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Permanent solutions for some oscillatory motions of fluids with power-law dependence of viscosity on the pressure and shear stress on the boundary

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Abstract: In this paper, we provide simple expressions for the permanent solutions corresponding to some oscillatory motions of two classes of Newtonian fluids with power-law dependence of viscosity on the pressure between two infinite horizontal parallel plates. The fluid motion is generated by the lower plate that applies an oscillatory shear stress to the fluid. Such solutions, which are lack in the existing literature, can be useful both for those who want to eliminate the transients from their experiments and as tests to verify numerical schemes that are developed to study complex unsteady flow problems of these fluids. The similar solutions corresponding to the motion due to a constant shear stress on the boundary are also determined and, contrary to our expectations, the shear stresses are constant on the whole flow domain although the associated velocity fields depend both of the spatial variable and the dimensionless pressure-viscosity coefficient. Finally, for validation, some comparative graphical illustrations are included and the convergence of starting solutions to the permanent solutions is graphically proved. Spatial profiles of starting solutions are also provided.

Keywords: oscillatory motions; permanent solutions; pressure-dependent viscosity; shear stress on the boundary.

1 Introduction

The fact that the fluid viscosity can depend on the pressure was early recognized by Stokes [1] and many experimental studies have certified this dependence (see for instance the

authoritative book of Bridgman [2] for the pertinent literature prior to 1931 and Cutler et al. [3], Johnson and Cameron [4], Johnson and Tewaarwerk [5] and Bair and Winer [6] with the therein references for later). The viscosity-dependence of pressure is important in many applications like the food processing, pharmaceutical tablet manufacturing, fuel oil pumping, fluid film lubrication, microfluidics, polymeric materials, geophysics and it is a direct result of gravity. Unfortunately, in the existing literature, there are very few studies containing exact solutions for unsteady motions of fluids with pressure-dependent viscosity in which the gravity to be taken into consideration and the first exact solutions for steady motions of such fluids seem to be those of Hron et al. [7].

Kannan and Rajagopal [8] studied the motion of such fluids between two parallel plates and found that the gravity has a significant influence on the flow characteristics as the viscosity changes with the depth. Le Roux [9] also found that the boundary layers develop in an orthogonal rheometer when slip boundary conditions are taken into consideration. Such boundary layers also develop when such fluids flow along an inclined plane and the gravity effects are taken into consideration (see Rajagopal [10]). Kannan and Rajagopal [11] shown that, due to the gravity, the viscosity of the rock glacier meaningfully varies with the depth. Interesting expressions for permanent solutions (steady-state or long time solutions) of the motion of fluids with pressure-dependent viscosity have been established by Rajagopal [12] and Prusa [13]. Prusa also showed that the starting solutions, that have been numerically obtained, converge to the steady-state solutions presented in terms of Kelvin functions.

The most general solutions for motions of the fluids with exponential or power-law dependence of viscosity on the pressure between two infinite horizontal parallel plates when the gravity effects are taken into consideration have been established by Rajagopal et al. [14] under series form in terms of the eigenfunctions of some suitable boundary value problems. The fluid motion is generated by both plates that are moving in their planes. Some uniqueness and qualitative results regarding the nature of obtained solutions are also provided. Other exact solutions for steady flows of such fluids in cylindrical or spherical domains have been established by Kalagirou et al. [15], respectively Housiadas et al. [16]. A steady extension to the couple stress fluids with exponential

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dependence of viscosity on the pressure has been presented by Lin et al. [17].

It is worth pointing out the fact that in all motion problems that have been above mentioned the velocity is given on the boundary although in many practical situations what is specified is the shear stress on the boundary (see Renardy [18–20]). In addition, some polymeric fluids can slip or slide on the boundary and the no-slip boundary condition is not applicable. Consequently, boundary conditions on stresses are meaningful and Renardy [20] showed how well-posed boundary value problems can be formulated. Actually, in Newtonian mechanics force is the cause and kinematics is the effect (see Rajagopal [21] for a detailed discussion on the problem) and prescribing the shear stress on the plate is tantamount to prescribe the force (shear) applied to move it. The first exact solutions for motions of fluids in which the shear stress is given on the boundary have been early enough obtained by Waters and King [22]. During the time a lot of similar solutions have been established (see for instance Jamil and Fetecau [23], Fetecau et al. [24], Zafar et al. [25] and therein references) but none of them for fluids with pressure dependent viscosities.

The purpose of this note is to provide exact expressions for the permanent components of the starting solutions corresponding to some motions with engineering applications of two classes of fluids with power-law dependence of viscosity on the pressure. The fluid motion between two infinite horizontal parallel plates is generated by the lower plate that applies oscillatory shear stresses to the fluid. The obtained solutions are independent of the initial condition but satisfy the boundary conditions and governing equations. To validate the results that have been obtained, three limiting cases are considered and some comparative graphical representations are included. Furthermore, the similar solutions for the motion induced by the lower plate that applies a constant shear stress to the fluid are also determined. In both types of fluids, the dimensionless steady shear stress corresponding to this motion is constant on the whole flow domain although the adequate velocity fields are functions both of the spatial variable y and the pressure-viscosity coefficient α .

2 Constitutive and governing equations

The constitutive equation of incompressible Newtonian fluids with pressure-dependent viscosity is given by (see for instance Rajagopal [12] and [10])

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} = -p\mathbf{I} + \eta(p)\mathbf{A}, \quad (1)$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ is the extra stress tensor, $\mathbf{A}$ is the first Rivlin–Ericksen tensor, $\mathbf{I}$ is the unit tensor, p is the hydrostatic pressure and $\eta(p)$ is the fluid viscosity which, for the present study, will have anyone of the forms

$$\eta(p) = \mu[\alpha(p - p_h) + 1]^{1/2} \text{ or } \eta(p) = \mu[\alpha(p - p_h) + 1]^2. \quad (2)$$

In the last two relations, μ is the fluid viscosity at the reference pressure p_h while the positive constant α is the dimensional pressure-viscosity coefficient. This coefficient, whose values have been obtained from measurements, is a measure of the pressure dependence of the viscosity of the liquid in elastohydrodynamic lubrication. Making $\alpha \rightarrow 0$ in Eq. (2), $\eta(p) \rightarrow \mu$ and Eq. (1) reduces to the constitutive equation of the ordinary incompressible Newtonian fluids. Furthermore, both relations (2) implies $\eta(p) \rightarrow \infty$ if $p \rightarrow \infty$, a feature that has been experimentally proved (see Bridgman [2]).

In the following, we shall determine simple exact expressions for the permanent solutions (steady-state or long time solutions) corresponding to some oscillatory motions of fluids defined by Eqs. (1) and (2)₁ or (2)₂. Let us assume that such a fluid is at rest between two infinite horizontal parallel plates at the distance h apart. At the moment $t = 0^+$, the lower plate begins to apply an oscillatory shear stress $S \cos(\omega t)$ or $S \sin(\omega t)$ to the fluid. Here, S and ω are the amplitude, respectively the frequency of oscillations. Due to the shear, the fluid is gradually moved and following Prusa [13] and Rajagopal et al. [14] we are looking for a solution of the form

$$\mathbf{v} = \mathbf{v}(y, t) = u(y, t)\mathbf{i}, \quad p = p(y), \quad (3)$$

where $\mathbf{v}$ is the velocity vector and $\mathbf{i}$ is the unit vector along the x -direction of a suitable Cartesian coordinate system whose upward y -axis is perpendicular to the plates.

