 Z. Naturforsch. 2020; 75(8)a: 713–725

Guenbo Hwang*

Initial-boundary value problems for the one-dimensional linear advection–dispersion equation with decay

https://doi.org/10.1515/zna-2020-0106
Received April 13, 2020; accepted June 18, 2020; published online July 31, 2020

Abstract: Initial-boundary value problems for the one-dimensional linear advection–dispersion equation with decay (LAD) are studied by utilizing a unified method, known as the Fokas method. The method takes advantage of the spectral analysis of both parts of Lax pair and the global algebraic relation coupling all initial and boundary values. We present the explicit analytical solution of the LAD equation posed on the half line and a finite interval with general initial and boundary conditions. In addition, for the case of periodic boundary conditions, we show that the solution of the LAD equation is asymptotically t-periodic for large t if the Dirichlet boundary datum is periodic in t. Furthermore, it can be shown that if the Dirichlet boundary value is asymptotically periodic for large t, then so is the unknown Neumann boundary value, which is uniquely characterized in terms of the given asymptotically periodic Dirichlet boundary datum. The analytical predictions for large t are compared with numerical results showing the excellent agreement.

Keywords: advection–dispersion equation; convection–diffusion equation; initial-boundary value problem; spectral analysis.

1 Introduction

In this paper, we study initial-boundary value problems for the one-dimensional linear advection–dispersion equation with decay (LAD) [1]

$$q_t = Dq_{xx} - cq_x - r_c q, \tag{1}$$

where $D > 0$, $c \geq 0$ and $r_c \geq 0$ represents the dispersion, advection coefficients and the first order decay rate. This equation is also known as the convection–diffusion equation with a reaction, according to scientific contexts [2]. The LAD equation is a combination of the linear dispersion and advection equations plus the decay, but which is also an analytical model, describing physical, chemical or biological diffusive transport phenomena in science and engineering [3–6]. For example, it describes a mathematical model for one-dimensional contaminant transport in porous medium systems with first-order decay. The LAD equation without the decay term was formally solved in [7, 8]. Analytical solutions of the LAD equation, including the decay term, have been developed in several literatures [1, 3, 6] (see also [9] and references therein). It should be noted that most proposed analytical solutions are for the LAD equation with relatively limited initial and boundary conditions, such as homogeneous, constant or exponential boundary conditions in time. Moreover, these solutions can be involved with the complementary error functions. A few explicit analytical solutions are available if more general or complicated initial and boundary conditions are prescribed.

We propose here explicit analytical solutions for initial-boundary value problems of the LAD equation with general boundary conditions by utilizing a unified method, also known as the Fokas method [10, 11] (see the monograph [12] and the pedagogical literature of the method [13]). The Fokas method has been introduced to analyze boundary value problems for nonlinear integrable systems, which can be considered as a significant extension of the inverse scattering transform. Moreover, it has been shown that the method can be extensively applicable to a large class of partial differential equations (PDEs); for example, nonlinear integrable systems and linear evolution equations [11, 14–16] and linear and nonlinear elliptic PDEs [17–21] as well as linear and integrable nonlinear discrete equations [22, 23].

The presented method has several advantages. For linear cases, the Fokas method is relatively simple, but very effective, to implement. The main steps of the Fokas method can be summarized as follows: (i) simultaneous analysis of the both parts of Lax pair, which can be considered a novel type of separability [24]; (ii) analysis of the global relation which is an algebraic equation that involves all initial and boundary values. This global relation can be used to determine the unknown boundary values, known as the generalized Dirichlet-to-Neumann map [25] (see also [26, 27]

***Corresponding author: Guenbo Hwang**, Department of Mathematics and Institute of Natural Sciences, Daegu University, Gyeongsan, Gyeongbuk, 38453, Korea, E-mail: ghwang@daegu.ac.kr

for the recent application of the global relation). Moreover, the Fokas method presents an integral representation of the solution with explicit exponential (x, t)-dependence. Thus, it enables to characterize the asymptotic behaviors of the solution and the unknown boundary values for large t [28–33].

The purpose of the work is to present new explicit analytical solutions for initial-boundary value problems of the LAD equation with general initial and boundary conditions. More specifically, letting $t = t'/D$ and dropping prime for simplicity, we consider the following rescaled equation

$$q_t = q_{xx} - bq_x - aq, \quad a, b \geq 0. \tag{2}$$

Note that one can reduce LAD equation (2) to the LAD equation without decay term by changing variable $q(x,t) = \mathrm{e}^{-at}u(x,t)$. The equation without decay term has been studied and solved by using the Fokas method in [34]. However, the form of equation (2) has the advantage of studying periodic asymptotics of the solution and boundary values for large t. We will derive the explicit integral representation of the solution for the Dirichlet boundary value problem of equation (2) posed on half line via the Fokas method. In addition, we will discuss the asymptotic behaviors of the solution and boundary values for large t. We will show that if the Dirichlet boundary value is periodic, the solution of the LAD is asymptotically t-periodic as $t \to \infty$. Moreover, it can be shown that if the Dirichlet boundary value is asymptotically periodic as $t \to \infty$, then so is the unknown Neumann boundary value, which can be uniquely characterized in terms of the asymptotically periodic Dirichlet boundary value. It should be noted that the Fokas method can be effectively applicable to solve boundary value problems with more general and complicated boundary conditions. In this respect, we will further address the explicit solutions for equation (2) posed on the half line with the Neumann and Robin boundary conditions as well as the solution of equation (2) formulated on a finite interval. In addition, we will present several examples including comparison of the analytical and numerical results, as applications.

2 The LAD equation on the half line

In Section 2, we first study the Dirichlet boundary value problem for the LAD equation posed on the half line

$$q_t = q_{xx} - bq_x - aq, \quad x > 0, \quad t > 0, \tag{3}$$

with the initial condition $q(x, 0) = q_0(x)$ and the Dirichlet boundary condition, denoted by

$$q(0, t) = g_0(t).$$

Also, we denote the unknown Neumann boundary value by $q_x(0, t) = g_1(t)$. We assume that $q_0(x)$ is sufficiently smooth and decays rapidly as $x \to \infty$ and $g_0(t)$ is sufficiently smooth.

The LAD equation can be written as an overdetermined linear system, known as a Lax pair

$$\mu_x - ik\mu = q, \tag{4a}$$

$$\mu_t + \omega(k)\mu = (ik - b)q + q_x, \tag{4b}$$

where $\mu = \mu(x, t, k)$ is the eigenfunction with the spectral parameter $k \in \mathbb{C}$ and the dispersion relation is given by

$$\omega(k) = k^2 + ibk + a.$$

Indeed, the LAD equation (2) is the compatible condition of the Lax pair equation (4) in the sense that $\mu_{tx} = \mu_{xt}$ implies that q solves equation (2) if the spectral parameter k is independent of x and t. Note that $\mathrm{Re}\,\omega(k) = k_1^2 - k_2^2 - bk_2 + a$ for $k = k_1 + ik_2 \in \mathbb{C}$ $(k_1, k_2 \in \mathbb{R})$. We introduce the region $D = \{k \in \mathbb{C} | \mathrm{Re}\,\omega(k) < 0\}$ and we decompose the region D into $D = D^+ \cup D^-$, where $D^+ = \{k \in \mathbb{C} | \mathrm{Re}\,\omega(k) < 0$ and $\mathrm{Im}\, z > 0\}$ and $D^- = \{k \in \mathbb{C} | \mathrm{Re}\,\omega(k) < 0$ and $\mathrm{Im}\, z < 0\}$ (see Figure 1). The Lax pair equation (4) can be written in the divergence form

$$\left(\mathrm{e}^{-ikx+\omega(k)t}q\right)_t = \left(\mathrm{e}^{-ikx+\omega(k)t}Q\right)_x, \tag{5}$$

where $Q(x, t, k) = (ik - b)q + q_x$. To analyze equation (5), we introduce the Fourier transform pair:

$$\hat{q}(k, t) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-ikx} q(x, t), \quad \mathrm{Im}\, k \leq 0,$$

and

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx} \hat{q}(k, t).$$

Then, taking $\int_0^\infty \mathrm{d}x$ in equation (5) yields

$$\left(\mathrm{e}^{\omega(k)t}\hat{q}(k, t)\right)_t = -\mathrm{e}^{\omega(k)t} Q(0, t, k). \tag{6}$$

Let $\hat{q}(k, 0) = \hat{q}_0(k)$ and we define the t-transform as

$$\hat{g}_j(k, t) = \int_0^t \mathrm{d}s\, \mathrm{e}^{\omega(k)s}\, \partial_x^j\, q(0, s), \quad j = 0, 1. \tag{7}$$

Applying $\int_0^t \mathrm{d}s$ in equation (6), we find the global relation given by

$$\mathrm{e}^{\omega(k)t}\hat{q}(k, t) = \hat{q}_0(k) - \hat{Q}(k, t), \quad \mathrm{Im}\, k \leq 0, \tag{8}$$

