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Dynamics of a discrete-time system with Z-type control

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Abstract: In community ecology, the stability of a predator–prey system is a considerably desired issue; as a result, population control of a predator–prey system is very important. The dynamics of continuous-time models with Z-type control is studied earlier. But, the effectiveness of the Z-type control mechanism in a discrete-time set-up is lacking. First, we consider a Lotka–Volterra type discrete-time predator–prey model. We observe that without control, the system exhibits rich dynamical behaviors including chaotic oscillations. We apply the Z-control mechanism in both direct and indirect ways to the system and observe that in both cases, controllers have the property to drive the populations of the system to the desired state. We conduct numerical simulation as supporting evidence of our analytical results.

Keywords: chaos control; discrete-time system; ecosystem conservation; stability; Z-type control.

1 Introduction

Ecology is the branch of biology, which addresses the full scale of life of species together with their physical environment that spans the entire planet [1]. The study of ecology provides information about the interactions among living and non-living things and the energy transfer from one trophic level to another trophic level through which the organisms survive in the world. Community ecology studies the interactions between the populations, for example, competition and predation [2]. These relationships may be represented through a food web. A food web consists of several food chains and a food chain shows how the organisms are related to each other by their food [3]. Each level of a food chain represents a different trophic level and food energy is transferred from one trophic level to another trophic level by which the ecological balance is maintained [4]. But recently, the ecological balance is disturbed due to the changing climate, habitat loss, degradation, etc., [5]. For these reasons, animal and plant species are moving in the way of destruction, which changes the ecosystem significantly. So, the preservation of ecological balance is very important and it will be possible if all species remain in the desired state.

To describe and analyze the predator–prey relationships, mathematical model has become a very important tool for its huge applications, where the models are usually framed by a set of relations and variables [6]. In mathematical modeling, there are mainly two alternative frameworks, discrete-time set-up, and continuous-time set-up, within which model variables develop over time. Continuous-time systems are described by differential equations. On the other hand, discrete-time systems are described by difference equations [7]. Difference equations are used to compute the population size at discrete points in time. Discrete-time models are more appropriate in small population size or when the population having no overlapping generation. Many annual plants and insect species (like ants, grasshoppers, budmoths, and cicadidae, etc.,) have no overlap in their successive generations, so their populations obey discrete-time behavior [8, 9]. On the other hand populations with overlapping generations are modeled by continuous-time model [10]. However, we can get rich and complex dynamics in just one-dimensional discrete-time model. For instance, chaos can be exhibited in one-dimensional discrete-time model [9]. Unfortunately, empirical evidence to support this theoretical possibility is scarce. Turchin and Taylor (1992) [11] fitted a minimum of 18 year's time-series data for 14 insect and 22 vertebrate populations by using single species difference equation model. They observed chaos in only one insect population (*Phyllaphis fagi*). They concluded that the complete spectrum of dynamical behaviors, ranging from stability to chaos, is likely to be found among natural populations. Costantino et al. (1995) [12] reported a joint theoretical and experimental study to test the hypothesis that changes in demographic parameters may change the predictability of

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population fluctuations. They predicted by using single species difference equation model (three-dimensional map for larva, pupa, and adult), that changes in adult mortality rate would produce a substantial shift in population dynamic behavior. By experimentally manipulating the adult mortality rate of flour beetle *Tribolium*, they also observed changes in the dynamics from stable fixed points to periodic cycles and aperiodic oscillations that corresponded to the prediction of the mathematical model. On the other hand, for showing chaotic behavior in a continuous-time autonomous model minimum three species are needed [13, 14]. Discrete-time models are easy to understand, develop, and simulate. For modeling experimental data, which are almost always discrete, it is mostly suitable [15].