Substituting Eq. (3) in (1) and the result in the balance of linear momentum, we attain to the following relevant partial or ordinary differential equations

$$\begin{aligned} \tau(y, t) = \eta(p) \frac{\partial u(y, t)}{\partial y}, \quad \rho \frac{\partial u(y, t)}{\partial t} &= \frac{\partial \tau(y, t)}{\partial y}, \\ \frac{dp(y)}{dy} + \rho g &= 0. \end{aligned} \quad (4)$$

Into above relations, $\tau(y, t) = S_{yx}(y, t)$ is the non-trivial component of the extra-stress tensor $\mathbf{S}$, ρ is the fluid density and g is the acceleration due to the gravity. In order to get Eq. (4)₃, we used the fact that the specific body force is $-g\mathbf{j}$, with $\mathbf{j}$ the unit vector along the y -axis. Of course, the continuity equation is identically verified while Eq. (4)₃ implies

$$p = p(y) = \rho g(h - y) + p_h, \quad (5)$$

where $p_h = p(h)$ is the dimensional pressure at the upper plate.

Now, replacing $p(y)$ from Eq. (5) in (2), introducing the results in Eq. (4)₁ and eliminating $\tau(y, t)$ we find the governing equations for the velocity field corresponding to the motion of the two types of fluids with pressure-dependent viscosity, namely

$$\begin{aligned} \mu \sqrt{\alpha \rho g (h - y) + 1} \frac{\partial^2 u(y, t)}{\partial y^2} - \frac{\mu \alpha \rho g}{2 \sqrt{\alpha \rho g (h - y) + 1}} \frac{\partial u(y, t)}{\partial y} \\ = \rho \frac{\partial u(y, t)}{\partial t}; \quad 0 < y < h, t > 0, \end{aligned} \quad (6)$$

respectively

$$\begin{aligned} \mu [\alpha \rho g (h - y) + 1]^2 \frac{\partial^2 u(y, t)}{\partial y^2} - 2 \mu \alpha \rho g [\alpha \rho g (h - y) + 1] \frac{\partial u(y, t)}{\partial y} \\ = \rho \frac{\partial u(y, t)}{\partial t}; \quad 0 < y < h, t > 0. \end{aligned} \quad (7)$$

The non-trivial shear stresses corresponding to these motions of fluids in consideration can be obtained using the relations

$$\begin{aligned} \tau(y, t) &= \mu \sqrt{\alpha \rho g (h - y) + 1} \frac{\partial u(y, t)}{\partial y} \quad \text{or} \quad \tau(y, t) \\ &= \mu [\alpha \rho g (h - y) + 1]^2 \frac{\partial u(y, t)}{\partial y}. \end{aligned} \quad (8)$$

In addition, the initial condition

$$u(y, 0) = 0; \quad 0 \leq y \leq h \quad (9)$$

and the following boundary conditions

$$\begin{aligned} \tau(0, t) &= \mu \sqrt{\alpha \rho g (h - y) + 1} \frac{\partial u(y, t)}{\partial y} \Big|_{y=0} \\ &= S \cos(\omega t), \quad u(h, t) = 0; \quad t > 0, \quad \text{or} \\ \tau(0, t) &= \mu \sqrt{\alpha \rho g (h - y) + 1} \frac{\partial u(y, t)}{\partial y} \Big|_{y=0} \\ &= S \sin(\omega t), \quad u(h, t) = 0; \quad t > 0, \end{aligned} \quad (10)$$

respectively

$$\begin{aligned} \tau(0, t) &= \mu [\alpha \rho g (h - y) + 1]^2 \frac{\partial u(y, t)}{\partial y} \Big|_{y=0} \\ &= S \cos(\omega t), \quad u(h, t) = 0; \quad t > 0, \quad \text{or} \\ \tau(0, t) &= \mu [\alpha \rho g (h - y) + 1]^2 \frac{\partial u(y, t)}{\partial y} \Big|_{y=0} \\ &= S \sin(\omega t), \quad u(h, t) = 0; \quad t > 0, \end{aligned} \quad (11)$$

have to be satisfied.

Introducing the next non-dimensional variables, functions and parameter

$$y^* = \frac{y}{h}, \quad t^* = \frac{S}{\mu} t, \quad u^* = \frac{\mu}{hS} u, \quad \tau^* = \frac{\tau}{S}, \quad \alpha^* = \alpha \rho g h \quad (12)$$

and dropping out the star notation, the governing Eqs. (6) and (7) for the fluid velocity take the dimensionless forms

$$\begin{aligned} \sqrt{\alpha(1-y)+1} \frac{\partial^2 u(y, t)}{\partial y^2} - \frac{\alpha}{2 \sqrt{\alpha(1-y)+1}} \frac{\partial u(y, t)}{\partial y} \\ = \text{Re} \frac{\partial u(y, t)}{\partial t}; \quad 0 < y < 1, t > 0, \end{aligned} \quad (13)$$

$$\begin{aligned} [\alpha(1-y)+1]^2 \frac{\partial^2 u(y, t)}{\partial y^2} - 2\alpha[\alpha(1-y)+1] \frac{\partial u(y, t)}{\partial y} \\ = \text{Re} \frac{\partial u(y, t)}{\partial t}; \quad 0 < y < 1, t > 0, \end{aligned} \quad (14)$$

where $\text{Re} = hV/\nu$ ($V = hS/\mu$ being a characteristic velocity) is the Reynolds number.

The corresponding initial and boundary conditions are

$$u(y, 0) = 0; \quad 0 \leq y \leq 1, \quad (15)$$

$$\frac{\partial u(y, t)}{\partial y} \Big|_{y=0} = \frac{\cos(\omega t)}{\sqrt{\alpha+1}} \quad \text{or} \quad \frac{\partial u(y, t)}{\partial y} \Big|_{y=0} = \frac{\sin(\omega t)}{\sqrt{\alpha+1}}, \quad u(1, t) = 0; \quad t > 0, \quad (16)$$

respectively,

$$\frac{\partial u(y, t)}{\partial y} \Big|_{y=0} = \frac{\cos(\omega t)}{(\alpha+1)^2} \quad \text{or} \quad \frac{\partial u(y, t)}{\partial y} \Big|_{y=0} = \frac{\sin(\omega t)}{(\alpha+1)^2}, \quad u(1, t) = 0; \quad t > 0. \quad (17)$$

The adequate non-dimensional shear stresses, which are given by

$$\begin{aligned} \tau(y, t) &= \sqrt{\alpha(1-y)+1} \frac{\partial u(y, t)}{\partial y} \quad \text{or} \quad \tau(y, t) \\ &= [\alpha(1-y)+1]^2 \frac{\partial u(y, t)}{\partial y}, \end{aligned} \quad (18)$$

allow us to determine the dimensionless permanent frictional forces per unit area exerted by the fluid on the upper plate.

3 Permanent solutions

It is well known the fact that the starting solutions $u_c(y, t)$ and $u_s(y, t)$ corresponding to oscillatory motions induced by the lower plate that applies shear stresses of the form $S \cos(\omega t)$, respectively $S \sin(\omega t)$ to the fluid can be written as sums of the permanent and transient components, namely

$$u_c(y, t) = u_{cp}(y, t) + u_{ct}(y, t), \quad u_s(y, t) = u_{sp}(y, t) + u_{st}(y, t). \quad (19)$$

Some time after the motion initiation the fluid moves according to the starting solutions $u_c(y, t)$ or $u_s(y, t)$. After this time, when the transients disappear or can be neglected, the fluid motion is characterized by the permanent solutions $u_{cp}(y, t)$, respectively $u_{sp}(y, t)$. This is the required time to reach the permanent state. It is important for those who want to eliminate the transients from their experiments. Generally, in order to determine this time for different fluid motions, it is necessary to know exact expressions at least for their permanent solutions because the corresponding starting solutions can be numerically determined.

This is the reason that, in the following, we shall establish exact expressions for the permanent components $u_{cp}(y, t)$ and $u_{sp}(y, t)$ corresponding to the motions of the two types of fluids with pressure-dependent viscosity. Their expressions are independent of the initial condition (15) but satisfy the boundary conditions and governing equations. In order to do determine them in a simple way, we introduce the dimensionless complex velocity field

$$v_p(y, t) = u_{cp}(y, t) + iu_{sp}(y, t), \quad (20)$$

where i is the imaginary unit. In order to determine $v_p(y, t)$ for the oscillatory motions of the two types of fluids with power-law dependence of viscosity on the pressure, we shall use suitable changes of the spatial variable and the method of separating variables.

