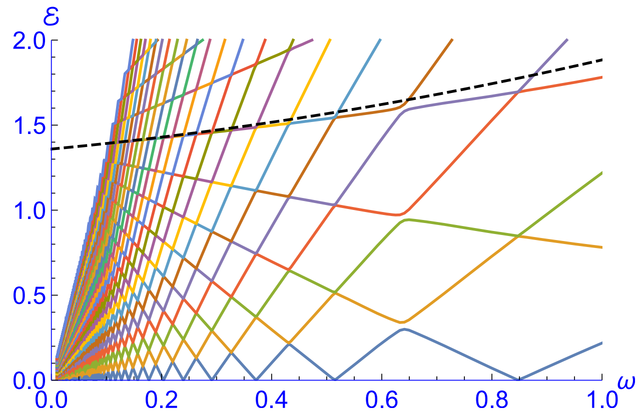

Figure 8: Various branches of the quasienergy $\mathcal{E}(\omega_0, F, G, \omega)$ as a function of ω for fixed values of $\omega_0=1$, $F=3$ and $G=2$. The different branches are generated by adding integer multiples of ω to $\pm\mathcal{E}$ and can be distinguished by their color. The black-dashed curve represents the adiabatic approximation $\mathcal{E}_0+\omega\,\mathcal{E}_1+\omega^2\,\mathcal{E}_2$ according to (168), (172) and (182).

10.2 Limit case $F, G \to 0$

For sake of comparison with the analogous results in [57], we rewrite the equation of motion (8) in the form

$$\frac{dX}{dt} = \lambda G\cos(\omega t)Z - \lambda F\sin(\omega t)Y, \tag{183}$$

$$\frac{dY}{dt} = \lambda F\sin(\omega t)X - \omega_0 Z, \tag{184}$$

$$\frac{dZ}{dt} = \omega_0 Y - \lambda G\cos(\omega t)X, \tag{185}$$

where λ is a formal expansion parameter that is ultimately set to $\lambda=1$.

In the case $\lambda=0$, there are only two normalized solutions of the classical Rabi problem that are T-periodic for all $T>0$, namely $\mathbf{X}(t)=\pm(1,0,0)^\top$. Hence, for infinitesimal λ, we expect that we still have $X(t)=\pm1+O(\lambda^2)$ but $(Y(t),Z(t))$ will describe an infinitesimal ellipse, i.e., $Y(t)=A\cos\omega t+O(\lambda^3)$ and $Z(t)=B\sin\omega t+O(\lambda^3)$, such that A and B depend linearly on λF and λG. These considerations and numerical investigations suggest the following Fourier–Taylor (FT) series ansatz, not yet normalized,

$$X(t) = \sum_{n=0}^{\infty}\lambda^{2n}\sum_{m=0}^{n}R_{n,m}(F,G,\omega,\omega_0)\cos 2m\omega t, \tag{186}$$

$$Y(t) = \sum_{n=0}^{\infty}\lambda^{2n+1}\sum_{m=0}^{n}S_{n,m}(F,G,\omega,\omega_0)\cos(2m+1)\omega t, \tag{187}$$

$$Z(t) = \sum_{n=0}^{\infty}\lambda^{2n+1}\sum_{m=0}^{n}T_{n,m}(F,G,\omega,\omega_0)\sin(2m+1)\omega t. \tag{188}$$

Inserting these series into the differential Equations (183)–(185) and collecting powers of λ yields recurrence relations for the functions $R_{n,m}, S_{n,m}$ and $T_{n,m}$. As initial conditions, we use the following choices that result from the above considerations and the lowest orders λ^0 and λ^1 of the differential Equations (183)–(185):

$$R_{0,0}(F,G,\omega,\omega_0) = 1\,, \tag{189}$$

$$R_{n,0}(F,G,\omega,\omega_0) = 0 \quad \text{for } n = 1,2,\ldots\,, \tag{190}$$

$$S_{0,0}(F,G,\omega,\omega_0) = -\frac{F\omega + G\omega_0}{(\omega-\omega_0)(\omega+\omega_0)}\,, \tag{191}$$

$$T_{0,0}(F,G,\omega,\omega_0) = -\frac{F\omega_0 + G\omega}{(\omega-\omega_0)(\omega+\omega_0)}\,. \tag{192}$$