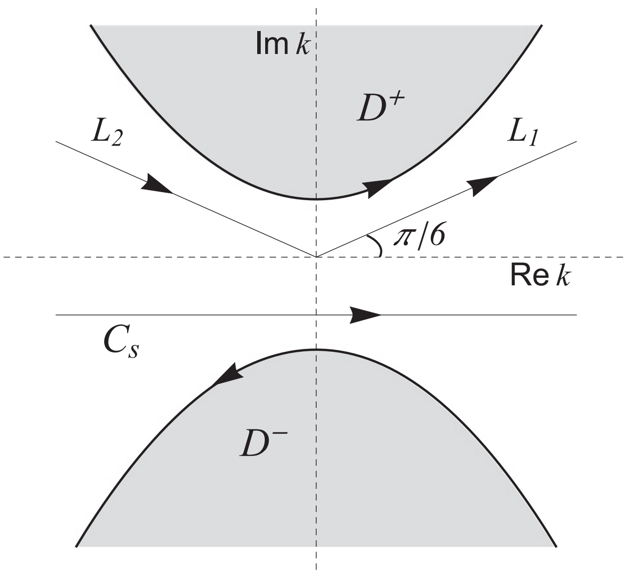

Figure 1: The region $D = D^+ \cup D^-$ (shaded) in the complex k-plane, where $\mathrm{Re}\,\omega(k) < 0$ with $\omega(k) = k^2 + ibk + a$ and the vertices at $k = i(-b \pm \sqrt{b^2 + 4a})/2$. The steepest descent contour C_s passing through the point $k = -ib/2$. Efficient contour $L = L_1 \cup L_2$ for numerical scheme (see the text for some details).

where

$$\widehat{Q}(k,t) = (ik - b)\widehat{g}_0(k,t) + \widehat{g}_1(k,t). \tag{9}$$

Employing the inverse Fourier transform in equation (8), the solution $q(x, t)$ can be recovered as

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx-\omega(k)t}\widehat{q}_0(k) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx-\omega(k)t}\widehat{Q}(k,t). \tag{10}$$

Since the integrand in the second integral of equation (10) is analytic and bounded in the region, where $\mathrm{Im}\,k > 0$ and $\mathrm{Re}\,\omega(k) > 0$, by the Cauchy theorem, we deform the contour $(-\infty, \infty)$ to ∂D^+ (see Figure 1). Thus, we find the reconstruction formula for the solution $q(x, t)$ as

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx-\omega(k)t}\widehat{q}_0(k) - \frac{1}{2\pi}\int_{\partial D^+} \mathrm{d}k\, e^{ikx-\omega(k)t}\widehat{Q}(k,t). \tag{11}$$

Note that the representation of the solution equation (11) involves the unknown function $\widehat{g}_1(k,t)$ via the unknown Neumann boundary value $q_x(0, t)$. Thus, for the explicit representation of the solution, it is necessary to determine this unknown boundary value. To this end, we use the global relation equation (8), which couples all initial and boundary values.

Note that if $\omega(k) = \omega(\nu(k))$, the transformation $k \to \nu(k)$ leaves $\omega(k)$ invariant. As a consequence, $\widehat{g}_j(k,t)$ ($j = 0, 1$) is invariant under this transformation. The equation $\omega(k) = \omega(\nu(k))$ has a trivial root $\nu(k) = k$ and a nontrivial root $\nu(k) = -k - ib$. Note that if $k \in D^+$, then $-k - ib \in D^-$. Hence, replacing $k \to -k - ib$ in equation (8) and solving the resulting equation for $\widehat{g}_1(k,t)$, the unknown boundary value is given by

$$\widehat{g}_1(k,t) = \widehat{q}_0(-k-ib) + ik\widehat{g}_0(k,t) - e^{\omega(k)t}\widehat{q}(-k-ib,t). \tag{12}$$

Substituting equation (12) into equation (9), we find

$$\begin{aligned}\widehat{Q}(k,t) = {} & \widehat{q}_0(-k-ib) + (2ik - b)\widehat{g}_0(k,t) \\ & - e^{\omega(k)t}\widehat{q}(-k-ib,t).\end{aligned} \tag{13}$$

The right-hand-side of equation (13) contains the unknown function $\widehat{q}(-k-ib,t)$, which is the transform of the solution $q(x, t)$. However, when this term is inserted into equation (11), the integrand is analytic and bounded in D^+. Hence, the integral of the term involving $\widehat{q}(-k-ib,t)$ over ∂D^+ vanishes by the Cauchy theorem. Finally, the explicit representation for the solution of equation (3) is given by

$$\begin{aligned}q(x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx-\omega(k)t}\widehat{q}_0(k) \\ & - \frac{1}{2\pi}\int_{\partial D^+} \mathrm{d}k\, e^{ikx-\omega(k)t}\big[\widehat{q}_0(-k-ib) \\ & + (2ik - b)\widehat{g}_0(k,t)\big].\end{aligned} \tag{14}$$

Remark 2.1. We can derive the unknown Neumann boundary value directly from equation (12) as

$$\begin{aligned}g_1(t) = {} & -\frac{1}{2i\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, k e^{-\omega(k)t}\widehat{q}_0(k) \\ & + \frac{1}{2i\pi}\int_{\partial D^+} \mathrm{d}k\, k e^{-\omega(k)t}\big[\widehat{q}_0(-k-ib) + (2ik - b)\widehat{g}_0(k,t)\big].\end{aligned} \tag{15}$$

Indeed, the effective portion of $\widehat{g}_1(k,t)$ in equation (12) is given by $\widehat{g}_1(k,t) = \widehat{q}_0(-k-ib) + ik\widehat{g}_0(k,t)$. Taking the inversion formula for the t-transform equation (a.3) discussed in Appendix, we find

$$\begin{aligned}g_1(t) = {} & \frac{1}{2i\pi}\int_{\partial D^+} \mathrm{d}k\, e^{-\omega(k)t}(2k + ib)\widehat{q}_0(-k-ib) \\ & + \frac{1}{2i\pi}\int_{\partial D^+} \mathrm{d}k\, e^{-\omega(k)t}k(2ik - b)\widehat{q}_0(-k-ib).\end{aligned} \tag{16}$$

Note that letting $k \to -k - ib$ (and then $-k - ib \in D^-$ for $k \in D^+$), we find

$$\frac{1}{2i\pi}\int_{\partial D^+} \mathrm{d}k\, \mathrm{e}^{-\omega(k)t}(k+ib)\widehat{q}_0(-k-ib) = \frac{1}{2i\pi}\int_{\partial D^-} \mathrm{d}k\, k\mathrm{e}^{-\omega(k)t}\widehat{q}_0(k).$$

Then, we deform the contour ∂D^- to $(-\infty, \infty)$ in the negative direction by using the Cauchy theorem. Thus, substituting the resulting equation into the first integral of equation (16), we find the unknown Neumann value $g_1(t)$ given by equation (15). Note that equation (15) is identical with the expression obtained from differentiating equation (14) with respect to x and evaluating the resulting equation at $x = 0$.

The representation of the solution given in equation (14) has the explicit (x, t)-dependence of the exponential form. Thus, it is possible to study the appropriate asymptotics of the solution for large t. Below, we will show that the solution for the LAD equation with a periodic Dirichlet boundary datum is asymptotically periodic for large t. The similar result of the Proposition 2.1 also can be found in [33] for the linearized Korteweg–de Vries equation.

Proposition 2.1. Let $q(x, t)$ be the solution given by equation (14). Assume that $q_0(x) \in \mathcal{S}[0, \infty)$ and $g_0(t) \in L^\infty[0, \infty)$, where $\mathcal{S}$ denote the space of Schwartz functions. Suppose that $g_0(t)$ is a periodic function with period τ. Then for any $x > 0$,

$$\lim_{n\to\infty} q(x, n\tau + t) = q_\infty(x, t), \tag{17}$$

where $q_\infty(x, t)$ is a periodic function with period τ and $q_\infty(0, t) = g_0(t)$. In particular, for any $x > 0$, $q(x, t)$ is asymptotically periodic such that

$$q(x, \tau + t) - q(x, t) = O\left(\mathrm{e}^{-at}\right), \quad t \to \infty. \tag{18}$$

Proof. Note that $\mathrm{e}^{-\omega(k)t}\widehat{q}_0(-k-ib)$ is analytic and bounded in the region, where $\mathrm{Im}k > 0$ and $\mathrm{Re}\omega(k) > 0$. Also note that by integration by parts, $(2ik - b)\widehat{g}_0(k, t) = O(1/k)$ as $k \to \infty$. Thus by the Cauchy theorem, we can deform the contour ∂D^+ to $(-\infty, \infty)$ in the second integral of equation (14) and hence, we write the representation of the solution given in equation (14) as the sum of three integrals $q(x, t) = Q^{(1)}(x, t) + Q^{(2)}(x, t) + Q^{(3)}(x, t)$, where

$$Q^{(1)}(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx-\omega(k)t}\widehat{q}_0(k),$$

$$Q^{(2)}(x, t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx-\omega(k)t}\widehat{q}_0(-k-ib),$$

$$Q^{(3)}(x, t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx-\omega(k)t}\,(2ik - b)\widehat{g}_0(k).$$