3.1 Case $\eta(p) = \mu\sqrt{\alpha(p - p_h) + 1}$

In this case, the corresponding dimensionless complex velocity field $v_p(y, t)$ has to satisfy the partial differential equation (see Eq. (13))

$$\begin{aligned} \sqrt{\alpha(1-y)+1} \frac{\partial^2 v_p(y, t)}{\partial y^2} - \frac{\alpha}{2\sqrt{\alpha(1-y)+1}} \frac{\partial v_p(y, t)}{\partial y} \\ = \operatorname{Re} \frac{\partial v_p(y, t)}{\partial t}; \quad 0 < y < 1, t \in R, \end{aligned} \quad (21)$$

and the boundary conditions

$$\left. \frac{\partial v_p(y, t)}{\partial y} \right|_{y=0} = \frac{1}{a} e^{i\omega t}, \quad v_p(1, t) = 0; \quad t \in R, \quad (22)$$

where $a = \sqrt{\alpha + 1}$.

Making the change of independent variable

$$y = \frac{\alpha + 1 - r^2}{\alpha} \quad \text{or equivalently} \quad r = \sqrt{\alpha(1-y)+1}, \quad (23)$$

the partial differential Eq. (21) takes the simpler form

$$\frac{\alpha^2}{4r} \frac{\partial^2 v_p(r, t)}{\partial r^2} = \operatorname{Re} \frac{\partial v_p(r, t)}{\partial t}; \quad 1 < r < a, \quad t \in R, \quad (24)$$

while the boundary conditions (22) become

$$v_p(1, t) = 0, \quad \left. \frac{\partial v_p(r, t)}{\partial r} \right|_{r=a} = -\frac{2}{a} e^{i\omega t}; \quad t \in R. \quad (25)$$

For the mixed boundary value problems (24) and (25), we are looking for a separable solution, namely

$$v_p(r, t) = V(r)T(t). \quad (26)$$

Introducing $v_p(r, t)$ from Eq. (26) in (24), we find that

$$\frac{\alpha^2}{\operatorname{Re}} \frac{d^2 V(r)}{dr^2} - 4\lambda r V(r) = 0, \quad \frac{dT(t)}{dt} - \lambda T(t) = 0, \quad (27)$$

where λ is constant. Bearing in mind, the boundary conditions (25) it results that $\lambda = i\omega$ and the function $V(r)$ has to satisfy the conditions

$$V(1) = 0, \quad \left. \frac{dV(r)}{dr} \right|_{r=a} = -\frac{2}{a}. \quad (28)$$

Eq. (27)₁ is an ordinary differential equation of Airy type whose general solution is (see for instance Zill [26], the exercise 34 on the page 251)

$$\begin{aligned} V(r) = \sqrt{r} \left[C_1 J_{1/3}(br\sqrt{r}) + C_2 Y_{1/3}(br\sqrt{r}) \right], \\ b = \frac{4}{3a} \sqrt{-i\omega \operatorname{Re}}, \end{aligned} \quad (29)$$

where $J_{1/3}(\cdot)$ and $Y_{1/3}(\cdot)$ are standard Bessel functions of the first and second kind of order $1/3$ while C_1 and C_2 are constants. Using the boundary conditions (28) and the property (A1) from Appendix we find that

$$V(r) = \frac{4\sqrt{r}}{3aba} \frac{J_{1/3}(b)Y_{1/3}(br\sqrt{r}) - Y_{1/3}(b)J_{1/3}(br\sqrt{r})}{Y_{1/3}(b)J_{-2/3}(ab\sqrt{a}) - J_{1/3}(b)Y_{-2/3}(ab\sqrt{a})}. \quad (30)$$

Finally, bearing in mind the definition of the complex velocity $v_p(y, t)$ and the previous analysis, it results that the dimensionless permanent solutions $u_{cp}(y, t)$ and $u_{sp}(y, t)$ corresponding to the two oscillatory motions of the fluids under consideration are given by

$$\begin{aligned} u_{cp}(y, t) = \frac{\sqrt[4]{\alpha(1-y)+1}}{\sqrt{\omega \operatorname{Re}(\alpha+1)}} \\ \times \operatorname{Real} \left\{ \frac{J_{1/3} Y_{1/3} \left(b \sqrt[4]{\alpha(1-y)+1} \right) - Y_{1/3} (b) J_{1/3} \left(b \sqrt[4]{\alpha(1-y)+1} \right)}{Y_{1/3} (b) J_{-2/3} \left(b \sqrt[4]{\alpha+1} \right) - J_{1/3} (b) Y_{-2/3} \left(b \sqrt[4]{\alpha+1} \right)} e^{i(\omega t + \pi/4)} \right\}, \end{aligned} \quad (31)$$

$$u_{sp}(y, t) = \frac{\sqrt{\alpha(1-y)+1}}{\sqrt{\omega \operatorname{Re}(\alpha+1)}} \times \operatorname{Im} \left\{ \frac{J_{1/3}(b)Y_{1/3}\left(b\sqrt[3]{\alpha(1-y)+1}\right) - Y_{1/3}(b)J_{1/3}\left(b\sqrt[3]{\alpha(1-y)+1}\right)}{Y_{1/3}(b)Y_{-2/3}\left(b\sqrt[3]{\alpha+1}\right) - J_{1/3}(b)Y_{-2/3}\left(b\sqrt[3]{\alpha+1}\right)} e^{i(\omega t + \pi/4)} \right\}, \quad (32)$$

where Im denotes the imaginary part of that which follows.

These solutions, as we already mentioned before, are independent of the initial condition (15) but satisfy the boundary conditions (16) and the governing Eq. (13). In order to determine the exact expressions of the dimensionless permanent frictional forces per unit area exerted by the fluid on the upper plate, namely

$$\tau_{cp}(1, t) = \frac{1}{\sqrt{\alpha+1}} \operatorname{Real} \left\{ \frac{J_{1/3}(b)Y_{-2/3}(b) - Y_{1/3}(b)J_{-2/3}(b)}{J_{1/3}(b)Y_{-2/3}\left(b\sqrt[3]{\alpha+1}\right) - Y_{1/3}(b)J_{-2/3}\left(b\sqrt[3]{\alpha+1}\right)} e^{i\omega t} \right\}, \quad (33)$$

$$\tau_{sp}(1, t) = \frac{1}{\sqrt{\alpha+1}} \operatorname{Im} \left\{ \frac{J_{1/3}(b)Y_{-2/3}(b) - Y_{1/3}(b)J_{-2/3}(b)}{J_{1/3}(b)Y_{-2/3}\left(b\sqrt[3]{\alpha+1}\right) - Y_{1/3}(b)J_{-2/3}\left(b\sqrt[3]{\alpha+1}\right)} e^{i\omega t} \right\}, \quad (34)$$

we have introduced Eqs. (31) and (32) in (18)₁ and took $y = 1$.

3.2 Case $\eta(p) = \mu[\alpha(p - p_h) + 1]^2$

The complex velocity field $v_p(y, t)$ corresponding to this case has to satisfy the partial differential equation (see Eq. (14))

$$\begin{aligned} & [\alpha(1-y)+1]^2 \frac{\partial^2 v_p(y, t)}{\partial y^2} - 2\alpha[\alpha(1-y)+1] \frac{\partial v_p(y, t)}{\partial y} \\ &= \operatorname{Re} \frac{\partial v_p(y, t)}{\partial t}; \quad 0 < y < 1, t \in R, \end{aligned} \quad (35)$$

with the boundary conditions

$$\left. \frac{\partial v_p(y, t)}{\partial y} \right|_{y=0} = \frac{1}{(\alpha+1)^2} e^{i\omega t}, \quad v_p(1, t) = 0; \quad t \in R. \quad (36)$$

Making the change of independent variable

$$y = \frac{\alpha+1-e^r}{\alpha} \quad \text{or equivalently} \quad r = \ln[\alpha(1-y)+1], \quad (37)$$

Eq. (35) reduces to the next partial differential equation with constant coefficients

$$\frac{\partial^2 v_p(r, t)}{\partial r^2} + \frac{\partial v_p(r, t)}{\partial r} - \frac{\operatorname{Re}}{\alpha^2} \frac{\partial v_p(r, t)}{\partial t}; \quad 0 < r < c, \quad t \in R, \quad (38)$$

where $c = \ln(\alpha+1)$.