For $n > 0$, the FT coefficients $R_{n,m}$, $S_{n,m}$ and $T_{n,m}$ can be recursively determined by means of the following relations:

$$R_{n+1,m} = \frac{1}{4m\omega}\left(F\,S_{n,m-1} - F\,S_{n,m} - G\,T_{n,m-1} - G\,T_{n,m}\right) \quad \text{for } 1 \le m \le n+1\,, \tag{193}$$

$$S_{n,m} = \frac{1}{2\left(((2m+1)\omega)^2 - \omega_0^2\right)}\left((-G\omega_0 - F(2m+1)\omega)R_{n,m} + (-G\omega_0 + F(2m+1)\omega)R_{n,m+1}\right) \quad \text{for } 0 \le m \le n\,, \tag{194}$$

$$T_{n,m} = \frac{1}{2\left(((2m+1)\omega)^2 - \omega_0^2\right)}\left((-G(1+2m)\omega - F\omega_0)R_{n,m} + (-G(1+2m)\omega + F\omega_0)R_{n,m+1}\right) \quad \text{for } 0 \le m \le n\,. \tag{195}$$

where, of course, we have to set $R_{n,n+1} = S_{n,n+1} = 0$ in (193)–(195). It follows that $R_{n,m}(F,G,\omega,\omega_0)$, $S_{n,m}(F,G,\omega,\omega_0)$ and $T_{n,m}(F,G,\omega,\omega_0)$ are rational functions of their arguments.

We will show the first few terms of the FT series for $X(t)$, $Y(t)$ and $Z(t)$:

$$X(t) = 1 - \frac{(F-G)(F+G)}{4(\omega^2-\omega_0^2)}\cos 2\,\omega t \; + \; -\frac{(F-G)(F+G)\left(3F^2\omega^2 + 3G^2\omega^2 - 4FG\omega\omega_0 - F^2\omega_0^2 - G^2\omega_0^2\right)}{8(\omega^2-\omega_0^2)^2(9\omega^2-\omega_0^2)}\cos 2\,\omega t + \left(\frac{3(F-G)^2(F+G)^2}{64(\omega^2-\omega_0^2)(9\omega^2-\omega_0^2)}\right)\cos 4\,\omega t + O(\lambda^6)\,, \tag{196}$$

$$Y(t) = \left(-\frac{F\omega + G\omega_0}{(\omega^2-\omega_0^2)}\right)\cos\,\omega t + \left(-\frac{(F-G)(F+G)(F\omega - G\omega_0)}{8(\omega^2-\omega_0^2)^2}\right) \times \cos 3\,\omega t + O(\lambda^5)\,, \tag{197}$$

$$Z(t) = \left(-\frac{G\omega + F\omega_0}{(\omega^2-\omega_0^2)}\right)\sin\,\omega t + \left(-\frac{(F-G)(F+G)(-G\omega + F\omega_0)}{8(\omega^2-\omega_0^2)^2}\right) \times \sin 3\,\omega t + O(\lambda^5)\,, \tag{198}$$

where λ stands for any linear combination of F and G. We note that the coefficients contain denominators of the form $\omega^2 - \omega_0^2$ and $9\omega^2 - \omega_0^2$ due to the denominator $(2m+1)^2\omega^2 - \omega_0^2$ in the recursion relations (194) and (195). Hence, the FT series breaks down at the resonance frequencies $\omega_{res}^{(m)} = \frac{\omega_0}{2m-1}$. This is the more plausible since according to the above ansatz $z_0 = 1$ which is not compatible with the resonance condition $z_0 = 0$ mentioned above.