We will estimate each integral in the above equations. For $Q^{(1)}(x, t)$, we write

$$Q^{(1)}(x, t) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}y\, q_0(y)\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ik(x-y)-\omega(k)t}.$$

Then, using the following identity,

$$\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx-\omega(k)t} = \sqrt{\frac{\pi}{t}}\exp\left[-at - \frac{(x-bt)^2}{4t}\right], \tag{19}$$

we find

$$\left|Q^{(1)}(x, t)\right| \le M_1\frac{\mathrm{e}^{-at}}{\sqrt{t}}\|q_0\|_{L^1[0,\infty)} \tag{20}$$

for some constant $M_1 > 0$. For $Q^{(2)}(x, t)$, we write

$$Q^{(2)}(x, t) = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}y\, q_0(y)\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx+(ik-b)y-\omega(k)t}.$$

From the identity,

$$\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx+(ik-b)y-\omega(k)t} = \sqrt{\frac{\pi}{t}}\exp\left[-at - \frac{(x-bt)^2 + 2(x+bt)y + y^2}{4t}\right].$$

it follows that

$$\left|Q^{(2)}(x, t)\right| \le M_2\frac{\mathrm{e}^{-at}}{\sqrt{t}}\|q_0\|_{L^1[0,\infty)} \tag{21}$$

for some constant $M_2 > 0$.

Regarding $Q^{(3)}(x, t)$, we write $Q^{(3)}(x, t)$ as

$$Q^{(3)}(x, t) = -\frac{1}{2\pi}\int_0^t \mathrm{d}s\, g_0(s)\Phi(x, t-s),$$

where

$$\Phi(x, t) = \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx-\omega(k)t}\,(2ik - b).$$

Note that since $\omega'(k) = 2k + ib$,

$$\begin{aligned}\Phi(x, t) &= \frac{1}{it}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx}\frac{\mathrm{d}}{\mathrm{d}k}\left(\mathrm{e}^{-\omega(k)t}\right)\\ &= -\frac{x}{t}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{ikx-\omega(k)t}.\end{aligned}$$

Then using equation (19), we find

$$\Phi(x, t) = -\frac{x\sqrt{\pi}}{t^{3/2}}\exp\left[-at - \frac{(x-bt)^2}{4t}\right]. \tag{22}$$

In order to derive equation (17), let $\{q_n(x, t)\}_1^\infty$ be the sequence given by

$$\begin{aligned}q_n(x, t) &= q(x, n\tau + t)\\ &= Q_n^{(1)}(x, t) + Q_n^{(2)}(x, t) + Q_n^{(3)}(x, t),\end{aligned} \tag{23}$$

where $Q_n^{(j)}(x,t) = Q^{(j)}(x, n\tau + t)$ for $j = 1, 2, 3$. From equations (20) and (21), it follows that

$$\left|Q_n^{(1)}(x,t) + Q_n^{(2)}(x,t)\right| \le (M_1 + M_2)\frac{e^{-a(n\tau+t)}}{\sqrt{n\tau+t}}\|q_0\|_{L^1[0,\infty)}.$$

Hence, we find

$$\lim_{n\to\infty}\left[Q_n^{(1)}(x,t) + Q_n^{(2)}(x,t)\right] = 0.$$

For $Q_n^{(3)}(x,t)$, we consider the following series

$$\sum_{m=1}^{\infty}\left[Q_{m+1}^{(3)}(x,t) - Q_m^{(3)}(x,t)\right]. \tag{24}$$

Since $g_0(t)$ is a periodic function with period τ, we know that

$$Q_{m+1}^{(3)}(x,t) - Q_m^{(3)}(x,t) = -\frac{1}{2\pi}\int_{m\tau+t}^{(m+1)\tau+t} ds\, g_0(t-s)\Phi(x,s),$$

which implies that by equation (22),

$$\left|Q_{m+1}^{(3)}(x,t) - Q_m^{(3)}(x,t)\right| \le \frac{M_3 e^{-a(m\tau+t)}}{(m\tau+t)^{3/2}}\|g_0\|_{L^\infty[0,\infty)}$$

for some constant $M_3 > 0$. Thus, the series given in equation (24) converges and so does the sequence $Q_n^{(3)}(x,t)$ as $n\to\infty$. Thus, the limit of the sequence $\{q_n(x,t)\}$ given in equation (23) exists for any $x > 0$ and we denote this limit by $q_\infty(x,t)$. Note that

$$\begin{aligned} q_\infty(x,\tau+t) - q_\infty(x,t) = {} & \left[q_\infty(x,\tau+t) - q_n(x,\tau+t)\right] \\ & + \left[q_{n+1}(x,t) - q_\infty(x,t)\right]. \end{aligned}$$

Then, taking $n\to\infty$ in the right-hand-side of the above equation, we find $q_\infty(x,\tau+t) = q_\infty(x,t)$. Thus, $q_\infty(x,t)$ is a periodic function with period τ. Moreover, $q_n(0,t) = g_0(t)$ yields $q_\infty(0,t) = g_0(t)$. □

Next, we study the asymptotically periodic boundary data for large t. We assume that the Dirichlet boundary value is asymptotically periodic for large t in the sense that [29]

$$g_0(t) = \tilde{g}_0(t) + O\left(t^{-7/2}\right), \quad t\to\infty,$$

where $\tilde{g}_0(t)$ is a periodic function with period τ. Without loss of generality, we consider the case of $\tau = 2\pi$. In what follows, we will show that the unknown Neumann boundary value $q_x(0,t)$ is also asymptotically periodic for large t with the same period τ.

We first note that by integration by parts, we write the function $\widehat{g}_0(k,t)$ as

$$\begin{aligned} \widehat{g}_0(k,t) = {} & \frac{1}{\omega(k)}\left(e^{\omega(k)t}g_0(t) - g_0(0)\right) \\ & - \frac{1}{\omega(k)}\int_0^t ds\, e^{\omega(k)s}\dot{g}_0(s)ds. \end{aligned}$$

Deforming the contour ∂D^+ to pass to the right of the singularity $k = \frac{i}{2}(-b + \sqrt{b^2+4a})$, which is denoted by ∂D_0^+, Equation (14) can be written as

$$\begin{aligned} q(x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx-\omega(k)t}\widehat{q}_0(k) \\ & - \frac{1}{2\pi}\int_{\partial D^+} dk\, e^{ikx-\omega(k)t}\widehat{q}_0(-k-ib) \\ & + g_0(t) + \frac{1}{2\pi}\int_{\partial D_0^+} dk\, e^{ikx-\omega(k)t}\frac{2ik-b}{\omega(k)} \\ & \times\left[g_0(0) + \int_0^t ds\, e^{\omega(k)s}\dot{g}_0(s)\right], \end{aligned} \tag{25}$$

where we have used the identity: by the residue theorem,

$$\frac{1}{2\pi}\int_{\partial D_0^+} dk\, e^{ikx}\frac{2ik-b}{\omega(k)}g_0(t) = -g_0(t).$$

Then, differentiating equation (25) with respect to x and evaluating the resulting equation at $x = 0$, the function $g_1(t)$ is given by

$$\begin{aligned} g_1(t) = {} & -\frac{1}{2i\pi}\int_{-\infty}^{\infty} dk\, k e^{-\omega(k)t}\widehat{q}_0(k) \\ & + \frac{1}{2i\pi}\int_{\partial D^+} dk\, k e^{-\omega(k)t}\widehat{q}_0(-k-ib) \\ & - \frac{1}{2i\pi}\int_{\partial D_0^+} dk\, e^{-\omega(k)t}\frac{k(2ik-b)}{\omega(k)} \\ & \times\left[g_0(0) + \int_0^t ds\, e^{\omega(k)s}\dot{g}_0(s)\right]. \end{aligned} \tag{26}$$

Proposition 2.2. Assume that $\tilde{g}_0(t)$ is a smooth periodic function with period $\tau = 2\pi$ and $q(0,\cdot) - \tilde{g}_0 \in \mathcal{S}[0,\infty)$. Suppose that $\tilde{g}_0(t)$ has the Fourier series given by

$$\tilde{g}_0(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \quad t \ge 0, \tag{27}$$

where $a_n \in \mathbb{C}$ with $a_0 = 0$. Then, there is a unique periodic function $\tilde{g}_1(t)$ with period τ such that

$$q_x(0,t) - \tilde{g}_1(t) = O\left(t^{-1}\right), \quad t\to\infty, \tag{28}$$

and its Fourier series is given by

$$\tilde{g}_1(t) = \sum_{n=-\infty}^{\infty} (ik_n a_n) e^{int}, \tag{29}$$

with

$$k_n = \begin{cases} -\dfrac{ib}{2} - \dfrac{1}{2}\kappa_n e^{i\Theta_n/2}, & n > 0, \\ -\dfrac{ib}{2} + \dfrac{1}{2}\kappa_n e^{i\Theta_n/2}, & n < 0, \end{cases} \tag{30}$$

where

$$\kappa_n = \left((b^2 + 4a)^2 + 16n^2 \right)^{\frac{1}{4}}$$

and $\Theta_n = \operatorname{Arg}(-b^2 - 4a - 4in)$ is the principal value of the argument.