Looking again for a separable solution as in the previous case, we find that the corresponding function $V(r)$ has to satisfy the ordinary differential equation

$$\frac{d^2 V(r)}{dr^2} + \frac{dV(r)}{dr} - \frac{i\omega \operatorname{Re}}{\alpha^2} V(r) = 0, \quad (39)$$

whose general solution is given by

$$V(r) = C_1 e^{r_1 r} + C_2 e^{r_2 r}; \quad r_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 + 4i\omega \operatorname{Re}}}{2\alpha}. \quad (40)$$

Coming back to the original variables, it results that

$$v_p(y, t) = \{C_1[\alpha(1-y)+1]^{r_1} + C_2[\alpha(1-y)+1]^{r_2}\} e^{i\omega t} \quad (41)$$

and using the boundary conditions (36) we find that

$$v_p(y, t) = \frac{1}{\alpha(\alpha+1)} \frac{[\alpha(1-y)+1]^{r_2} - [\alpha(1-y)+1]^{r_1}}{r_1(\alpha+1)^{r_1} - r_2(\alpha+1)^{r_2}} e^{i\omega t}. \quad (42)$$

Consequently, the dimensionless permanent solutions corresponding to this case are

$$\begin{aligned} u_{cp}(y, t) &= \frac{1}{\alpha(\alpha+1)} \operatorname{Real} \left\{ \frac{[\alpha(1-y)+1]^{r_2} - [\alpha(1-y)+1]^{r_1}}{r_1(\alpha+1)^{r_1} - r_2(\alpha+1)^{r_2}} e^{i\omega t} \right\}, \\ & \quad (43) \end{aligned}$$

$$u_{sp}(y, t) = \frac{1}{\alpha(\alpha+1)} \operatorname{Im} \left\{ \frac{[\alpha(1-y)+1]^{r_2} - [\alpha(1-y)+1]^{r_1}}{r_1(\alpha+1)^{r_1} - r_2(\alpha+1)^{r_2}} e^{i\omega t} \right\}, \quad (44)$$

while the permanent frictional forces per unit area exerted by the fluid on the upper plate are

$$\tau_{cp}(1, t) = \frac{1}{\alpha(\alpha+1)} \operatorname{Real} \left\{ \frac{\sqrt{\alpha^2 + 4i\omega \operatorname{Re}}}{r_1(\alpha+1)^{r_1} - r_2(\alpha+1)^{r_2}} e^{i\omega t} \right\}, \quad (45)$$

$$\tau_{sp}(1, t) = \frac{1}{\alpha(\alpha+1)} \operatorname{Im} \left\{ \frac{\sqrt{\alpha^2 + 4i\omega \operatorname{Re}}}{r_1(\alpha+1)^{r_1} - r_2(\alpha+1)^{r_2}} e^{i\omega t} \right\}. \quad (46)$$

Finally, we would like to point out the fact that interesting results regarding the start-up of the stress-controlled oscillatory flow have been theoretically and experimentally obtained by Lee et al. [27] and Hassager [28]. In both papers, the shear stress on the boundary oscillates sinusoidal, i.e. $\tau(0, t) = S \sin(\omega t + \varphi)$. The phase φ of the applied stress varies between 0 and 2π . Our boundary conditions can be obtained for $\varphi = 0$ or $\pi/2$. An analytical expression for the non-trivial value around which the resulting strain oscillates is determined by Lee et al. [27], while a transiently decaying contribution is provided by Hassager [28] in the case of the theory of linear viscoelasticity.

4 Limiting cases

In order to certify the correctness of results that have been here obtained, as well as to get some physical insight of

them, we consider some particular cases whose solutions are well known in the existing literature or can be easily determined and include a few comparative graphical representations.

4.1 Case $\alpha = 0$ (Ordinary incompressible Newtonian fluids)

The dimensionless permanent velocity fields $u_{Ncp}(y, t)$ and $u_{Nsp}(y, t)$ corresponding to incompressible Newtonian fluids performing the same motions as in previous sections have the simple forms (see Javaid et al. [29, Eqs. (42) without porous and magnetic effects])

$$u_{Ncp}(y, t) = -\frac{1}{\sqrt{\omega Re}} \operatorname{Re} \left\{ \frac{\operatorname{sh}[(1-y)\sqrt{i\omega Re}]}{\operatorname{ch}(\sqrt{i\omega Re})} \frac{e^{i\omega t}}{\sqrt{i}} \right\}, \quad (47)$$

$$u_{Nsp}(y, t) = -\frac{1}{\sqrt{\omega Re}} \operatorname{Im} \left\{ \frac{\operatorname{sh}[(1-y)\sqrt{i\omega Re}]}{\operatorname{ch}(\sqrt{i\omega Re})} \frac{e^{i\omega t}}{\sqrt{i}} \right\}, \quad (48)$$

while the adequate frictional forces exerted by the fluid on the upper plate are given by

$$\begin{aligned} \tau_{Ncp}(1, t) &= \operatorname{Re} \left\{ \frac{1}{\operatorname{ch}(\sqrt{i\omega Re})} e^{i\omega t} \right\}, \\ \tau_{Nsp}(1, t) &= \operatorname{Im} \left\{ \frac{1}{\operatorname{ch}(\sqrt{i\omega Re})} e^{i\omega t} \right\}. \end{aligned} \quad (49)$$

Now, we show that the Newtonian solutions $u_{Ncp}(y, t)$ and $u_{Nsp}(y, t)$ can be obtained as limiting cases of $u_{cp}(y, t)$ and $u_{sp}(y, t)$ given by Eqs. (31), respectively (32) when $\alpha \rightarrow 0$. Indeed, using the asymptotic approximations (A2) from Appendix, we can easily prove that

$$\begin{aligned} u_{cp}(y, t) &\approx \frac{1}{\sqrt{\omega Re} \sqrt[4]{(\alpha+1)(\alpha(1-y)+1)}} \operatorname{Re} \left\{ \frac{\sin \left[b \left[1 - \sqrt[4]{\alpha(1-y)+1} \right] \right]}{\cos \left[b \left[1 - \sqrt[4]{\alpha+1} \right] \right]} \frac{e^{i\omega t}}{\sqrt{-i}} \right\}, \\ u_{sp}(y, t) &\approx \frac{1}{\sqrt{\omega Re} \sqrt[4]{(\alpha+1)(\alpha(1-y)+1)}} \operatorname{Im} \left\{ \frac{\sin \left[b \left[1 - \sqrt[4]{\alpha(1-y)+1} \right] \right]}{\cos \left[b \left[1 - \sqrt[4]{\alpha+1} \right] \right]} \frac{e^{i\omega t}}{\sqrt{-i}} \right\}, \end{aligned} \quad (50)$$

for small enough values of the non-dimensional pressure-viscosity coefficient α . Using the Maclaurin series expansions for $[1 + \alpha(1-y)]^{3/4}$ and $(1+\alpha)^{3/4}$ and the identities (A3) from Appendix and taking the limit of Eqs. (50) and (51) when $\alpha \rightarrow 0$ we recover Eqs. (47), respectively (48). Furthermore, following the same way as before, it is easy to show that the frictional forces $\tau_{Ncp}(1, t)$ and $\tau_{Nsp}(1, t)$ given by Eqs. (49) can be also obtained as limiting cases of $\tau_{cp}(1, t)$ and $\tau_{sp}(1, t)$ given by Eqs. (33), respectively (34) when $\alpha \rightarrow 0$.