Using the FT series solution (186)–(188), it is a straightforward task to calculate the quasienergy $\mathcal{E} = a_0$ as the time-independent part of the FT series of

$$\frac{1}{2}\left(\omega_0 + \frac{G\,\cos(\omega t)Y(t) + F\,\sin(\omega t)Z(t)}{R + Z(t)}\right) = a_0 + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n e^{\daleth n\omega t}\,, \tag{199}$$

according to (21). The first few terms of the result are given by

$$\mathcal{E} = \frac{\omega_0}{2} - \frac{2FG\omega + F^2\omega_0 + G^2\omega_0}{8\left(\omega^2 - \omega_0^2\right)} + \frac{4FG\left(F^2+G^2\right)\omega^3 + \left(F^4 + 22F^2G^2 + G^4\right)\omega^2\omega_0 + 12FG\left(F^2+G^2\right)\omega\omega_0^2 + \left(3F^4 + 2F^2G^2 + 3G^4\right)\omega_0^3}{128\left(\omega^2-\omega_0^2\right)^3} + O\left(\lambda^6\right). \tag{200}$$

This is in agreement with the result for linear polarization, see [57], Equation (198), if we set $G = 0$.

It will be instructive to check the first two terms of (200) by using the decomposition of the quasienergy into a dynamical and a geometrical part. In lowest order in λ, the classical RPE solution is a motion on an ellipse with semi axes

$$a = \frac{F\omega + G\omega_0}{\left|\omega^2 - \omega_0^2\right|}, \quad \text{and } b = \frac{G\omega + F\omega_0}{\left|\omega^2 - \omega_0^2\right|}. \tag{201}$$

Hence, the geometrical part of the quasienergy reads

$$\mathcal{E}_g = \frac{\omega}{4\pi}\,\pi\, a\, b + O\left(\lambda^4\right) = \frac{\omega\left(G\omega + F\omega_0\right)\left(F\omega + G\omega_0\right)}{4\left(\omega^2-\omega_0^2\right)^2} + O\left(\lambda^4\right). \tag{202}$$

The dynamical part is obtained as

$$\begin{aligned}\mathcal{E}_d &= \frac{\omega_0\, X + G\cos\overline{(\omega t)\, Y + F\sin(\omega t)\, Z}}{2R}\\ &= \frac{\omega_0}{2} + \frac{-4FG\omega^3 - 3\left(F^2+G^2\right)\omega^2\omega_0 + \left(F^2+G^2\right)\omega_0^3}{8\left(\omega^2-\omega_0^2\right)^2}\\ &\quad + O\left(\lambda^4\right).\end{aligned} \tag{203}$$

The sum of both parts together correctly yields

$$\mathcal{E} = \mathcal{E}_d + \mathcal{E}_g = \frac{\omega_0}{2} - \frac{2FG\omega + \left(F^2+G^2\right)\omega_0}{8\left(\omega^2-\omega_0^2\right)} + O\left(\lambda^4\right). \tag{204}$$

Moreover, the slope relation (132) is satisfied in the considered order,

$$\frac{\partial\mathcal{E}}{\partial\omega} = \frac{\left(G\omega + F\omega_0\right)\left(F\omega + G\omega_0\right)}{4\left(\omega^2-\omega_0^2\right)^2} + O\left(\lambda^4\right) = \frac{\mathcal{E}_g}{\omega}, \tag{205}$$

in accordance with [57], Equation (202).

However, as mentioned above, the FT series for the quasienergy has poles at the values $\omega = \omega_{res}^{(m)} = \frac{1}{2m-1}, m = 1, 2, \ldots$ and hence the present FT series ansatz is not suited to investigate the Bloch–Siegert shift for small λ. We have thus chosen another approach in Section 9.