Various types of mathematical models have been developed to describe and analyze different types of biological phenomena by which one can predict the future state of a system. Historically, at first, a single species population model was formulated by Malthus [16]. Also, it is well known that many ecological models have been developed by incorporating logistic growth term for the prey species and Holling type-II functional response for the predator species after the pioneering work of Lotka and Volterra [17–19]. In recent years, the ecological balance is one of the most desirable and widely investigated issue in both continuous-time and discrete-time systems [8, 20]. If the population sizes of interacting species are controlled anyhow, then the ecological balance may be maintained. Different biological phenomena like imposition of a population floor [21, 22], addition of refugia [23], omnivory [24], intraspecific density dependence [25], toxic inhibition [26], spatial effect [27], dispersal [28–30], predator switching [31, 32], Allee effect [33], additional predator [34], additional food [35], harvesting of predator [36], fear effect [37, 38], etc., may increase the ecological stability. Although a system may be stable by incorporating these phenomena, these are not always capable to achieve desired population abundance. But there is an effective and powerful control strategy named Z-type control mechanism which is capable to achieve ecological balance and to drive the abundance of a species to the desired state. The Z-control mechanism can keep species away from extinction and improve ecosystem stability.

The Z-type control mechanism is a neural dynamic approach. It is an error-based method that can be used to control a system described in the form of a set of equations, which are termed as system equations. In this method, the design formula guarantees that each component of the error function converges to zero. This idea is performed by compelling the difference between the actual output and the desired output of the system to be zero. The Z-type control can be applied through both the direct and indirect ways in a system. In a direct way, control can be applied to all the variables simultaneously for controlling the dynamics of the system. In indirect control, if one variable needs to be controlled, then the control is applied to another variable.

Many researchers have investigated different types of continuous-time models with the Z-type control mechanism. Zhang et al. [39] considered the Lotka–Volterra predator–prey model with the Z-control mechanism in both direct and indirect ways to drive the prey and predator populations to the desired states for preventing species from extinction. Lacitignola et al. [40] studied a generalized predator–prey system with Z-type control. Nadim et al. [41] explored the possible applications of fear in the prey population by changing the density of the predator population and observed that after incorporating the Z-control mechanism the system reaches to a stable steady-state. Alzahrani et al. [42] studied an eco-epidemiological model with Z-type controller on the predator population and observed that the disease, as well as the chaotic oscillations, can be removed from the system. Samanta et al. [43] proposed and analyzed an epidemic model with the Z-control mechanism. Recently, Lacitignola et al. [44] applied the Z-type approach to control backward bifurcation phenomena in a SIR model. Very recently, Senapati et al. [45] studied an SI type disease model employing Z-control approach through the removal of the population.

So many continuous-time predator–prey models equipped with the Z-type controller have been developed successfully, but as far as our knowledge goes, there is no such investigation on the predator–prey model with Z-type control in the discrete-time system. This is the first liberal attempt to investigate the predator–prey model with the Z-type control mechanism in a discrete-time set-up.

Here, we consider a Lotka–Volterra type predator–prey model in discrete-time set-up and apply Z-type control mechanism in both direct and indirect ways for controlling the populations. The rest of the paper is ordered as follows. In Section 2, a direct Z-control mechanism is applied to a predator–prey model. In Section 3, an indirect Z-controller on the predator population is applied to control the prey population density. In Section 4, we come to the end of the paper with final remarks, in which we discuss how the inclusion and/or exclusion of species into or from the system depending on the sign of the update parameter helps to get a desired state of the system.

2 Direct control of populations

In this section, we discuss briefly the general design procedure of direct Z-control laws for a discrete-time system. To do this, a controller group is considered for the simultaneous control of prey and predator populations by taking the exogenous measures for both species and applying the Z-type dynamic method. We also check the convergence performance of the controller group explicitly.

2.1 Controller-group design

Here, we consider a general two-dimensional model in discrete-time set-up as follows:

$$\begin{cases} x_{n+1} = f\,(x_n, y_n), \\ y_{n+1} = g\,(x_n, y_n). \end{cases} \tag{2.1}$$

After introducing two exogenous measures for interacting species, the above model takes the following form:

$$\begin{cases} x_{n+1} = f\,(x_n, y_n) - u_n x_n, \\ y_{n+1} = g\,(x_n, y_n) - w_n y_n, \end{cases} \tag{2.2}$$

where the exogenous measures u_n and w_n denote the direct control variables for species-I and species-II respectively. Now our aim is to drive the populations of species-I and species-II to the desired states simultaneously i. e., $[x_n, y_n]^T \to [p_n, q_n]^T$, where p_n and q_n denote the desired population densities for species-I and species-II respectively and the superscript T denotes the transpose of a vector or matrix. In this error-based method, to design the Z-type controller group the following steps are followed:

At first, a vector-valued error function e_n is considered as

$$e_n = \begin{bmatrix} e_n^1 \\ e_n^2 \end{bmatrix} = \begin{bmatrix} x_n - p_n \\ y_n - q_n \end{bmatrix}. \tag{2.3}$$

Secondly, the design formula is defined in such a way that the error function approaches to zero and it is adopted as:

$$e_{n+1} = \frac{1}{m} e_n, \tag{2.4}$$

where $m > 1$ is the design parameter, used to scale convergence rate.