As regards the dimensionless permanent solutions $u_{cp}(y, t)$ and $u_{sp}(y, t)$ given by Eqs. (43) and (44) corresponding to the same oscillatory motions of the second type of fluids with power-law dependence of viscosity on the pressure, we provide Figures 1 and 2. From these figures, as expected, it clearly results the convergence of these solutions to the Newtonian solutions $u_{Ncp}(y, t)$, respectively $u_{Nsp}(y, t)$ when the dimensionless pressure-viscosity coefficient $\alpha \rightarrow 0$. In addition, Figures 3 and 4 show that the diagrams of $\tau_{cp}(1, t)$ and $\tau_{sp}(1, t)$ given by Eqs. (45) and (46) tend to superpose over those of $\tau_{Ncp}(1, t)$, respectively $\tau_{Nsp}(1, t)$ given by Eq. (49)₁ and ₂ when the same parameter $\alpha \rightarrow 0$. As expected, in both cases the oscillations' amplitude is the same and the phase difference is obviously observed.

4.2 Case $\omega = 0$ (flow due to a constant shear stress S on the boundary)

Dimensionless permanent velocity fields corresponding to this motion of the two different types of fluids with $\eta(p) = \mu[\alpha(p-p_h)+1]^{1/2}$ or $\eta(p) = \mu[\alpha(p-p_h)+1]^2$, namely

$$\begin{aligned} u_{sp1}(y) &= \frac{2}{\alpha} \left[1 - \sqrt{\alpha(1-y)+1} \right], \text{ respectively} \\ u_{sp2}(y) &= \frac{1}{\alpha} \left[\frac{1}{\alpha(1-y)+1} - 1 \right], \end{aligned} \quad (52)$$

are immediately obtained solving the ordinary differential equations (see Eqs. (13) and (14))

$$\begin{aligned} \frac{d}{dy} \left[\sqrt{\alpha(1-y)+1} \frac{du_{sp}(y)}{dy} \right] &= 0, \text{ respectively} \\ \frac{d}{dy} \left[[\alpha(1-y)+1]^2 \frac{du_{sp}(y)}{dy} \right] &= 0, \end{aligned} \quad (53)$$

with the boundary conditions

$$\left. \frac{du_{sp}(y)}{dy} \right|_{y=0} = 1, \quad u_{sp}(1) = 0. \quad (54)$$

Figure 5 shows that, for decreasing values of the oscillations' frequency ω , the diagrams of the dimensionless velocity field $u_{cp}(y, t)$ given by Eq. (43) tend to superpose over the profile of the velocity $u_{sp2}(y)$ given by Eq. (52)₂ when $\omega \rightarrow 0$. More exactly, the dimensionless velocity field $u_{sp2}(y)$ corresponding to the motion induced by the lower plate that applies a constant shear stress S to the fluid can be obtained as a limiting case of $u_{cp}(y, t)$ given by Eq. (43) when $\omega \rightarrow 0$ and this convergence can be also proved by direct computations. In addition, the fluid velocity in absolute value smoothly decreases from

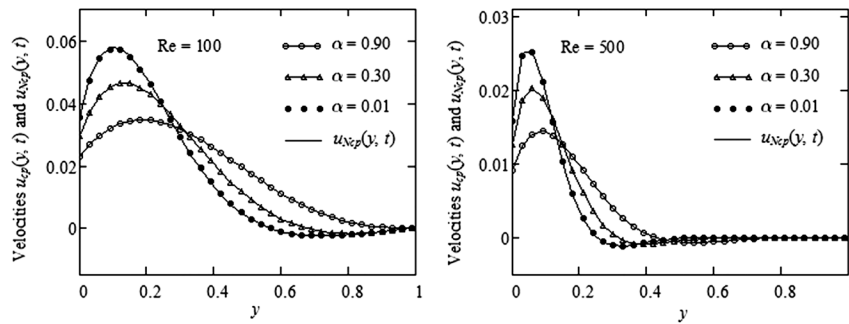


Figure 1: Profiles of the permanent solutions $u_{cp}(y, t)$ for three values of the pressure-viscosity coefficient α and $u_{Ncp}(y)$ given by Eqs. (43) and (47), for $\omega = \pi/6$ and $t = 10$.

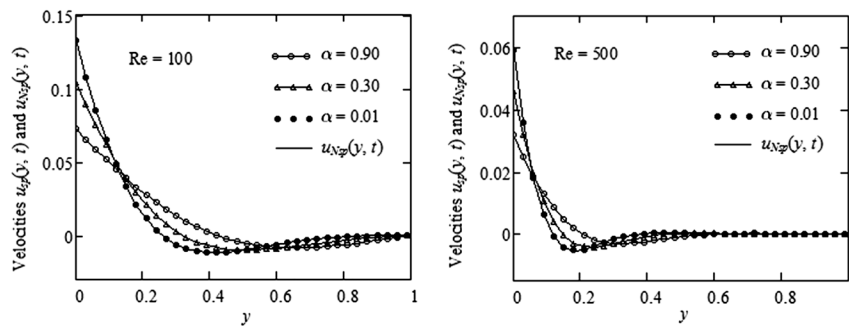


Figure 2: Profiles of the permanent solutions $u_{sp}(y, t)$ for three values of the pressure-viscosity coefficient α and $u_{Nsp}(y)$ given by Eqs. (44) and (48), for $\omega = \pi/6$ and $t = 10$.

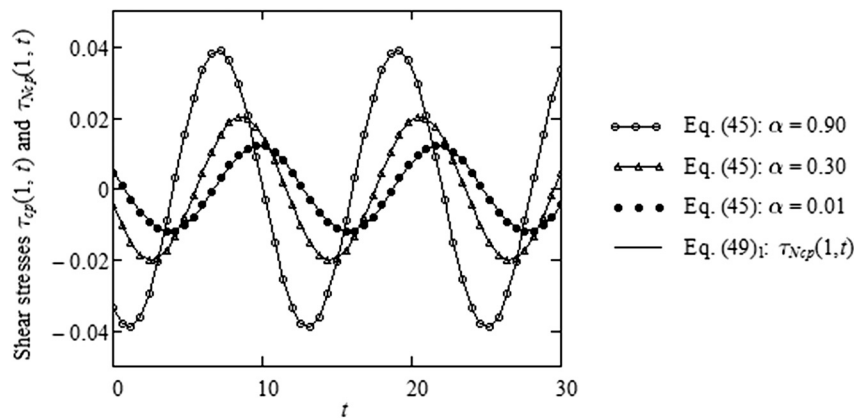


Figure 3: Time evolution of frictional forces $\tau_{cp}(1, t)$ for three values of the pressure-viscosity coefficient α and $\tau_{Ncp}(1, t)$ given by Eqs (45) and (49)₁, for $\omega = \pi/6$ and $Re = 100$.

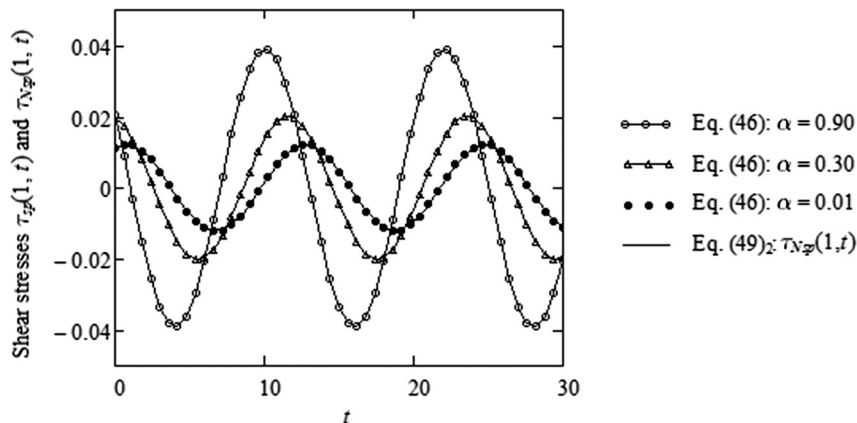


Figure 4: Time evolution of frictional forces $\tau_{sp}(1, t)$ for three values of the pressure-viscosity coefficient α and $\tau_{Nsp}(1, t)$ given by Eqs. (46) and (49)₂, for $\omega = \pi/6$ and $Re = 100$.

maximum values on the lower plate to the zero value on the stationary plate.