10.3 Limit case $\omega_0 \to 0$

It is well known, see, e.g., [57] or [53], that for $\omega_0 = 0$ and linear polarization ($F = 0$) the equation of motion (183) – (185) has the exact solution

$$X(t) = \cos\left(\frac{G}{\omega}\sin\omega t\right) = J_0\left(\frac{G}{\omega}\right) + 2\sum_{m=1}^{\infty} J_{2m}\left(\frac{G}{\omega}\right)\cos 2m\omega t, \tag{206}$$

$$Y(t) = 0, \tag{207}$$

$$Z(t) = -\sin\left(\frac{G}{\omega}\sin\omega t\right) = -2\sum_{n=0}^{\infty} J_{2m+1}\left(\frac{G}{\omega}\right)\sin\,(2m+1)\omega t, \tag{208}$$

where the $J_k(\ldots)$ denote the Bessel functions of the first kind and the series representation results from the Jacobi–Anger expansion. Upon inserting the Taylor series of $J_k(x)$, that starts with the lowest power x^k, into (206) and (208), we would obtain the FT series of $X(t)$ and $Z(t)$. On the other hand, we have considered an FT series of $X(t)$, $Y(t)$ and $Z(t)$ in Section 2 that can be specialized to $\omega_0 = F = 0$. (We will indicate the specialization to $\omega_0 = F = 0$ by using the notation $\mathbb{X}(t), \mathbb{Y}(t)$ and $\mathbb{Z}(t)$) The only difference is normalization: The solution (206)–(208) is already normalized and satisfies $\overline{\mathbb{X}(t)} = J_0\left(\frac{G}{\omega}\right)$, whereas the ansatz (186)–(188) assumes $\overline{\mathbb{X}(t)} = 1$. It follows that the FT series (186)–(188), specialized to $\omega_0 = F = 0$ is identical with the FT series obtained by (206)–(208) upon division by $J_0\left(\frac{G}{\omega}\right)$. We have checked this for a couple of examples. Especially, it follows that

$$\mathbb{R}_{n,m}(g) = \left[\frac{2J_{2m}(g)}{J_0(g)}\right]_{2n} g^{2n}, \tag{209}$$

where $g \equiv \frac{G}{\omega}$ and $[f(x)]_n$ denotes the coefficient a_n of the Taylor series $f(x) = \sum_n a_n x^n$.

Unfortunately, it does not seem possible to generalize the above $\omega_0 = 0$ solution obtained for the linear polarization case to the elliptical case. However, its FT series is already known: We have only to specialize (186)–(188) to the case $\omega_0 = 0$. But unlike in the case of $\omega_0 = F = 0$, the summation over n involved in this FT series cannot be performed to obtain a result in closed form.

10.3.1 Limit case $\omega_0 = 0$ and $F \to 0$

We can only get a result for "almost linear" polarization, i.e., in the lowest linear order of F. To achieve this result, we first note that for $\omega_0 = 0$ the functions $X(t)$ and $Z(t)$ will be even functions of F and $Y(t)$ will be an odd one. This is compatible with the above-mentioned fact that $\mathbb{Y}(t)$ vanishes for $F = 0$ and can be shown by induction over n using the recurrence relations (193)–(195). It follows that the linear part $Y_1(t)$ of $Y(t) = Y_1(t)F + Y_3(t)F^3 + \ldots$ can be obtained by applying the recurrence relation (194) that reduces to

$$S_{n,m} = \frac{F}{2((2m+1)\omega)}(R_{n,m+1} - R_{n,m}) \tag{210}$$

$$\overset{(209)}{=} \frac{Fg^{2n}}{2((2m+1)\omega)}\left(\left[\frac{2J_{2(m+1)}(g)}{J_0(g)}\right]_{2n} - \left[\frac{2J_{2m}(g)}{J_0(g)}\right]_{2n}\right). \tag{211}$$