Substituting Eq. (2.3) in Eq. (2.4), we get

$$\begin{bmatrix} x_{n+1} - p_{n+1} \\ y_{n+1} - q_{n+1} \end{bmatrix} = \frac{1}{m} \begin{bmatrix} x_n - p_n \\ y_n - q_n \end{bmatrix}. \tag{2.5}$$

Now from Eq. (2.2) and (2.5), finally we obtain the analytical expression of the control variables as:

$$\begin{cases} u_n = \dfrac{1}{x_n}\left[f\,(x_n, y_n) - \dfrac{1}{m}\,(x_n - p_n) - p_{n+1}\right], \\ w_n = \dfrac{1}{y_n}\left[g\,(x_n, y_n) - \dfrac{1}{m}\,(y_n - q_n) - q_{n+1}\right], \end{cases} \tag{2.6}$$

which is the Z-type controller group for the simultaneous control of species-I and species-II.

2.2 Theoretical analysis

In this subsection, the convergence performance of the Z-type controller group for the above generalized two-dimensional model is examined theoretically. Here, it is shown that each component of the tracking error vector for the above model (2.2) converges to zero.

Theorem 1: For a bounded desired state $[p_n, q_n]^T$, starting from a positive initial state $[x_0, y_0]^T$, the tracking error vector e_n of the above model (2.2) furnished with the Z-type controller group (2.6) converges to zero.

Proof: According to the design procedure of Z-type controller group (2.6), we have

$$\begin{cases} e_{n+1}^1 = \dfrac{1}{m} e_n^1, \\ e_{n+1}^2 = \dfrac{1}{m} e_n^2, \end{cases} \tag{2.7}$$

where, the design parameter is $m\,(>1)$.

Now, we evaluate the Jacobian matrix (J) for the above system of equations (2.7) and it is given as follows:

$$J = \begin{pmatrix} \dfrac{1}{m} & 0 \\ 0 & \dfrac{1}{m} \end{pmatrix}.$$

The eigenvalues of the matrix J are $\frac{1}{m}, \frac{1}{m}$. It is clear that, $\left|\frac{1}{m}\right| < 1$ as $m > 1$, which shows that the system (2.7) converges to the fixed point $(0, 0)$ i. e. the tracking errors become zero. Hence, we can conclude that both the species-I and species-II of the system (2.2) converge to the respective desired states.

2.3 Example of direct control

In this subsection, we consider a Lotka–Volterra type predator–prey model in discrete-time set-up as follows:

$$\begin{cases} x_{n+1} = x_n \exp\left(r\left(1 - \dfrac{x_n}{k}\right) - \lambda y_n\right), \\ y_{n+1} = y_n \exp\left(c\lambda x_n - d\right), \end{cases} \tag{2.8}$$

where, x_n and y_n denote the prey and predator population densities respectively. Here, r and k denote the birth rate and the carrying capacity of prey species respectively, λ is the maximum predation rate of predator species, c is the conversion rate and d is the natural death rate of predator species. After introducing exogenous measures on both species the above model takes the following form

$$\begin{cases} x_{n+1} = x_n \exp\left(r\left(1-\frac{x_n}{k}\right) - \lambda y_n\right) - u_n x_n, \\ y_{n+1} = y_n \exp\left(c\lambda x_n - d\right) - w_n y_n, \end{cases} \tag{2.9}$$