Finally, it is worth pointing out the fact the dimensionless shear stresses corresponding to this motion of the two types of fluids with power-law dependence of viscosity on the pressure are constants on the whole flow domain. They are equal with the non-dimensional shear stress applied by the lower plate to the fluid, namely $\tau_{Sp1} = \tau_{Sp2} = 1$, although the adequate velocity fields given by Eqs. (52)₁ and ₂ are functions both of the spatial variable y and the dimensionless pressure-viscosity coefficient α .

4.3 Case $\alpha = \omega = 0$ (flow of an incompressible Newtonian fluid induced by a constant shear stress S on the boundary)

Making $\omega \rightarrow 0$ in Eq. (47) or $\alpha \rightarrow 0$ in anyone of the equalities (52)₁ and ₂, and bearing in mind the final remark from the previous subsection, we recover the dimensionless permanent velocity and shear stress fields (see Javaid et al. [29])

$$u_{Nsp}(y) = y - 1, \quad \tau_{Nsp} = 1, \quad (55)$$

corresponding to the motion of incompressible Newtonian fluids induced by the lower plate that applies a constant shear stress S to the fluid. The sign difference between the present results and those of Javaid et al. [29] appears due to the boundary conditions on the lower plate which are taken with opposite signs. The convergence of the velocity fields given by Eqs. (52)₁ and ₂ to $u_{Nsp}(y)$ is also shown in Figure 6. Indeed, from these figures, it is clearly results that for $\alpha \rightarrow 0$ the diagrams of the two solutions tend to the same straight line whose equation is given by Eq. (55)₁. Moreover, as expected, the fluid velocity in absolute value is a decreasing function with regard to the pressure-viscosity coefficient α . This is

normal because if α increases the viscosity increases and the fluid flows slower.

5 Conclusions

In this paper, we established exact expressions for the dimensionless permanent solutions corresponding to some unsteady motions of two classes of incompressible Newtonian fluids with power-law dependence of viscosity on the pressure between two infinite horizontal parallel plates. The fluid motion is induced by the lower plate that applies an oscillatory shear stress to the fluid and the gravity effects are taken into consideration. These solutions, which are new in the literature, are presented in simple forms under the polynomial form or in terms of some standard Bessel functions of the first and second kind. They can be useful for those who want to eliminate the transients from their experiments or can be used as tests to verify different numerical methods that are developed to study more complex motion problems. In addition, these solutions have been already used here to determine dimensionless permanent frictional forces per unit area exerted by the fluid on the upper plate. Another relevant physical entity such as the vorticity can be also determined.

In order to demonstrate the correctness of results that have been obtained, as well as to get some physical insight of them, three special cases are considered and some comparative graphical representations are presented in Figures 1–6. From Figures 1 and 2, it clearly results that the diagrams of velocity fields $u_{cp}(y, t)$ and $u_{sp}(y, t)$ given by Eqs. (43) and (44) tend to superpose over those of $u_{Ncp}(y, t)$, respectively $u_{Nsp}(y, t)$ given by Eqs. (47) and (48) when the dimensionless pressure-viscosity coefficient $\alpha \rightarrow 0$. More precisely, as it was to be expected, the dimensionless permanent velocity fields corresponding to oscillatory motions of fluids with power-law dependence of viscosity on the pressure converge to those of ordinary Newtonian fluids performing the same motions if $\alpha \rightarrow 0$. The

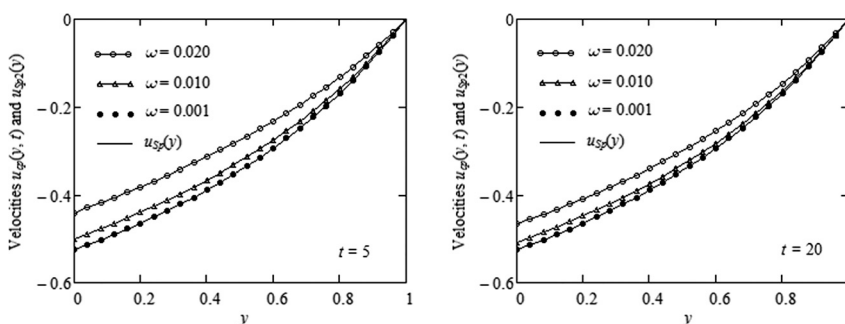


Figure 5: Profiles of the permanent solutions $u_{cp}(y, t)$ for three values of the frequency ω and $u_{sp2}(y)$ given by Eqs. (43), respectively (52)₂, for $\alpha = 0.9$ and $Re = 100$.

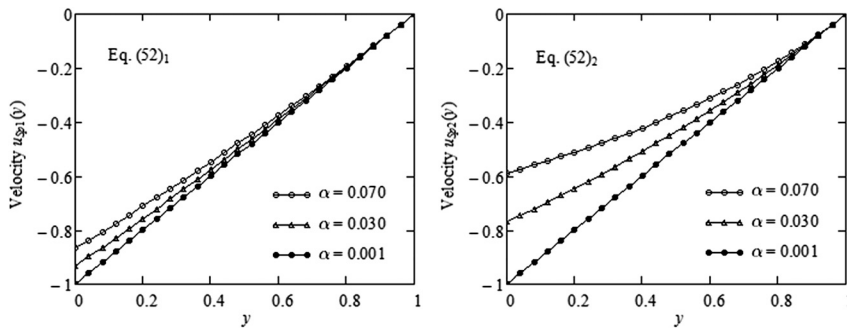


Figure 6: Profiles of the permanent solutions $u_{sp1}(y)$ and $u_{sp2}(y)$ given by Eq. (52)₁, respectively (52)₂ for three values of the pressure-viscosity coefficient α .

convergence of the first two solutions $u_{cp}(y, t)$ and $u_{sp}(y, t)$ given by Eqs. (31) and (32) to $u_{Ncp}(y, t)$, respectively $u_{Nsp}(y, t)$ has been analytically proved.

The convergence of the upper wall shear stresses $\tau_{cp}(1, t)$ and $\tau_{sp}(1, t)$ given by Eqs. (45) and (46) to the corresponding Newtonian solutions $\tau_{Ncp}(1, t)$ and $\tau_{Nsp}(1, t)$ given by Eq. (49)₁, respectively (49)₂ has been graphically proved by Figures 3 and 4 for $\alpha \rightarrow 0$. In these figures, the oscillatory characteristics of the motion are better underlined. The oscillations' amplitudes, of the same order of magnitude in both motions, diminish for decreasing values of the pressure-viscosity coefficient α and the phase difference is easily observed. Figure 5 is also devoted to the convergence of the velocity field $u_{cp}(y, t)$ given by Eq. (43) to $u_{sp}(y)$ from Eq. (52)₂ when $\omega \rightarrow 0$. Figure 6 shows that the diagrams of both velocity fields given by Eqs. (52)₁ and ₂ tend to a straight line if $\alpha \rightarrow 0$. This straight line, as expected, represents just the profile of the velocity field $u_{Nsp}(y)$ given by Eq. (55)₁. From this figure, it also results that the fluid velocity in absolute value grows for decreasing values of α . Consequently, the ordinary fluids flow faster as compared to fluids with pressure-dependent viscosity. This is possible since the fluid viscosity diminishes if the coefficient α decreases.