Now we can perform the sum over $n = 0, \ldots, \infty$ without any problems:

$$\sum_{n=0}^{\infty} S_{n,m} = \sum_{n=0}^{\infty}\frac{Fg^{2n}}{2((2m+1)\omega)}\left(\left[\frac{2J_{2(m+1)}(g)}{J_0(g)}\right]_{2n} - \left[\frac{2J_{2m}(g)}{J_0(g)}\right]_{2n}\right) \tag{212}$$

$$= \frac{F(J_{2(m+1)}(g) - J_{2m}(g))}{(1+2m)\omega J_0(g)}, \tag{213}$$

which, finally, yields

$$\begin{aligned} Y(t) &= Y_1(t)F + O(F^3) \\ &= \frac{F}{\omega}\sum_{m=0}^{\infty}\frac{J_{2(m+1)}(g) - J_{2m}(g)}{(1+2m)}\cos(2m+1)\omega t + O(F^3), \end{aligned} \tag{214}$$

where we have multiplied the result by $J_0(g)$ in order to obtain a normalized solution. The analytical approximation given by (206), (214) and (208) is surprisingly of good quality even for relative large values of, say, $\frac{F}{G} \sim 1/4$, see Figure 9.

10.3.2 Limit case $\omega_0 = 0$ and $F \to G$

The Rabi problem with circular polarization ($F = G$) has two simple periodic (not yet normalized) solutions of (183)–(185), namely

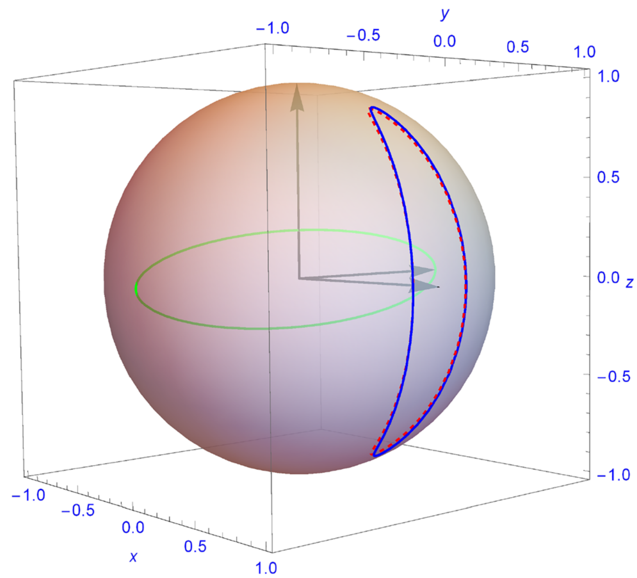

Figure 9: The periodic solution of the equation of motion (183)–(185) with the values of the parameters $F = 1/4$, $G = \omega = 1$, $\omega_0 = 0$ according to numerical integration (blue curve) and analytical approximation (206), (208) and (214) (dashed red curve). The green curve represents the ellipse in the $y-z$-plane swept by the magnetic field vector.

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \pm\begin{pmatrix} \omega - \omega_0 \\ -F\cos\omega t \\ -F\sin\omega t \end{pmatrix}, \tag{215}$$

see, e.g., [57], Equation (69). Let us consider the special solution for $\omega_0 = 0$

$$\begin{pmatrix} X_c(t) \\ Y_c(t) \\ Z_c(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{F}{\omega}\cos\omega t \\ -\frac{F}{\omega}\sin\omega t \end{pmatrix}, \tag{216}$$

and look for corrections in linear order of the parameter δ describing eccentricity, namely

$$\delta \equiv F - G. \tag{217}$$

To this end, we insert $\omega_0 = 0$ and $G = F - \delta$ into the FT series solution (186)–(188) and expand the FT series coefficients up to terms linear in δ. The $\delta = 0$ parts of the coefficients satisfy

$$R_{0,0} = 1, \tag{218}$$

$$S_{0,0} = -\frac{F}{\omega}, \tag{219}$$

$$T_{0,0} = -\frac{F}{\omega}, \tag{220}$$

in accordance with (216). The δ-linear parts are given by

$$R_{n,1} = -\frac{3^{1-n}}{2}\frac{F^{2n-1}}{\omega^{2n}}\delta, \quad \text{for } n = 1, 2, \dots, \tag{221}$$

$$S_{n,0} = -\frac{3^{1-n}}{4}\frac{F^{2n}}{\omega^{2n-1}}\delta, \quad \text{for } n = 1, 2, \dots, \tag{222}$$

$$T_{0,0} = \frac{\delta}{\omega}, \tag{223}$$

$$T_{n,0} = \frac{3^{1-n}}{4}\frac{F^{2n}}{\omega^{2n+1}}\delta, \quad \text{for } n = 1, 2, \dots, \tag{224}$$

$$S_{n,1} = \frac{3^{-n}}{4}\frac{F^{2n}}{\omega^{2n+1}}\delta, \quad \text{for } n = 1, 2, \dots. \tag{225}$$