where, the exogenous measures u_n and w_n denote the Z-controllers (or update parameters) for prey and predator species respectively, which vary over time. The positive values of u_n and w_n imply that we should remove (emigration, harvesting, culling) prey and/or predators from the system, whereas, the negative values imply that we should add (immigration) prey and/or predators into the system to obtain the desired prey and predator population densities. Now our aim is to attain the respective desired states, i. e., $x_n \to p_n$ and $y_n \to q_n$. For this purpose, we consider the error function and adopt the design formula as described in subsection 2.1. Finally, we obtain the control variables u_n and w_n for the above model (2.9) as follows:

$$\begin{cases} u_n = \frac{1}{x_n}\left[x_n \exp\left(r\left(1-\frac{x_n}{k}\right) - \lambda y_n\right) - \frac{1}{m}\left(x_n - p_n\right) - p_{n+1}\right], \\ w_n = \frac{1}{y_n}\left[y_n \exp\left(c\lambda x_n - d\right) - \frac{1}{m}\left(y_n - q_n\right) - q_{n+1}\right], \end{cases} \tag{2.10}$$

which is the controller group for the simultaneous control of prey and predator populations.

Therefore, using the error function and the design formula we can obtain the explicit expressions for inputs u_n and w_n, which act as the exogenous measure, and these input variables (update parameters) depend on the parameter values, state variables, and the Z control parameter. Therefore, if we change the prey and predator population abundance following the update parameters u_n and w_n, then the error functions $e_n^1 = x_n - p_n$ and $e_n^2 = y_n - q_n$ will converge to zero and the desired prey and predator densities can be achieved.

2.4 Numerical simulation

In this subsection, explanatory simulation results are given to show the feasibility of the controller group (2.10) for the simultaneous control of prey and predator populations. Here, our main goal is to show numerically, how the different dynamical behavior of the uncontrolled system (2.8) (chaotic oscillation, periodic oscillation, and stable dynamics) are controlled by the Z-type controller group (2.10) and also to show both prey and predator populations reach to the desired states in each case. Here, we consider the value of the parameters as

$$k = 1, d = 0.1, \lambda = 2, c = 0.1 \tag{2.11}$$

with initial condition $(0.8, 0.5)$. In Figure 1 it is seen that the uncontrolled system shows chaotic dynamics with an increase of the value of the parameter r via period-doubling bifurcation. For better visualization of switching of the dynamics, we consider three different values of r $(3, 5, 6)$ and draw time-series solutions for the prey population (Figure 2). Now we apply direct Z-type control in the uncontrolled system (2.8), where the value of the design parameter $m = 3$ and the desired population densities are $p_n = p_{n+1} = 2$ and $q_n = q_{n+1} = 5$. We consider three different values of r, for which the uncontrolled system (2.8) shows chaotic oscillation, periodic oscillation and stable dynamics respectively. We observe that in all these three cases, Z-type controllers drive the populations to the desired states and these results are plotted in Figure 3, Figure 4 and Figure 5 respectively. Figure 3 shows the chaotic or irregular oscillation of the uncontrolled system (2.8) for $r = 6$ together with how the Z-type controller group (2.10) removes the irregular oscillation from the controlled system (2.9). In Figure 4, we see that the uncontrolled system (2.8) shows periodic oscillation for $r = 5$ whereas the controlled system shows stable dynamics. For $r = 3$ both the uncontrolled and controlled systems show stable dynamics, which is exhibited in Figure 5 and also for the controlled system (2.9) the desired population densities are achieved through the Z-type controllers (2.10). In the last row of each figure, the first one reveals the time-series evaluation of tracking errors e_n^1 and e_n^2, from which it is seen that they converge to zero. It ensures about the success of Z-type control mechanism. The last one of each figure exhibits the time-series evaluation of control variables u_n and w_n.

3 Indirect control of population

In this section, the design procedure of indirect Z-control laws for a discrete-time system is discussed. Here we intend to control the prey population density by taking control measure on the predator population and applying indirect Z-type control mechanism. Also, this section provides