The time variations of the midplane velocity fields $u_{cp}(y, t)$ and $u_{sp}(y, t)$ given by Eqs. (43), respectively (44) are presented in Figures 7 and 8 for two values of Reynolds number Re . From these figures, as well as from Figures 3 and 4, the oscillatory feature of the motion is better captured. The amplitude of the oscillations, which is of the same order of magnitude in both motions, is an increasing function with respect to the dimensionless pressure-viscosity coefficient α and decreases for increasing values of the Reynolds number Re . The phase difference between the profiles of the velocities corresponding to the two motions due to cosine or sine oscillations of the shear stress on the boundary is obviously observed.

Figures 9 and 10 have been presented to provide approximate values for the required time to reach the

permanent state for oscillatory motions of the two types of fluids with pressure-dependent viscosities induced by cosine oscillations of the shear stress on the boundary. This is the time after which the fluid moves according to the permanent solutions. More exactly, it is the time after which the diagrams of starting solutions are almost identical to those of the corresponding permanent solutions. As it results from these figures, this time decreases for increasing values of the Reynolds number Re . It also diminishes for increasing values of the pressure-viscosity coefficient α (see Vieru et al. [30]). In addition, as it was to be expected, the convergence of the starting solutions $u_c(y, t)$ (numerical solutions) to the corresponding permanent components $u_{cp}(y, t)$ can be easily observed from these figures.

Spatial profiles of the dimensionless starting solutions corresponding to both oscillatory motions of fluids with power-law dependence of viscosity on the pressure are presented in Figures 11 and 12 for fixed values of the parameters α and ω and the Reynolds number Re . From these graphical representations, it is easy to see that the initial condition (15), as well as the boundary condition on the upper plate, is clearly satisfied. Furthermore, the oscillatory characteristics of the fluid motion, as well as the phase difference between motions due to cosine or sine oscillations of the shear stress on the boundary, are more easily observed. For these graphs, "the Maple pde-solver" has been used.

In Figures 13 and 14, the three-dimensional distribution of the velocity fields $u_c(y, t)$ and $u_s(y, t)$ has been visualized with the help of the two-dimensional contour graphs (see for instance Fullard and Wake [31]) for $Re = 100$, $\alpha = 0.9$ and $\omega = \pi/6$. They are based on the numerical solutions of the partial differential Eqs. (13), respectively (14) with the initial condition (15) and the corresponding boundary conditions. The trajectories of contour graphs are marked with the black color. Light and dark black colors are representing the maximum, respectively minimum values of the solutions while the middle value is coded with medium black. Their values between maximum and minimum are

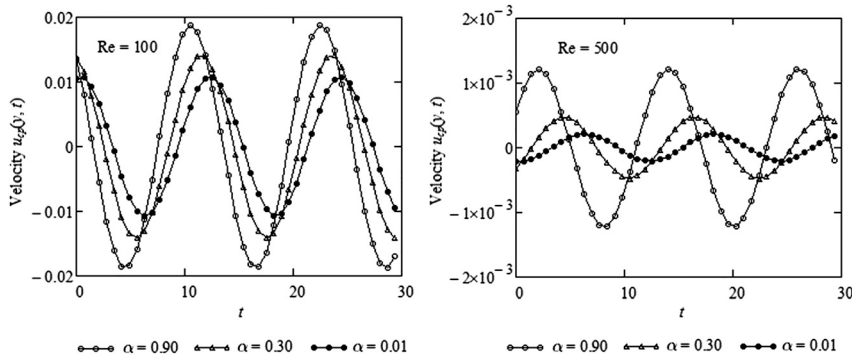


Figure 7: Midplane velocity as a function of time corresponding to $u_{cp}(y, t)$ given by Eq. (43) for three values of the pressure-viscosity coefficient α , for $\omega = \pi/6$ and $\gamma = 0.5$.

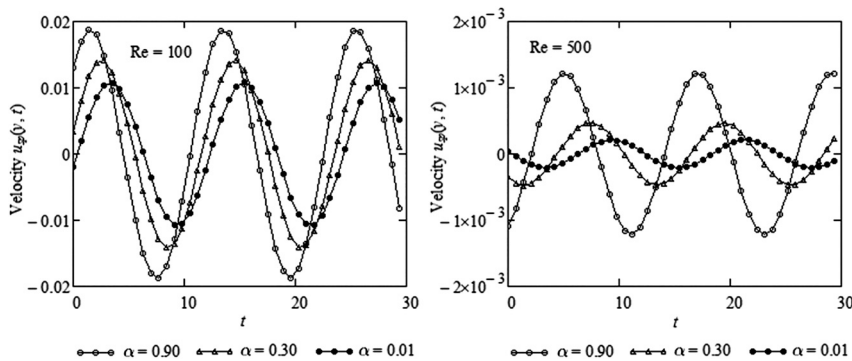


Figure 8: Midplane velocity as a function of time corresponding to $u_{sp}(y, t)$ given by Eq. (44) for three values of the pressure-viscosity coefficient α , for $\omega = \pi/6$ and $\gamma = 0.5$.

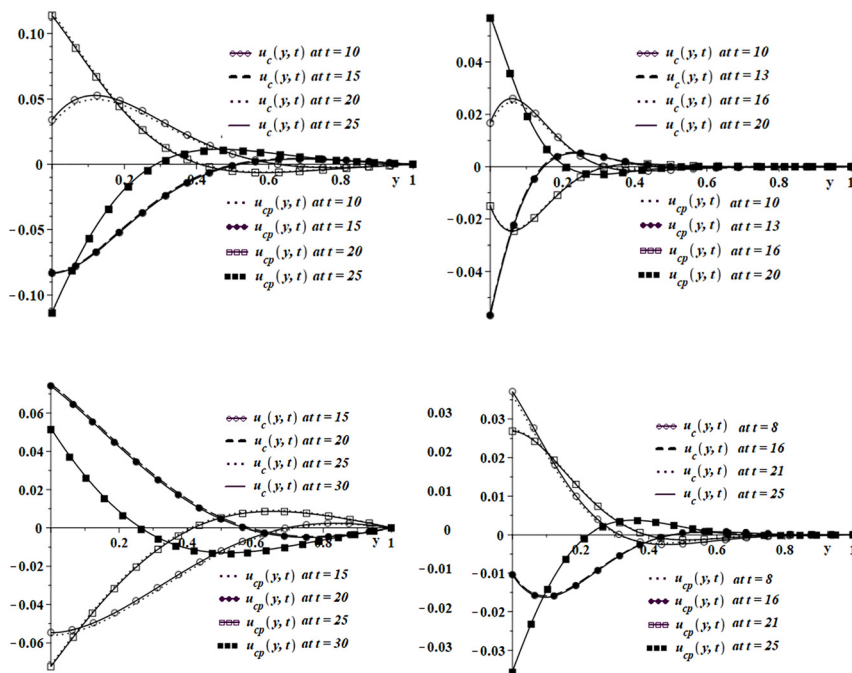


Figure 9: Comparison of the permanent solution $u_{cp}(y, t)$ given by Eq. (31) with the corresponding starting solution (numerical solution), for $\alpha = 0.9$, $\omega = \pi/6$.

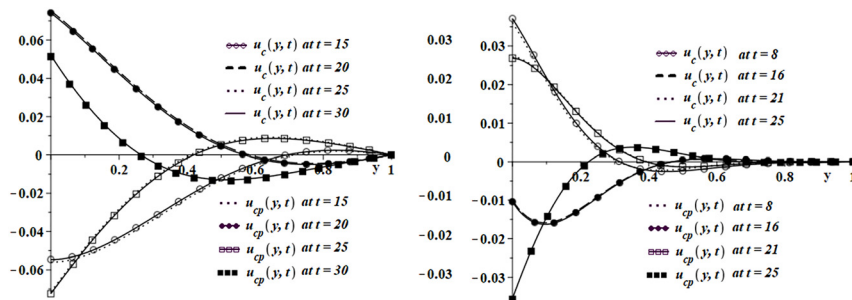


Figure 10: Comparison of the permanent solution $u_{cp}(y, t)$ given by Eq. (43) with the corresponding starting solution (numerical solution), for $\alpha = 0.9$, $\omega = \pi/6$.

viewed with the gradient of the color between light and dark black. The oscillatory behavior of the solutions is observed by viewing the alternation of the two distinct sets of almost closed trajectories along the time t . Along the y -axis, the color of all curves tends to become middle black for increasing values of y . It means that, as expected, the

amplitude of oscillations decreases for increasing values of the spatial variable y .