Proof. Let $q(x,t)$ be the solution given in equation (25) with $g_0(t) = \tilde{g}_0(t)$. We first prove equation (28). We write equation (26) as the sum of three integrals $g_1(t) = I_1(t) + I_2(t) + I_3(t)$, where

$$I_1(t) = -\frac{1}{2i\pi}\int_{-\infty}^{\infty} dk\, k e^{-\omega(k)t}\hat{q}_0(k),$$

$$I_2(t) = \frac{1}{2i\pi}\int_{\partial D^+} dk\, k e^{-\omega(k)t}\hat{q}_0(-k-ib),$$

$$I_3(t) = -\frac{1}{2i\pi}\int_{\partial D_0^+} dk\, e^{-\omega(k)t}\frac{k(2ik-b)}{\omega(k)} \times \left[g_0(0) + \int_0^t ds\, e^{\omega(k)s}\dot{g}_0(s) \right].$$

Note that using integration by parts, we find

$$\begin{aligned} I_1(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-\omega(k)t} q_0(0) + \frac{1}{2\pi}\int_0^{\infty} dx\, \dot{q}_0(x) \int_{-\infty}^{\infty} dk\, e^{-ikx-\omega(k)t} \\ &= \frac{1}{2\sqrt{\pi t}} e^{-at}\left[e^{-\frac{b^2 t}{4}} q_0(0) + \int_0^{\infty} dx\, e^{-\frac{(x+bt)^2}{4t}} \dot{q}_0(x) \right], \end{aligned}$$

which implies that $I_1(t) = O(e^{-at})$ as $t \to \infty$. For the integral $I_2(t)$, we write $I_2(t)$ as

$$I_2(t) = \frac{1}{2i\pi}\int_0^{\infty} dx\, q_0(x) \int_{\partial D_0^+} dk\, k e^{(ik-b)x-\omega(k)t}.$$

By the Cauchy theorem, we deform the contour ∂D_0^+ to $(-\infty, \infty)$ in the above equation, that is,

$$\begin{aligned} \int_{\partial D_0^+} dk\, k e^{(ik-b)x-\omega(k)t} &= \int_{-\infty}^{\infty} dk\, k e^{(ik-b)x-\omega(k)t} \\ &= \sqrt{\frac{\pi}{t}} \exp\left[-at - \frac{(x+bt)^2}{4t} \right]. \end{aligned}$$

Thus, we know that $I_2(t) = O(e^{-at})$ as $t \to \infty$.

Regarding the integral $I_3(t)$, by integration by parts, we write $I_3(t)$ as

$$\begin{aligned} I_3(t) = -\frac{1}{2i\pi}\int_{\partial D_0^+} dk\, \frac{k(2ik-b)}{\omega(k)} \times \Big[g_0(t) \\ - \omega(k)\int_0^t ds\, e^{-(t-s)\omega(k)} g_0(s) \Big]. \end{aligned} \tag{31}$$

Then, substituting the Fourier expansion (27) into (31), we find

$$\begin{aligned} I_3(t) &= -\frac{1}{2i\pi}\int_{\partial D_0^+} dk\, k(2ik-b) \times \sum_{n\neq 0} a_n \left(\frac{e^{int}}{\omega(k)} - \int_0^t ds\, e^{-(t-s)\omega(k)+ins} \right) \\ &= -\frac{1}{2i\pi}\int_{\partial D_0^+} dk\, k(2ik-b) \times \sum_{n\neq 0} a_n \left(\frac{e^{int}}{\omega(k)} - \frac{e^{int} - e^{-\omega(k)t}}{\omega(k)+in} \right). \end{aligned} \tag{32}$$

For each $n \in \mathbb{Z}$, k_n defined in equation (30) solves the equation $\omega(k) + in = 0$ in ∂D^+. Thus, before we split the integral equation (32), we deform the contour ∂D_0^+ to pass to the right of the singularities $k = k_n$, denoted by $\partial\hat{D}_0^+$. As a consequence, equation (32) can be written as

$$\begin{aligned} I_3(t) = &-\frac{1}{2i\pi}\sum_{n\neq 0} a_n e^{int} \int_{\partial\hat{D}_0^+} dk\, k(2ik-b) \\ &\times \left(\frac{1}{\omega(k)} - \frac{1}{\omega(k)+in} \right) \\ &- \frac{1}{2i\pi}\sum_{n\neq 0} a_n \int_{\partial\hat{D}_0^+} dk\, e^{-\omega(k)t}\frac{k(2ik-b)}{\omega(k)+in}. \end{aligned} \tag{33}$$

Note that as $k \to \infty$

$$k(2ik-b)\left(\frac{1}{\omega(k)} - \frac{1}{\omega(k)+in} \right) = -\frac{2n}{k^2} + O\left({}^1/_{k^3} \right).$$

Thus, by the residue theorem, we evaluate the first integral in equation (33) as

$$\begin{aligned} &-\frac{1}{2i\pi}\sum_{n\neq 0} a_n e^{\mathrm{int}} \int_{\partial\hat{D}_0^+} dk\, k(2ik-b) \times \left(\frac{1}{\omega(k)} - \frac{1}{\omega(k)+in} \right) \\ &= \sum_{n\neq 0} ik_n a_n e^{\mathrm{int}} = \tilde{g}_1(t). \end{aligned}$$

For the second integral in equation (33), we use the steepest descent method for asymptotics of integrals. Note that $-\omega(k)$ has a critical point at $k = -ib/2$ and

$$\frac{k(2ik-b)}{\omega(k)+in} = O\left(k + \frac{ib}{2} \right), \quad k \to -\frac{ib}{2}.$$

Hence, we introduce the steepest descent contour C_S, where C_S is the line passing through the point $k = -ib/2$ and parallel to the real k-axis (depicted in Figure 1). Deforming the contour $\partial\widehat{D}_0^+$ to the contour C_S and using the steepest descent method, we know that

$$\int_{\partial\widehat{D}_0^+} dk\, e^{-\omega(k)t}\frac{k(2ik-b)}{\omega(k)+in} = O(t^{-1}), \quad t \to \infty.$$

Therefore, $g_1(t) = \tilde{g}_1(t) + O(t^{-1})$ as $t \to \infty$. Moreover, the uniqueness of $\tilde{g}_1(t)$ follows from the fact that $u(0,\cdot) - \tilde{g}_0(t) \in \mathcal{S}[0,\infty)$ (see [31] for some details). □

Remark 2.2. It should be noted that the Fokas method can be applied effectively well to solve boundary problems with more general boundary conditions, such as the Neumann and Robin boundary conditions, namely, (i) Neumann boundary: $q_x(0,t) = g_1(t)$; (ii) Robin boundary: $\alpha q(0,t) + q_x(0,t) = h(t)$, $\alpha \in \mathbb{R}$.

Neumann boundary condition. Solving equation (12) for $\widehat{g}_0(k,t)$, the effective portion of the function $\widehat{g}_0(k,t)$ is given by

$$\widehat{g}_0(k,t) = -\frac{1}{ik}\left[\widehat{q}_0(-k-ib) - \widehat{g}_1(k,t)\right]. \tag{34}$$

Thus, substituting equation (34) into equation (9), we find $\widehat{Q}(k,t)$ for equation (11) as

$$\widehat{Q}(k,t) = -\frac{1}{ik}\left[(ik-b)\widehat{q}_0(-k-ib) - (2ik-b)\widehat{g}_1(k,t)\right].$$

Robin boundary condition. The Robin boundary value can be expressed in terms of the transforms

$$\alpha\widehat{g}_0(k,t) + \widehat{g}_1(k,t) = \widehat{h}(k,t), \tag{35}$$

where $\widehat{h}(k,t) = \int_0^t ds\, e^{\omega(k)s}h(s)$. Substituting eqauation (12) into equation (35), the effective portion of $\widehat{g}_0(k,t)$ is given by

$$\widehat{g}_0(k,t) = -\frac{1}{ik+\alpha}\left[\widehat{q}_0(-k-ib) - \widehat{h}(k,t)\right],$$

and then

$$\widehat{g}_1(k,t) = \frac{1}{ik+\alpha}\left(\alpha\widehat{q}_0(-k-ib) + ik\widehat{h}(k,t)\right).$$

Using the above equations in equation (9), the effective portion for $\widehat{Q}(k,t)$ is given by

$$\widehat{Q}(k,t) = -\frac{1}{ik+\alpha}\Big[(ik-b-\alpha)\widehat{q}_0(-k-ib) - (2ik-b)\widehat{h}(k,t)\Big].$$