It is straightforward to perform the summations over n and to insert the results into (186)–(188) thus obtaining the analytical approximations

$$X_a(t) = 1 + \frac{3\delta F}{2(F^2 - 3\omega^2)}\cos(2\omega t), \tag{226}$$

$$Y_a(t) = \left(-\frac{F}{\omega} + \frac{3\delta F^2}{4\omega(F^2 - 3\omega^2)}\right)\cos(\omega t) - \frac{\delta F^2}{4\omega(F^2 - 3\omega^2)}\cos(3\omega t), \tag{227}$$

$$Z_a(t) = \left(-\frac{F}{\omega} + \frac{\delta}{F}\left(\frac{1}{\omega} - \frac{3F^2}{4\omega(F^2 - 3\omega^2)}\right)\right)\sin(\omega t) - \frac{\delta F^2}{4\omega(F^2 - 3\omega^2)}\sin(3\omega t). \tag{228}$$

The quality of these approximations is surprisingly high, see Figure 10, where a deviation between analytical approximation and numerical integration is only visible for $\delta \sim \frac{1}{4}$.

11 Application: work performed on a two level system

As an application of the results obtained in the preceding sections, we consider the work performed on a TLS by an elliptically polarized magnetic field during one period. For a related experiment, see [63]. In contrast to classical physics, this work is not just a number but, following [64], has to be understood in terms of two subsequent energy measurements. At the time $\tau = 0$, the TLS is assumed to be in a mixed state according to the canonical ensemble

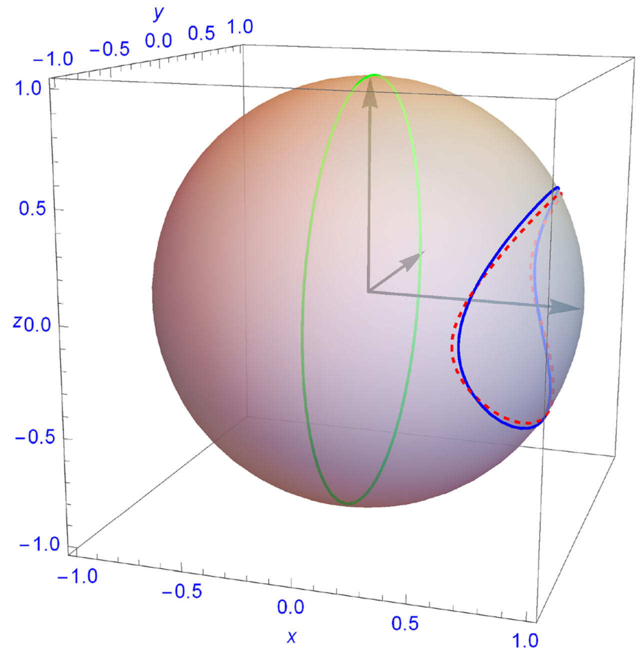

Figure 10: The periodic solution of the equation of motion (183) – (185) with the values of the parameters $F = 1, G = 3/4$, $\omega = 1, \omega_0 = 0$ according to numerical integration (blue curve) and (normalized) analytical approximations (226)–(228) (dashed red curve). The green curve represents the ellipse in the $y-z$-plane swept by the magnetic field vector.