The principal outcomes that can be brought to light from the previous presentation are:

- Permanent solutions corresponding to some motions of Newtonian fluids with power-law dependence of

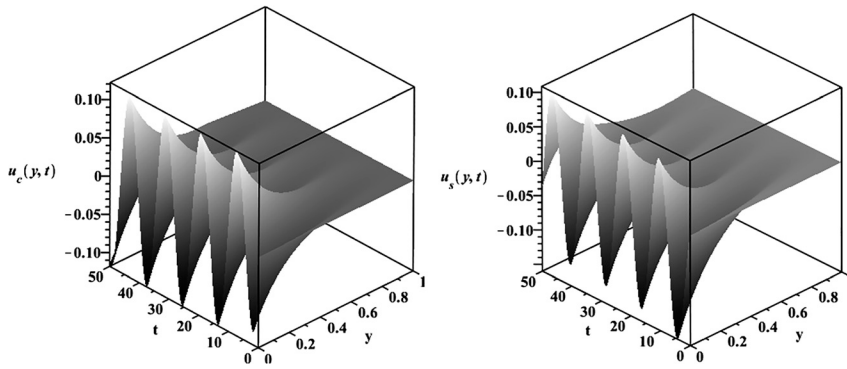


Figure 11: Spatial profiles of the velocity fields $u_c(y, t)$ and $u_s(y, t)$ (numerical solutions) whose permanent components $u_{cp}(y, t)$, respectively $u_{sp}(y, t)$ are given by Eqs. (31) and (32), for $Re = 100$, $\alpha = 0.9$, $\omega = \pi/6$.

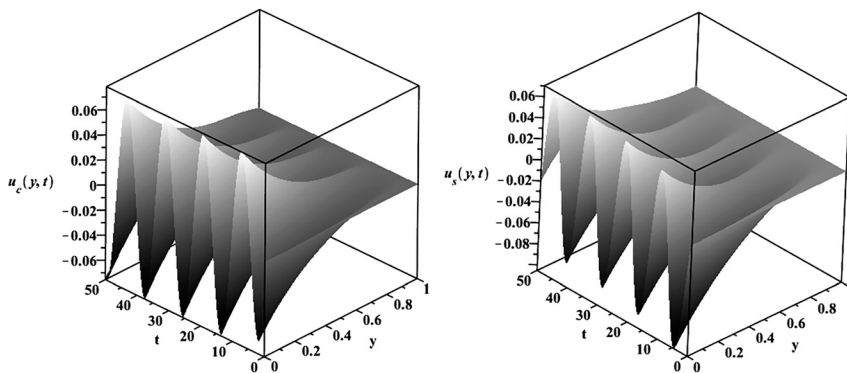


Figure 12: Spatial profiles of the velocity fields $u_c(y, t)$ and $u_s(y, t)$ (numerical solutions) whose permanent components $u_{cp}(y, t)$, respectively $u_{sp}(y, t)$ are given by Eqs. (43) and (44), for $Re = 100$, $\alpha = 0.9$, $\omega = \pi/6$.

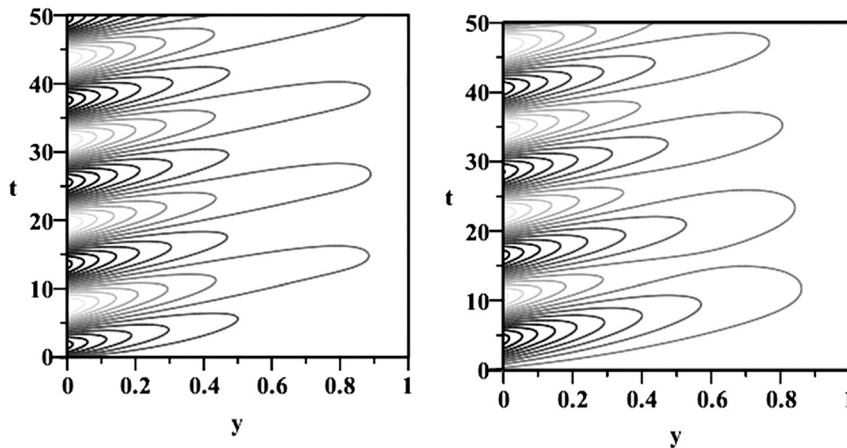


Figure 13: Contour profiles of the velocity fields $u_c(y, t)$ and $u_s(y, t)$ (numerical solutions) whose permanent components $u_{cp}(y, t)$, respectively $u_{sp}(y, t)$ are given by Eqs. (31) and (32), for $Re = 100$, $\alpha = 0.9$, $\omega = \pi/6$.

- viscosity on the pressure have been established in simple forms.
- The amplitude of oscillations of velocity and frictional force diminishes for decreasing values of the pressure-viscosity coefficient α or increasing values of Reynolds number Re .
- Required time to reach the permanent state has been approximated for motions due to cosine oscillations of the shear on the boundary. It diminishes for increasing values of Re .
- Similar solutions for the motion induced by a constant shear on the boundary have been also established.

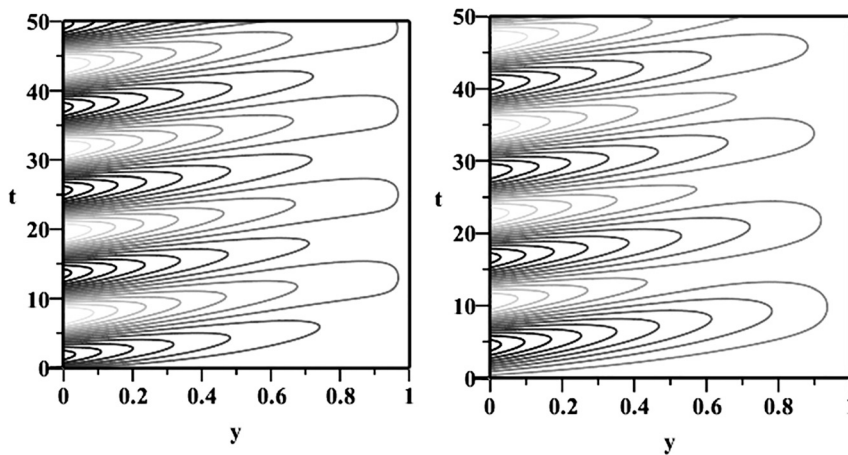


Figure 14: Contour profiles of the velocity fields $u_c(y, t)$ and $u_s(y, t)$ (numerical solutions) whose permanent components $u_{cp}(y, t)$, respectively $u_{sp}(y, t)$ are given by Eqs. (43) and (44), for $Re = 100$, $\alpha = 0.9$, $\omega = \pi/6$.

Ordinary fluids flow faster than fluids with pressure-dependent viscosities.

$$\sin(iz) = \operatorname{ish}(z), \quad \cos(iz) = \operatorname{ch}(z). \quad (\text{A3})$$

The shear stresses corresponding to this motion of both types of fluids are constant over the entire flow field although the adequate velocities depend on the spatial variable y and α .

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Appendix

$$zJ'_\nu(z) + \nu J_\nu(z) = zJ_{\nu-1}(z), \quad (\text{A1})$$

where $J_\nu(\cdot)$ and $Y_\nu(\cdot)$ are Bessel functions of the first and second kind of order ν .

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left[z - \frac{(2\nu+1)\pi}{4}\right],$$

$$Y_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left[z - \frac{(2\nu+1)\pi}{4}\right] \text{ for } |z| \gg 1, \quad (\text{A2})$$

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