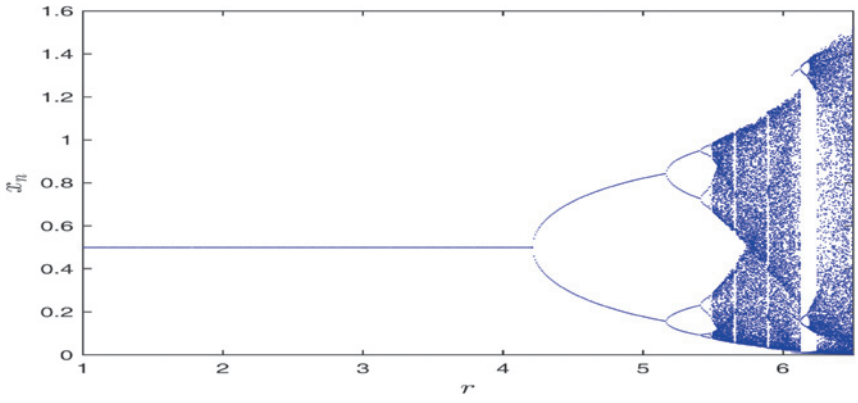

Figure 1: Figure shows the bifurcation diagram for the prey species of the uncontrolled system (2.8). Here bifurcating parameter is r and other parameter values are taken as $k = 1, d = 0.1, \lambda = 2$ and $c = 0.1$.

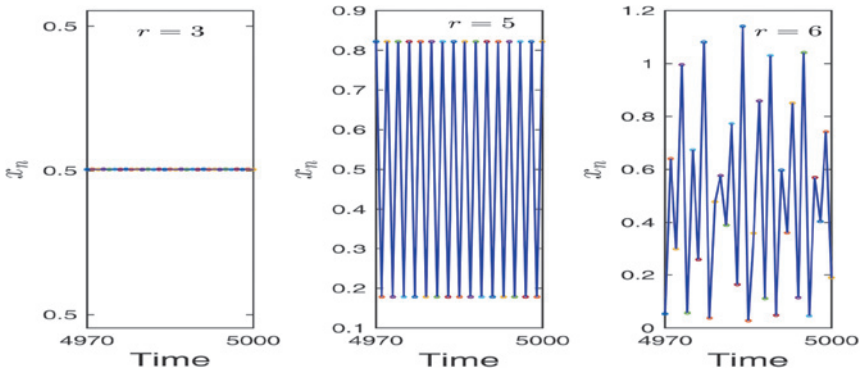

Figure 2: Switching of dynamics of the uncontrolled system (2.8) for different values of r. Left figure shows stable dynamics for $r = 3$, middle figure shows periodic oscillations for $r = 5$ and right figure shows chaotic oscillations for $r = 6$. Here other parameter values are taken as $k = 1, d = 0.1, \lambda = 2$ and $c = 0.1$.

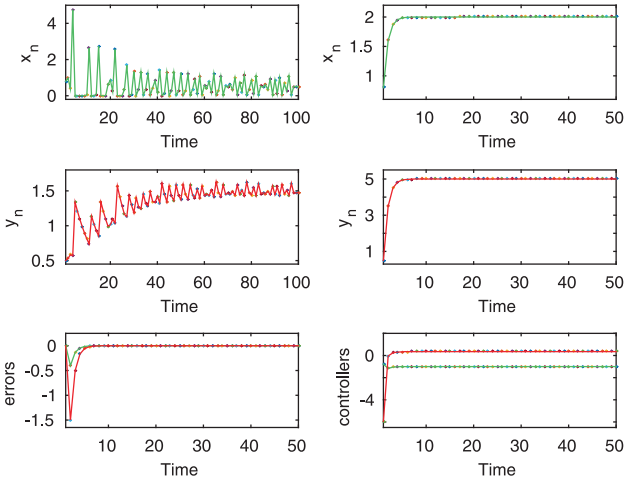

Figure 3: Here, the first two rows of left column show the chaotic oscillation for the uncontrolled system (2.8) when $r = 6$ and the first two rows of right column show the stable dynamics for the controlled system (2.9). Here $m = 3$ and the other parameter values are same as in equation (2.11). The desired states are $p_n = 2$ and $q_n = 5$. The last row displays the time-series evaluations of tracking errors and update parameters for the successful execution of Z-type control.