$$W = \exp(-\beta H(0))/\mathrm{Tr}(\exp(-\beta H(0))), \tag{229}$$

with dimensionless inverse temperature $\beta = \frac{\hbar\omega}{k_B T}$ and

$$H(0) = \frac{\nu}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{230}$$

Then at the time $\tau = 0$, one performs a Lüders measurement of the instantaneous energy $H(0)$ with the two possible outcomes $\pm\frac{\nu}{2}$. Hence after the measurement the system is in the pure state P_1 with probability $\mathrm{Tr}(P_1 W) = \frac{1}{Z}e^{-\beta\nu/2}$ or in the pure state P_2 with probability $\mathrm{Tr}(P_2 W) = \frac{1}{Z}e^{\beta\nu/2}$, where P_1 and P_2 are the projectors onto the eigenstates of $H(0)$, i.e.,

$$P_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{231}$$

and $Z = e^{-\beta\nu/2} + e^{\beta\nu/2}$.

After this measurement, the system evolves according to the Schrödinger Equation (1) with Hamiltonian $H(\tau)$. At the time $\tau = 2\pi$, the system hence is in the pure state $U(2\pi, 0)\,P_1\,U(2\pi, 0)^*$ with probability $\mathrm{Tr}(P_1 W)$ or in the pure state $U(2\pi, 0)\,P_2\,U(2\pi, 0)^*$ with probability $\mathrm{Tr}(P_2 W)$. Then a second measurement of the instantaneous energy

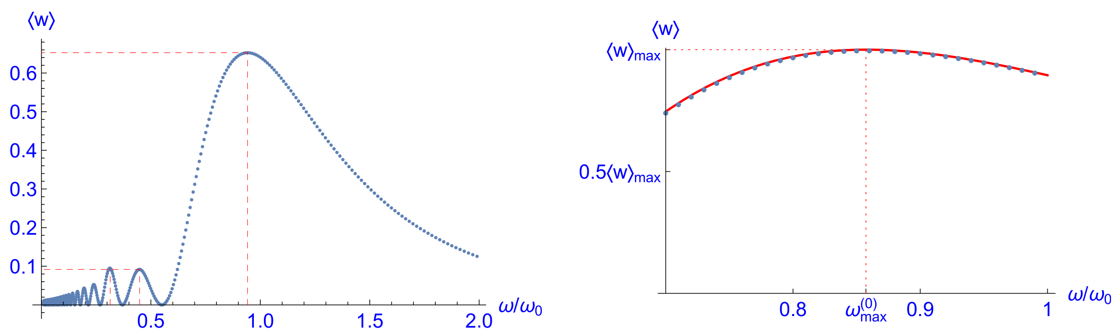

Figure 11: The mean value $\langle w\rangle$ of the work performed of a two level system (TLS) as a function of the normalized driving frequency ω/ω_0. The left panel contains the numerical results for the fixed values $F = 0.5$, $G = 0.1$, $\beta = 10$ and shows a prominent maximum at $\omega_{\max} \approx 0.941843\,\omega_0$, as well as a large number of smaller maxima. The right panel is devoted to the limit of small amplitudes and contains the numerical results for $F = 0.05$, $G = 0.01$, $\beta = 10$ (blue dots) that agree well with the analytical limit according to (235) (red curve). For the right panel, the work is maximal at $\omega^{(0)}_{\max} \approx 0.857295\,\omega_0$.

$H(2\pi) = H(0)$ is performed, again with the two possible outcomes $\pm\frac{\nu}{2}$. Both measurements together have four possible outcomes symbolized by pairs (i, j) where $i, j = 1, 2$ that occur with probabilities

$$p_{i,j} = \mathrm{Tr}\,(W\,P_i)\mathrm{Tr}\,(P_j\,U\,(2\pi, 0)\,P_i\,U\,(2\pi, 0)^*)\,, \qquad (232)$$

such that $\sum_{i,j=1}^{2} p_{i,j} = 1$. The differences of the outcomes of the energy measurements yield three possible values $w = \pm\nu, 0$ for the work performed on the system with respective probabilities that can be calculated by using the monodromy matrix (56). The result is identical to that obtained for the case of linear polarization in [53] since it depends only on the parameters α, r of the monodromy matrix. Using the above probabilities, it is straightforward to calculate the mean value of the performed work