3 The LAD equation on a finite interval

In Section 3, we discuss the Fokas method to solve the LAD equation posed on a finite interval, namely,

$$q_t = q_{xx} - bq_x - aq, \quad 0 < x < L, \ t > 0. \tag{36}$$

We denote the initial and boundary values as

$$q(x,0) = q_0(x), \quad q(0,t) = g_0(t), \quad q(L,t) = f_0(t), \quad q_x(0,t) = g_1(t), \quad q_x(L,t) = f_1(t).$$

We assume that the boundary values $g_0(t)$ and $f_0(t)$ are given, while the boundary values $g_1(t)$ and $f_1(t)$ are unknown. We introduce the Fourier transform pair and the t-transform as

$$\widehat{q}(k,t) = \int_0^L dx\, e^{-ikx}q(x,t),$$

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\widehat{q}(k,t)$$

and

$$\widehat{f}_j(k,t) = \int_0^t ds\, e^{\omega(k)s}\,\partial_x^j q(L,s), \quad j = 0, 1.$$

Taking $\int_0^L dx$ in equation (5) and employing $\int_0^t ds$ in the resulting equation, equation (5) yields the following global relation

$$e^{\omega(k)t}\widehat{q}(k,t) = \widehat{q}_0(k) - \widehat{G}(k,t) + e^{-ikL}\widehat{F}(k,t), \tag{37}$$

where

$$\widehat{F}(k,t) = (ik-b)\widehat{f}_0(k,t) + \widehat{f}_1(k,t), \tag{38a}$$

$$\widehat{G}(k,t) = (ik-b)\widehat{g}_0(k,t) + \widehat{g}_1(k,t). \tag{38b}$$

Applying the inverse Fourier transform, we find

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx-\omega(k)t}\widehat{q}_0(k) - \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx-\omega(k)t}\widehat{G}(k,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-L)-\omega(k)t}\widehat{F}(k,t). \tag{39}$$

By the Cauchy theorem, we deform the contour $(-\infty, \infty)$ to ∂D^+ for the integral involving the term $\widehat{G}(k,t)$ and to $-\partial D^-$ for the integral involving the term $\widehat{F}(k,t)$ (noting $x-L<0$), respectively. Then the reconstruction formula for $q(x,t)$ is given by

$$
\begin{aligned}
q(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx-\omega(k)t}\hat{q}_0(k) - \frac{1}{2\pi}\int_{\partial D^+} dk\, e^{ikx-\omega(k)t}\hat{G}(k,t) \\
&- \frac{1}{2\pi}\int_{\partial D^-} dk\, e^{ik(x-L)-\omega(k)t}\hat{F}(k,t).
\end{aligned} \tag{40}
$$

Note that it requires to determine the unknown functions $\hat{g}_1(t)$ and $\hat{f}_1(t)$ in the representation of the solution equation (40). As before, the global relation equation (37) plays a crucial role. For simplicity, we write equation (37) as

$$
\hat{g}_1(k,t) - e^{-ikL}\hat{f}_1(k,t) = N(k,t) - e^{\omega(k)t}\hat{q}(k,t), \tag{41}
$$

where

$$
N(k,t) = \hat{q}_0(k) - (ik-b)\hat{g}_0(k,t) + e^{-ikL}(ik-b)\hat{f}_0(k,t).
$$

Replacing $k \to -k - ib$ in equation (41), we find

$$
\hat{g}_1(k,t) - e^{(ik-b)L}\hat{f}_1(k,t) = N(-k-ib,t) - e^{\omega(k)t}\hat{q}(-k-ib,t). \tag{42}
$$

We solve the linear system of equations (41) and (42) for $\hat{g}_1(k,t)$ and $\hat{f}_1(k,t)$. As discussed in Section 2, note that the terms involving $e^{\omega(k)t}\hat{q}(k,t)$ and $e^{\omega(k)t}\hat{q}(-k-ib,t)$ in equations (41) and (42) do not contribute in equation (40) thanks to the Cauchy theorem. Thus, the effective portions for the unknown functions $\hat{g}_1(k,t)$ and $\hat{f}_1(k,t)$ can be expressed as

$$
\hat{g}_1(k,t) = \frac{1}{\Delta(k)}\left[e^{-ikL}N(-k-ib,t) - e^{(ik-b)L}N(k,t)\right], \tag{43}
$$

$$
\hat{f}_1(k,t) = \frac{1}{\Delta(k)}\left[N(-k-ib,t) - N(k,t)\right], \tag{44}
$$

where $\Delta(k) = e^{-ikL} - e^{(ik-b)L}$ is the determinant of the corresponding linear system. Finally, substituting equations (43) and (44) into equation (38), we find the functions $\hat{G}(k,t)$ and $\hat{F}(k,t)$ in equation (40) given by

$$
\begin{aligned}
\hat{G}(k,t) = &\ (ik-b)\hat{g}_0(k,t) + \frac{1}{\Delta(k)} \\
&\times\left[e^{-ikL}N(-k-ib,t) - e^{(ik-b)L}N(k,t)\right] \\
\hat{F}(k,t) = &\ (ik-b)\hat{f}_0(k,t) + \frac{1}{\Delta(k)}\left[N(-k-ib,t) - N(k,t)\right].
\end{aligned} \tag{45}
$$

Remark 3.1. Since the function $\Delta(k)$ has simple zeros at

$$
k_m = \frac{m\pi}{L} - \frac{ib}{2}, \quad m \in \mathbb{Z},
$$

the functions $\hat{F}(k,t)$ and $\hat{G}(k,t)$ given in equation (45) have singularities at these points, which can be used to obtain an alternative series representation of the solution [16].

4 Examples

In Section 4, we consider the following Dirichlet boundary value problem

$$
q_t = q_{xx} - q_x - q, \quad x > 0,\ t > 0, \tag{46}
$$

with the given initial and boundary values $q(x,0) = q_0(x)$ and $q(0,t) = g_0(t)$.

Example 4.1. It is well-known that the function $q(x,t) = e^{t-x}$ solves equation (46). In this case,

$$
q(x,0) = e^{-x}, \quad q(0,t) = e^{t}.
$$

We will derive the form of the solution directly from the above conditions. In terms of the transforms, we find

$$
\begin{aligned}
\hat{q}_0(k) &= \frac{1}{ik+1}, \\
\hat{g}_0(k,t) &= \frac{e^{(\omega(k)+1)t} - 1}{\omega(k)+1},
\end{aligned}
$$

where $\omega(k) = k^2 + ik + 1$. From equation (14), it follows that the solution is given by

$$
\begin{aligned}
q(x,t) = &\frac{1}{2\pi}\int_{\partial D^+} dk\, e^{ikx-\omega(k)t}\left(\frac{1}{ik+1} + \frac{1}{ik-2}\right) \\
&- \frac{1}{2\pi}\int_{\partial D^+} dk\, e^{ikx}(2ik-1)\frac{e^{t} - e^{-\omega(k)t}}{\omega(k)+1},
\end{aligned} \tag{47}
$$

where we have deformed the contour $(-\infty,\infty)$ to ∂D^+ by the Cauchy theorem. Note that the integrand of the first integral in equation (47) has a simple pole at $k = i$ in D^+, while the integrand of the second integral has a removable singularity at $k = i$ in D^+. Thus, by the residue theorem, we find $q(x,t) = e^{t-x}$, as desired.

Example 4.2. We consider the case

$$
q(x,0) = 0, \quad q(0,t) = C, \tag{48}
$$

where C is a real constant. The function $\hat{g}_0(k,t)$ is given by

$$
\hat{g}_0(k,t) = \frac{C(e^{\omega(k)t} - 1)}{\omega(k)},
$$

which yields

$$
q(x,t) = -\frac{C}{2\pi}\int_{\partial D^+} dk\, e^{ikx}(2ik-1)\frac{1 - e^{-\omega(k)t}}{\omega(k)}. \tag{49}
$$

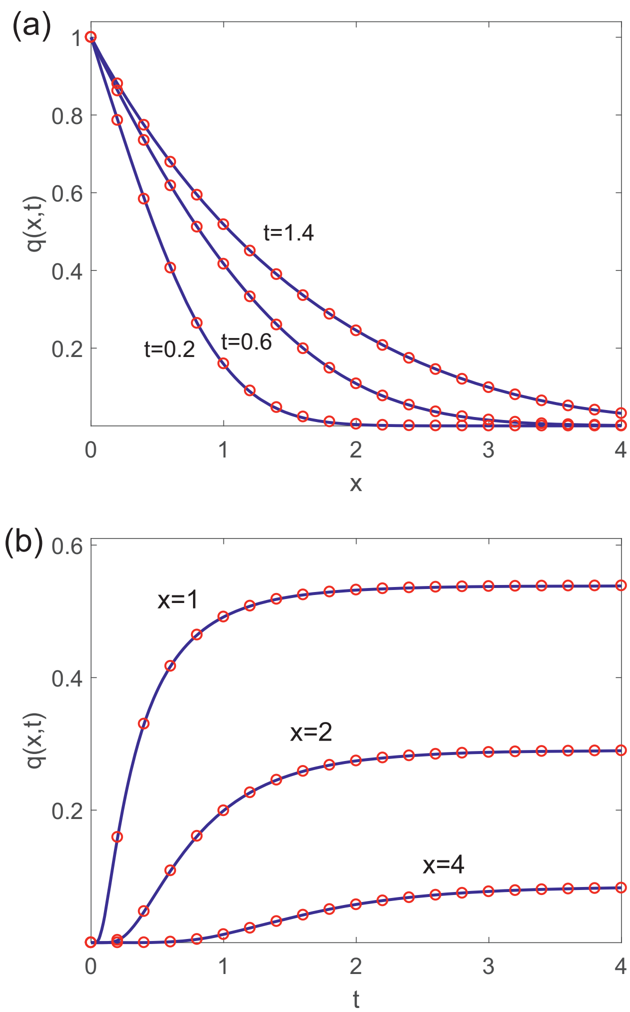

Figure 2: Comparison of analytical solution (blue solid curve) and numerical solution (red open dot) with $C = 1$ and the initial and boundary values given in Equation (48). (a) Analytical and numerical solutions *versus* x when $t = 0.2$, 0.6 and 1.4 (from lower curve to upper one, respectively). (b) Analytical and numerical solutions *versus* t when $x = 1$, 2 and 4 (from upper curve to lower one, respectively) (color online).