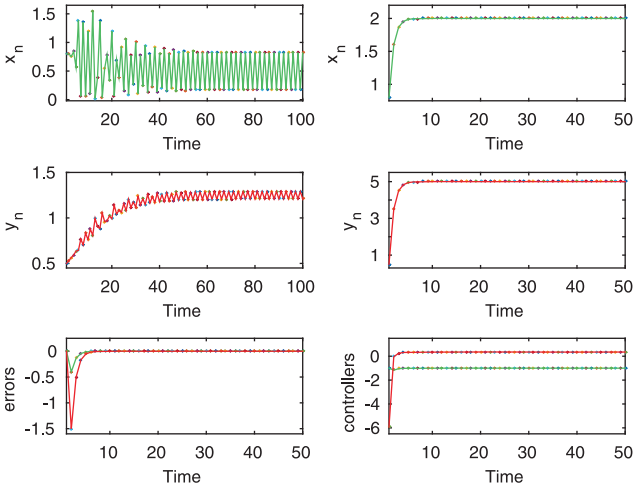

Figure 4: Here, the first two rows of left column exhibit periodic oscillation for the uncontrolled system (2.8), whereas the first two rows of the right column show stable dynamics of the controlled system (2.9). Here, $r = 5$, $m = 3$ and the other parameter values are same as equation in (2.11). The last row shows that tracking errors converge to zero and the variation of the update parameters.

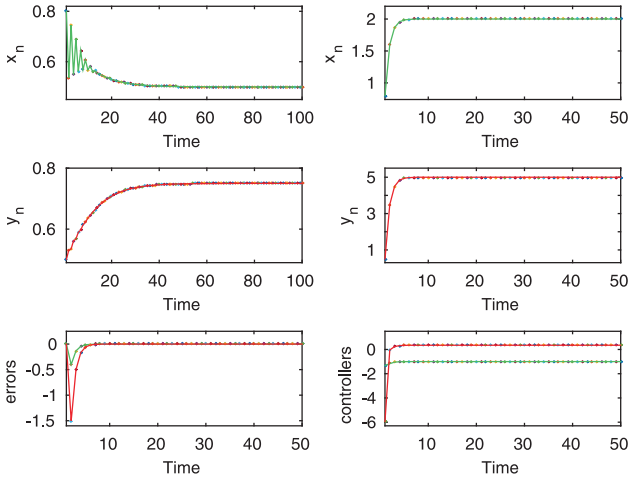

Figure 5: Here, both uncontrolled and controlled systems exhibit stable dynamics, which are shown in the first two rows for $r = 3$, $m = 3$ and the other parameter values are same as in equation (2.11).

theoretical analysis to show the effectiveness of indirect Z-type control law.

3.1 Controller design

Here, we apply indirect Z-type control on the predator–prey model (2.8). For this purpose we introduce an exogenous measure on the predator population and then the model (2.8) takes the following form:

$$\begin{cases} x_{n+1} = x_n \exp\left(r\left(1-\dfrac{x_n}{k}\right) - \lambda y_n\right), \\ y_{n+1} = y_n \exp\left(c\lambda x_n - d\right) - w_n y_n, \end{cases} \tag{3.1}$$

where, w_n denotes the indirect control variable for prey population.

Now, our aim is to find the analytical expression of the indirect control variable w_n.

For this, first we define an error function as

$$e_n^1 = v_n^1 = x_n - p_n. \tag{3.2}$$

Next we consider the design formula as

$$v_{n+1}^1 = \frac{1}{m} v_n^1, \tag{3.3}$$

where, $m(>1)$ is the design parameter. Now substituting (3.2) in (3.3), we get

$$x_{n+1} - p_{n+1} = \frac{1}{m}\left(x_n - p_n\right), \tag{3.4}$$

but equation (3.4) does not contain y_{n+1} in which the control variable w_n is included. Thus we have to define another error function. Let us consider the second error function as

$$v_n^2 = v_{n+1}^1 - \frac{1}{m} v_n^1, \tag{3.5}$$

and the behavior of v_n^2 is same as v_n^1 i.e. $v_{n+1}^2 = \frac{1}{m} v_n^2$.