$$\langle w\rangle = \omega_0\,(p_{2,1} - p_{1,2}) = 4\,\omega_0 r^2\,(1 - r^2)\sin^2\alpha\,\tanh\left(\frac{\beta\nu}{2}\right) \geq 0\,, \qquad (233)$$

see [53], Equation (55). A detailed investigation of the work statistics is beyond the scope of the present article. We will only give an example of the frequency dependence of $\langle w\rangle$ that exhibits resonance phenomena similar to those mentioned in Section 9, see Figure 11.

However, a clear difference to the situation dealt with in Section 9 is that for small amplitudes the frequency $\omega_{\max}$ where $\langle w\rangle$ is maximal does not approach the eigenfrequency ω_0 of the TLS but some other limit $\omega^{(0)}_{\max}$ in the interval

$$0.8\,\omega_0 < \omega^{(0)}_{\max} < 0.9\,\omega_0\,, \qquad (234)$$

depending on the eccentricity of the elliptic polarization. The small amplitude limit $\langle w\rangle^{(0)}$ of $\langle w\rangle$ can be calculated by using the lowest order approximation derived in Section 10.2 and reads:

$$\langle w\rangle^{(0)} = \frac{4\,\omega_0}{\left(\omega^2 - \omega_0^2\right)^2}\sin^2\left(\frac{\pi\omega_0}{\omega}\right)\tanh\left(\frac{\beta\nu}{2}\right)(F\omega + G\omega_0)^2\,, \qquad (235)$$

see Figure 11 for an example.

12 Summary and outlook

The time evolution of the TLS subject to a monochromatic, circularly polarized external field (RPC) can be solved in terms of elementary functions, and the analogous problem with linear polarization (RPL) leads to the confluent Heun functions. However, these two problems are only limit cases of the general Rabi problem with elliptical polarization (RPE), and it is a natural question to look for a solution of the latter valid in the realm where the rotating wave approximation breaks down. This is done in the present paper by performing the following steps:

(1) Reduction to the classical RPE,
(2) reduction of the classical time evolution to the first quarter period,
(3) transformation of the classical equation of motion to two third order differential equations, and
(4) solution of the latter by power series.

This strategy has been checked by comparison with the numerical integration of the equations of motion for an example. Moreover, we have calculated the various Fourier series of the components of the periodic solution and the corresponding quantum or classical Floquet exponent (or quasienergy). Further, we have obtained the first terms of the power series for the resonance frequencies w. r. t. the semi-axes F and G of the polarization ellipse. The latter were checked by comparison with the partially known results in the circular ($F = G$) and in the linear polarization limit ($G = 0$). This kind of result could not be obtained by a pure numerical treatment of RPE and thus justifies our analytical approach. Analogous remarks apply to the problem of how much work is performed on a TLS by the driving field. For a first overview numerical methods are sufficient, see Figure 11, but analytical methods yield more detailed results, e.g., for the small amplitude limit, see Section 11.

Other limit cases that can be discussed without recourse to the third-order differential equation are the adiabatic limit ($\omega \to 0$), the small amplitude limit ($F, G \to 0$) and the limit of vanishing energy splitting of the TLS ($\omega_0 \to 0$). In the latter case, it turns out that the exact solution of the special case $\omega_0 = F = 0$ cannot be transferred to the elliptical domain except for the limit cases $F \to 0$ and $F \to G$. Moreover, we have checked some general statements on the Rabi problem [57] like the slope relation (132) using our analytical approximations for some of these limit cases, as well as the power series solutions mentioned above.

It appears that this completes the set of problems related to the RPE that can be addressed with the present methods, with one exception: In principle, it would also be possible to solve the underlying $s = 1/2$ Schrödinger equation directly by a transformation into a third-order differential equation. However, we have omitted this topic, firstly because of lack of space, and secondly because it is not clear which new results would follow from the direct solution.

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