The integral in equation (49) can not be computed explicitly, so we evaluate integral numerically. To this end, we use the efficient analytical-numerical scheme proposed in [34]. By the Cauchy theorem, we deform the contour ∂D^+ to $L = L_1 \cup L_2$ (depicted in Figure 1), where

$$L_1 = \{k = re^{i\pi/6} | r \geq 0\},$$
$$L_2 = \{k = re^{5i\pi/6} | r \geq 0\}.$$

Note that along this contour L, the exponential terms in the integral decay rapidly for large k and hence numerical integration converges quickly. The accuracy and order of convergence for this hybrid analytical-numerical scheme has been discussed in [34]. Analytical and numerical solutions are shown in Figure 2 with $C = 1$. The analytical solution in Figure 2 (as blue solid curve) has been obtained by integrating equation (49) numerically along the contour L via the adaptive Gauss-Kronrod quadrature method with $|k| = 100$ (or equivalently $r = 100$). The numerical solution displayed in Figure 2 (as red open dot) has been found by using MATLAB's build-in function pdepe with $x = 100$. Figure 2 shows that the analytical solutions are in excellent agreement with numerical results.

Example 4.3. We consider the case of the periodic boundary value

$$q(x, 0) = 0, \quad q(0, t) = 2\sin 2\pi t. \tag{50}$$

In this case, the solution is given by

$$q(x, t) = -\frac{1}{2\pi}\int_{\partial D^+} dk\, e^{ikx-\omega(k)t}(2ik-1)\hat{g}_0(k, t), \tag{51}$$

where

$$\hat{g}_0(k, t) = \frac{2[2\pi + e^{\omega(k)t}(\omega(k)\sin 2\pi t - 2\pi\cos 2\pi t)]}{\omega^2(k) + 4\pi^2}.$$

The integral in equation (51) can be computed by a similar way as discussed in Example 4.2. Figure 3a, b show that the analytical solution (shown in the figure as blue solid curve) is asymptotically periodic for large t and is again in excellent agreement with the numerical results (marked in the figure as red open dots).

Example 4.4. We consider the case that the Dirichlet boundary value is asymptotically periodic, namely,

$$q(x, 0) = 0, \quad q(0, t) = \cos 2\pi t - \frac{1}{t^4 + 1}. \tag{52}$$

From equation (29), it follows that

$$\tilde{g}_1(t) = \frac{ik_1}{2}e^{2i\pi t} + \frac{ik_{-1}}{2}e^{-2i\pi t}, \tag{53}$$

where

$$k_{\pm 1} = -\frac{i}{2} \mp \frac{1}{2}(25 + 64\pi^2)^{1/4} e^{\frac{i}{2}\Theta_{\pm 1}}$$

with $\Theta_{\pm 1} = \mathrm{Arg}(-5 \mp 8i\pi)$. The asymptotic Neumann boundary value given in equation (53) is shown in Figure 4 by a blue solid curve. We have also compared the Neumann boundary value in this figure as a red dashed curve, which has been numerically obtained by the forward-difference derivative.

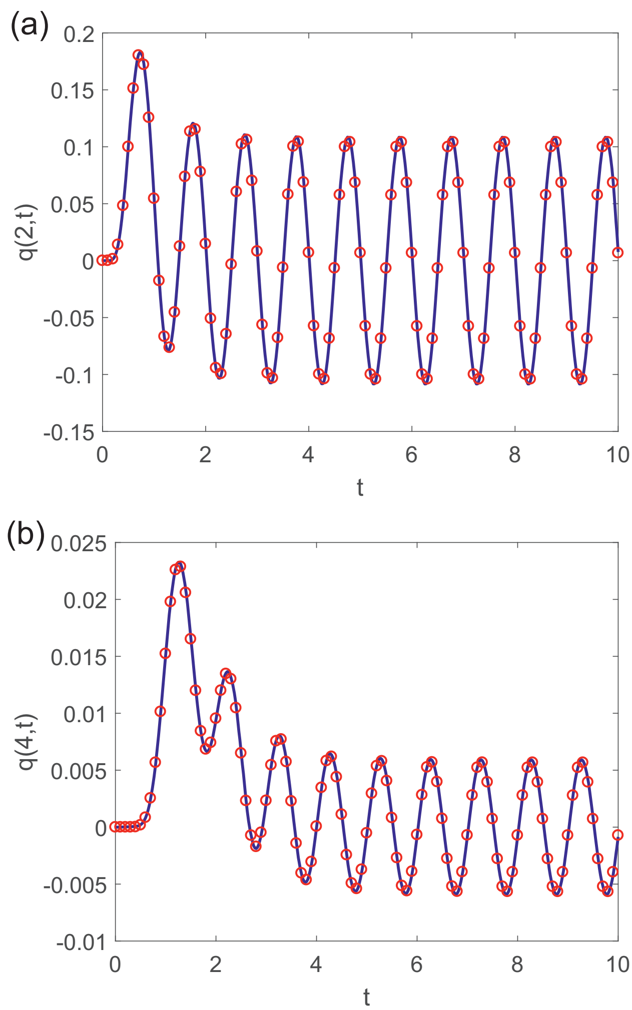

Figure 3: Comparison of analytical solutions (blue solid curves) and numerical solutions (red open dots) *versus* t with the initial and boundary values given in equation (50), where $x = 2$ in (a) and $x = 4$ in (b), respectively (color online).

Example 4.5. We consider the LAD equation posed on the finite interval,

$$q_t = q_{xx} - q_x - q, \quad 0 < x < 1, \;\; t > 0 \tag{54}$$

with

$$q(x, 0) = 0, \quad q(0, t) = 2\sin 2\pi t, \quad q(1, t) = 0.$$

In this case, $N(k, t) = -(ik - 1)\hat{g}_0(k, t)$, and hence

$$\hat{F}(k, t) = \frac{1}{\Delta(k)}(2ik - 1)\hat{g}_0(k, t),$$

$$\hat{G}(k, t) = \frac{1}{\Delta(k)}(2ik - 1)e^{-ik}\hat{g}_0(k, t),$$

where $\Delta(k) = e^{-ik} - e^{ik-1}$ and $\hat{g}_0(k, t)$ is given in Example 4.3. From equation (40), we find

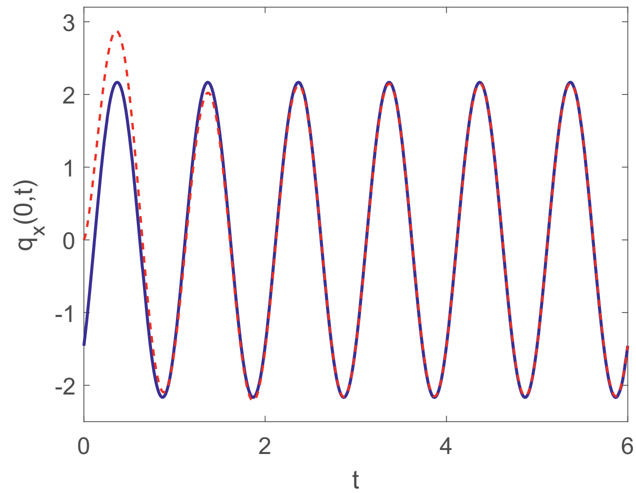

Figure 4: Comparison of numerically obtained Neumann boundary value (red dashed curve) and asymptotic formula (blue solid curve) given in equation (53) (color online).

$$q(x, t) = -\frac{1}{2\pi}\int_{\partial D^+} dk\, e^{ik(x-1)-\omega(k)t}\frac{2ik-1}{\Delta(k)}\hat{g}_0(k, t)$$
$$-\frac{1}{2\pi}\int_{\partial D^-} dk\, e^{ik(x-1)-\omega(k)t}\frac{2ik-1}{\Delta(k)}\hat{g}_0(k, t).$$

Using change of variable $k \to -k - i$ for the second integral, the solution $q(x, t)$ can be written as

$$q(x, t) = -\frac{1}{2\pi}\int_{\partial D^+} dk\, e^{-\omega(k)t}\left(e^{ik(x-1)} - e^{-(ik-1)(x-1)}\right) \times \frac{2ik-1}{\Delta(k)}\hat{g}_0(k, t). \tag{55}$$

The solution $q(x, t)$ given in equation (55) is shown in Figure 5(a). We also have displayed the solution curves in Figure 5(b) as blue solid curves for $x = 0.1$, 0.5 and 0.8, which are in excellent agreement with the numerical solutions (red dashed curve).