Using $v_{n+1}^2 = \frac{1}{m} v_n^2$ in equation (3.5), we get the following expression

$$v_{n+2}^1 - \frac{2}{m} v_{n+1}^1 + \frac{1}{m^2} v_n^1 = 0. \tag{3.6}$$

Now putting the value of v_n^1 in equation (3.6), we get

$$x_{n+2} - p_{n+2} - \frac{2}{m}\left(x_{n+1} - p_{n+1}\right) + \frac{1}{m^2}\left(x_n - p_n\right) = 0. \tag{3.7}$$

After that substituting the value of x_{n+2} in equation (3.7), we get

$$y_{n+1} = \frac{1}{\lambda}\left[r\left(1-\frac{x_{n+1}}{k}\right) - \log\left(\frac{2}{m}\left(x_{n+1} - p_{n+1}\right) - \frac{1}{m^2}\left(x_n - p_n\right) + p_{n+2}\right) + \log\left(x_{n+1}\right)\right] \tag{3.8}$$

and finally we obtain the control variable as

$$w_n = \exp\left(c\lambda x_n - d\right) - \frac{h}{y_n}, \tag{3.9}$$

where

$$h = \frac{1}{\lambda}\left[r\left(1-\frac{x_{n+1}}{k}\right) - \log\left(\frac{2}{m}\left(x_{n+1} - p_{n+1}\right) - \frac{1}{m^2}\left(x_n - p_n\right) + p_{n+2}\right) + \log\left(x_{n+1}\right)\right]. \tag{3.10}$$

Here equation (3.9) gives the indirect controller for the controlled system (3.1).

Therefore, using the error function and the design formula we obtain the analytical expression for input w_n, which acts as an exogenous measure and if we change the predator population abundance following the update parameter (w_n), then the error function $e_n^1 = x_n - p_n$ will converge to zero and the desired prey density can be achieved.

3.2 Theoretical analysis

In this subsection, the convergence performance of the Z-type controller (3.9) for the model (3.1) is examined theoretically. Here, we show that the error for the model (3.1) equipped with indirect Z-controller converges to zero.

Theorem 2: For a bounded desired state p_n, starting from a positive initial state $[x_0, y_0]^T$, the tracking error e_n^1 (or v_n^1) for

the model (3.1) furnished with Z-type controller (3.9) converges to zero.

Proof: According to the design procedure of Z-type controller (3.9), we obtain that

$$v_n^2 = v_{n+1}^1 - \frac{1}{m}v_n^1. \tag{3.11}$$

Now, substituting (3.11) into $v_{n+1}^2 = \frac{1}{m}v_n^2$ we get

$$v_{n+2}^1 - \frac{2}{m}v_{n+1}^1 + \frac{1}{m^2}v_n^1 = 0. \tag{3.12}$$

Solving the above difference equation (3.12) we get

$$v_n^1 = (c_1 + c_2 n)\left(\frac{1}{m}\right)^n,$$

where c_1, c_2 are arbitrary constants.

Using the initial condition, we obtain $c_1 = x_0 - p_0$ and

$$c_2 = m\left(x_0 \exp\left(r\left(1 - \frac{x_0}{k}\right) - \lambda y_0\right) - p_1\right) - (x_0 - p_0).$$

Therefore, the tracking error e_n^1 (or v_n^1) converges to zero for large values of n and $\forall\ m > 1$, which implies that the prey population x_n converges to the desired population p_n, i. e., $x_n \to p_n$. This completes the proof.

3.3 Numerical simulation

In this subsection, numerical simulations are offered to verify the success of the indirect Z-type controller (3.9). Here, we consider the same parameter values and initial conditions as taken in subsection 2.4. In a similar manner, as described in the direct case, we choose the value of the parameter r in such a way that the system (2.8) without Z-control shows chaotic, periodic and stable dynamics respectively. Then we apply indirect Z-type controller on the predator population for controlling the prey population density. Here we choose the value of the design parameter $m = 2$ and $p_n = p_{n+1} = 0.7$. From Figure 6, we see that the irregular oscillation has eliminated from the uncontrolled system (2.8) due to the controller (3.9) and also the controller drives the prey population to the desired state, where the desired prey population density is $p_n = 0.7$. From Figure 7, it is seen that the controller removes the periodic oscillation from the uncontrolled system (2.8) and the prey population density tends to the desired state $p_n = 0.7$. In Figure 8, we see that both the controlled as well as uncontrolled systems are stable and the desired prey population density is achieved for the controlled system due to the controller (3.9). Also in each figure, we observe that the tracking error converges to zero, which agrees with the theoretical analysis. The last one of each figure displays the time-series evaluation of the update parameter (or controller).