5 Concluding remarks

We have demonstrated the Fokas method to solve initial-boundary value problems for the LAD equation posed on the half line and a finite interval with general boundary conditions. In addition to solving the LAD equation, we have discussed the case of periodic boundary conditions, which commonly appears in physics and engineering. In particular, we have characterized the long time asymptotic behaviors of the solution and the unknown boundary value, which can be uniquely determined by the known asymptotically periodic boundary datum. These analytical predictions have been compared with numerical results showing the excellent agreement.

It should be noted that the presented method is relatively simple, but remarkably effective, to implement, finding an explicit integral representation of the solution. The method works equally well for boundary value problems for linear and nonlinear integrable systems. In contrast to classical transform methods such as the Fourier and Laplace transforms, the Fokas method has several advantages. The integral representation of the solution involves explicit (x, t)-dependence of the exponential form, and hence it allows to study the long time asymptotics of the solution. Also, from the integral representation of analytic functions, it makes possible to compute effectively the numerical solution [12, 20, 34]. More importantly, the method can be nonlinearizable in the sense that it is successfully used for analyzing nonlinear integrable PDEs as well as nonlinear lattices [35]. Recently, the method has been extended to study well-posedness, regularity and controllability of PDEs [36–38].

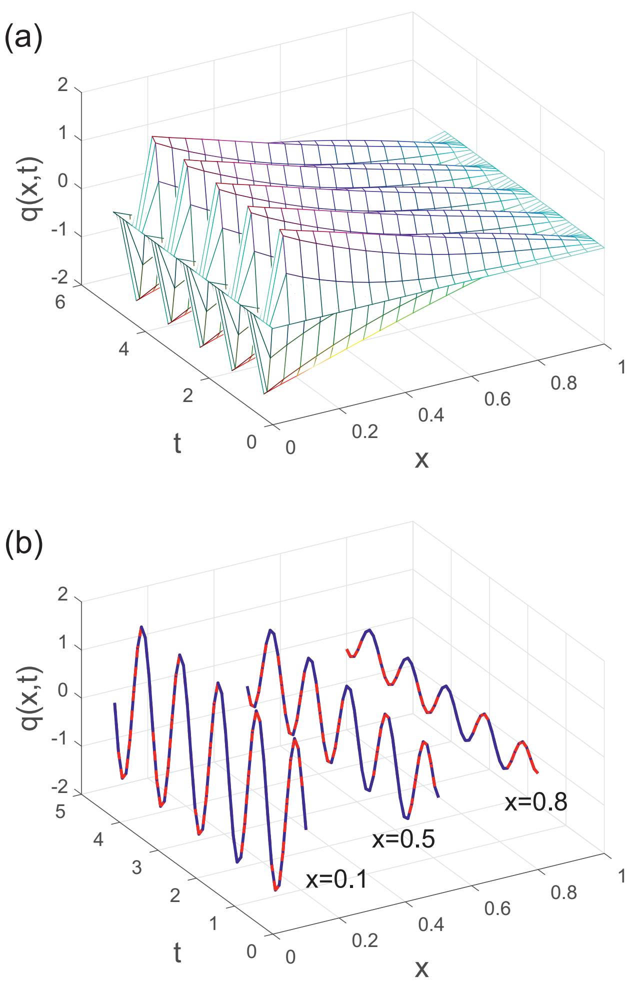

Figure 5: (a) Analytical solution of equation (54) given in equation (55). (b) Comparison of the analytical solution (blue solid curve) and the numerical solution (red dashed curve) for $x = 0.1$, 0.5 and 0.8. (color online).

The LAD equation can be considered as a generalization of the heat equation and hence we expect that it could be possible to study boundary value problems with more complicated boundary data as shown in [39–43]. Also, note that the explicit representations of the solutions in finite and infinite domains can be used to analyze the effect of the exit and inlet boundary conditions for small Péclet number as discussed in [44]. We will address regarding these issues in the near future.

Acknowledgments: The author thanks anonymous referees for their useful comments and constructive suggestions. The work is supported by the Daegu University Research Grant 2017.
Author contribution: All the authors have accepted responsibility for the entire content of this submitted manuscript and approved submission.
Research funding: This study was funded by Daegu University Research Grant 2017.
Conflict of interest statement: The authors declare no conflicts of interest regarding this article.

Appendix A

The inversion formula for the t-transform. The inversion formula for the spectral function equation (7) can be derived by performing spectral analysis of equation (4b). We consider the following spectral problem

$$\mu_t + \omega(k)\mu = g(t), \quad 0 < t < T, \tag{A.1}$$

where $\mu = \mu(t, k)$ and $\omega(k) = k^2 + ibk + a$. Multiplying by the integrating factor $e^{\omega(k)t}$, we found the Jost solutions given by

$$\mu_1(t, k) = \int_0^t ds\, e^{-\omega(k)(t-s)} g(s),$$
$$\mu_2(t, k) = -\int_t^T ds\, e^{-\omega(k)(t-s)} g(s).$$

Note that μ_1 and μ_2 are analytic and bounded in $\mathbb{C} \setminus D$ and D, respectively. Also, the jump condition, also known as a scalar Riemann-Hilbert problem, is given by

$$(\mu_2 - \mu_1)(t, k) = -e^{-\omega(k)t}\hat{g}(k, T), \tag{A.2}$$

where $\hat{g}(k, t) = \int_0^t ds\, e^{\omega(k)s} g(s)$. By integration by parts, we find $\mu_{1,2} = O(1/k^2)$ as $k \to \infty$, and hence the solution of the

Riemann-Hilbert problem equation (a.2) can be solved by the Plemelj formula [12]

$$\mu(t,k) = -\frac{1}{2i\pi}\int_{\partial D} d\zeta\, e^{-\omega(\zeta)t}\frac{\widehat{g}(\zeta,T)}{\zeta - k},$$

where $\int_{\partial D} d\zeta = \int_{\partial D^+} d\zeta + \int_{\partial D^-} d\zeta$. Substituting the above equation into equation (a.1), we find

$$g(t) = -\frac{1}{2i\pi}\int_{\partial D} d\zeta\, e^{-\omega(\zeta)t}\frac{\omega(k)-\omega(\zeta)}{\zeta - k}\widehat{g}(\zeta,T)$$
$$= \frac{1}{2i\pi}\int_{\partial D} d\zeta\, e^{-\omega(\zeta)t}(\zeta + ib + k)\widehat{g}(\zeta,T).$$

Letting $k \to -k - ib$ in the integral over ∂D^- (and noting that if $k \in D^-$, then $-k - ib \in D^+$), we know that

$$\int_{\partial D^-} d\zeta\, e^{-\omega(\zeta)t}(\zeta + ib + k)\widehat{g}(\zeta,T)$$
$$= \int_{\partial D^+} d\zeta\, e^{-\omega(\zeta)t}(\zeta - k)\widehat{g}(\zeta,T).$$

Thus, the reconstruction formula for $g(t)$ can be found as

$$g(t) = \frac{1}{2i\pi}\int_{\partial D^+} d\zeta\, e^{-\omega(\zeta)t}(2\zeta + ib)\widehat{g}(\zeta,T).$$

Note that we can replace $\widehat{g}(\zeta,T)$ by $\widehat{g}(\zeta,t)$ in the above equation. Indeed, the difference between these terms

$$\int_{\partial D^+} d\zeta \int_t^T ds\, e^{-\omega(\zeta)(t-s)}(2\zeta + ib)g(s)$$

vanishes by the Cauchy theorem, since $t - s < 0$ and the integrand is analytic and bounded in D^+. Therefore, we find the following inversion formula for the spectral function $\widehat{g}(k,t)$ as

$$g(t) = \frac{1}{2i\pi}\int_{\partial D^+} dk\, e^{-\omega(k)t}\omega'(k)\widehat{g}(k,t). \qquad \text{(A.3)}$$

References

[1] M. T. van Genuchten and W. J. Alves, Analytical solutions of the one-dimensional convective-dispersive solute transport equation, Washington US Department of Agriculture, Technical Bulletins 157268, United States Department of Agriculture, Economic Research Service, 1982.
[2] J. Zhong, C. Zeng, Y. Yuan, Y. Zhang, and Y. Zhang, “Numerical solution of the unsteady diffusion-convection-reaction equation based on improved spectral Galerkin method,” *AIP Adv.*, vol. 8, p. 045314, 2018.
[3] J. S. Pérez Guerrero, E. M. Pontedeiro, M. T. van Genuchten, and T. H. Skaggs, “Analytical solution of the one-dimensional advection–dispersion solute transport equation subject to time-dependent boundary conditions,” *Chem. Eng. J.*, vol. 221, pp. 487–491, 2013.
[4] I. Boztosun and A. Charafi, “An analysis of the linear advection–diffusion equation using mesh-free and mesh-dependent methods,” *Eng. Anal. Bound. Elem.*, vol. 26, pp. 889–895, 2002.
[5] A. Mojtabi and M. O. Deville, “One-dimensional linear advection–diffusion equation: analytical and finite element solutions,” *Comput. Fluids*, vol. 107, pp. 189–195, 2015.
[6] A. Moranda, R. Cianci, and O. Paladino, “Analytical solutions of one-dimensional contaminant transport in soils with source production-decay,” *Soil Syst.*, vol. 2, no. 40, pp. 1–16, 2018.
[7] D. H. Tang, E. O. Frind, and E. A. Sudicky, “Contaminant transport in fractured porous media: analytical solution for a single fracture,” *Water Resour. Res.*, vol. 17, pp. 555–564, 1981.
[8] G. Dagan, “Theory of solute transport by groundwater,” *Annu. Rev. Fluid Mech.*, vol. 19, pp. 183–215, 1987.
[9] J. S. Pérez Guerrero, L. C. G. Pimentel, T. H. Skaggs, and M. T. van Genuchten, “Analytical solution of the advection–diffusion transport equation using a change-of-variable and integral transform technique,” *Int. J. Heat Mass Tran.*, vol. 52, pp. 3297–3304, 2009.
[10] A. S. Fokas, “A unified transform method for solving linear and certain nonlinear PDEs,” *Proc. Math. Phys. Eng. Sci.*, vol. 453, pp. 1411–1443, 1997.
[11] A. S. Fokas, “Integrable nonlinear evolution equations on the half-line,” *Commun. Math. Phys.*, vol. 230, pp. 1–39, 2002.
[12] A. S. Fokas, “A unified approach to boundary value problems,” in *CBMS-NSF Regional Conf. Series in Applied Mathematics*, Philadelphia, SIAM, 2008, https://doi.org/10.10.1137/1.9780898717068.
[13] B. Deconinck, T. Trogdon, and V. Vasan, “The method of Fokas for solving linear partial differential equations,” *SIAM Rev.*, vol. 56, pp. 159–186, 2014.
[14] A. S. Fokas, “On the integrability of certain linear and nonlinear partial differential equations,” *J. Math. Phys.*, vol. 41, pp. 4188–4237, 2000.
[15] A. S. Fokas, “A new transform method for evolution partial differential equations,” *IMA J. Appl. Math.*, vol. 67, pp. 559–590, 2002.
[16] A. S. Fokas and B. Pelloni, “A transform method for linear evolution PDEs on a finite interval,” *IMA J. Appl. Math.*, vol. 70, pp. 564–587, 2005.
[17] M. J. Colbrook, N. Flyer, and B. Fornberg, “On the Fokas method for the solution of elliptic problems in both convex and non-convex polygonal domains,” *J. Comput. Phys.*, vol. 374, pp. 996–1016, 2018.
[18] A. S. Fokas, “Two dimensional linear PDEs in a convex polygon,” *Proc. Math. Phys. Eng. Sci.*, vol. 457, pp. 371–393, 2001.
[19] B. Pelloni and D. A. Pinotsis, “The elliptic sine-Gordon equation in a half plane,” *Nonlinearity*, vol. 23, pp. 77–88, 2010.
[20] M. J. Colbrook, A. S. Fokas, and P. Hashemzadeh, “A hybrid analytical-numerical technique for elliptic PDEs,” *SIAM J. Sci. Comput.*, vol. 41, pp. A1066–A1090, 2019.
[21] M. J. Colbrook, “Extending the unified transform: curvilinear polygons and variable coefficient PDEs,” *IMA J. Numer. Anal.*, vol. 40, pp. 976–1004, 2020.
[22] G. Biondini and G. Hwang, “Initial-boundary value problems for discrete evolution equations: discrete linear Schrödinger and integrable discrete nonlinear Schrödinger equations,” *Inverse Probl.*, vol. 24, pp. 1–44, 2008,065011.
[23] B. Moon and G. Hwang, “Discrete linear evolution equations in a finite lattice,” *J. Differ. Equ. Appl.*, vol. 25, pp. 630–646, 2019.
[24] A. S. Fokas, “Lax pair: a novel type of separability,” *Inverse Probl.*, vol. 25, p. 123007, 2009.

[25] A. S. Fokas, "The generalized Dirichlet-to-Neumann map for certain nonlinear evolution PDEs," *Commun. Pure Appl. Math.*, vol. LVIII, pp. 639–670, 2005.
[26] D. G. Crowdy and E. Luca, "Solving Wiener-Hopf problems without kernel factorization," *Proc. Math. Phys. Eng. Sci.*, vol. 470, p. 20140304, 2014.
[27] M. J. Colbrook, L. J. Ayton, and A. S. Fokas, "The unified transform for mixed boundary condition problems in unbounded domains," *Proc. R. Soc. A.*, vol. 475, p. 20180605, 2019.
[28] A. B. de Monvel, A. Kotlyarov, D. Shepelsky, and C. Zheng, "Initial boundary value problems for integrable systems: towards the long-time asymptotics," *Nonlinearity*, vol. 23, p. 2483, 2010.
[29] J. Lenells and A. S. Fokas, "The unified method on the half-line: II. NLS on the half-line with *t*-periodic boundary conditions," *J. Phys. Math. Theor.*, vol. 45, p. 195202, 2012.
[30] G. Hwang and A. S. Fokas, "The modified Korteweg–de Vries equation on the half-line with a sine-wave as Dirichlet datum," *J. Nonlinear Math. Phys.*, vol. 20, pp. 135–157, 2013.
[31] J. Lenells and A. S. Fokas, "The nonlinear Schrödinger equation with *t*-periodic data: II. Perturbative results," *Proc. Math. Phys. Eng. Sci.*, vol. 471, p. 20140926, 2015.
[32] G. Hwang, "The modified Korteweg–de Vries equation on the quarter plane with *t*-periodic data," *J. Nonlinear Math. Phys.*, vol. 24, pp. 620–634, 2017.
[33] B. Moon and G. Hwang, "The Korteweg–de Vries equation on the quarter plane with asymptotically *t*-periodic data via the Fokas method," *Asymptot. Anal.*, vol. 107, pp. 115–133, 2018.
[34] F. R. J. de Barros, M. J. Colbrook, and A. S. Fokas, "A hybrid analytical-numerical method for solving advection–dispersion problems on a half -line," *Int. J. Heat Mass Tran.*, vol. 139, pp. 482–491, 2019.
[35] M. Toda, "Theory of a nonlinear lattice," in *Springer Series in Solid-State Science*, vol. 20, Berlin, Springer-Verlag, 1989.
[36] A. S. Fokas, A. A. Himonas, and D. Mantzavinos, "The nonlinear Schrödinger equation on the half-line," *Trans. Amer. Math. Soc.*, vol. 369, pp. 681–709, 2017.
[37] A. A. Himonas, D. Mantzavinos, and F. Yan, "The Korteweg–de Vries equation on an interval," *J. Math. Phys.*, vol. 60, p. 051507, 2019.
[38] K. Kalimeris and T. Özsari, "An elementary proof of the lack of null controllability for the heat equation on the half line," *Appl. Math. Lett.*, vol. 104, p. 106241, 2020.
[39] D. Mantzavinos and A. S. Fokas, "The unified method for the heat equation: I. Non-separable boundary conditions and non-local constraints in one dimension," *Eur. J. Appl. Math.*, vol. 24, pp. 857–886, 2013.
[40] A. S. Fokas and B Pelloni, "Generalized Dirichlet to Neumann map for moving initial-boundary value problems," *J. Math. Phys.*, vol. 48, p. 013502, 2007.
[41] B. Pelloni and D. A. Smith, "Nonlocal and multipoint boundary value problems for linear evolution equations," *Stud. Appl. Math.*, vol. 141, pp. 46–88, 2018.
[42] B. Deconinck, B. Pelloni, and N. E. Sheils, "Non-steady-state heat conduction in composite walls," *Proc. Math. Phys. Eng. Sci.*, vol. 470, p. 20130605, 2014.
[43] N. E. Sheils and B. Deconinck, "Heat conduction on the ring: interface problems with periodic boundary conditions," *App. Math. Lett.*, vol. 37, pp. 107–111, 2014.
[44] M. Massabó, R. Cianci, and O. Paladino, "An analytical solution of the advection dispersion equation in a bounded domain and its application to laboratory experiments," *J. Appl. Math.*, vol. 2011, p. 493014, 2011.