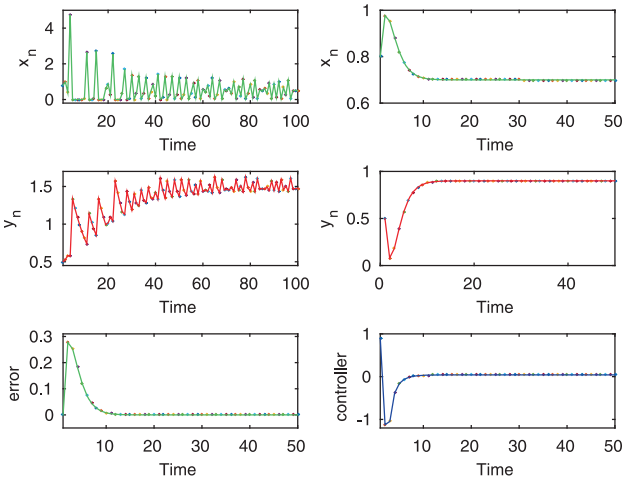

Figure 6: Here, the first two rows of left column show chaotic dynamics of the uncontrolled system (2.8) for $r = 6$, whereas the first two rows of right column exhibit stable dynamics for the controlled system (3.1). Here $m = 2$ and the other parameter values are taken as in equation (2.11) and desired prey population density $p_n = 0.7$. The last row shows the time-series evaluation of tracking error and update parameter for the successful execution of indirect Z-type control.

Now we draw the basin of attractions of the controlled system (3.1) for two different values of the design parameter m (2 and 2.5), with desired prey population density $p_n = 0.7$. Here, $r = 3$ and the other parameter values are

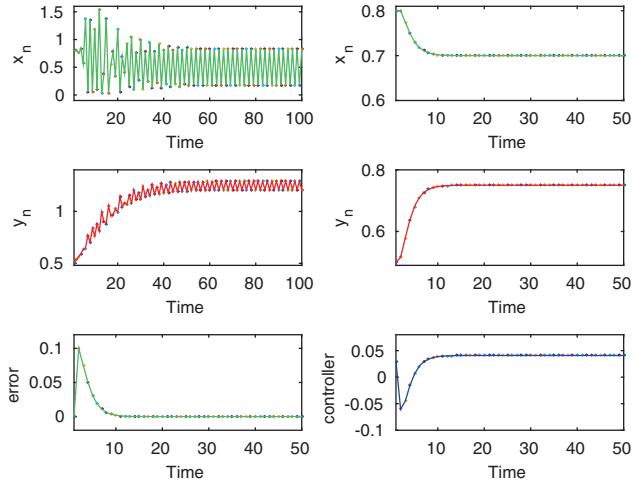

Figure 7: In this figure, the first two rows of left column show that the uncontrolled system (2.8) oscillates periodically, while the controlled system shows stable dynamics for $r = 5$, where $m = 2$ and the other parameters are same as in equation (2.11) with the desired state $p_n = 0.7$.

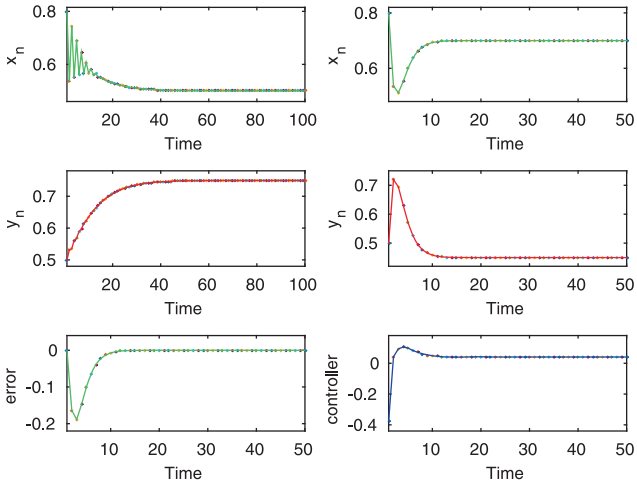

Figure 8: Here, both the controlled and uncontrolled systems show stable dynamics, which are executed in the first two rows of first and second column for $r = 3$, where $m = 2$, $p_n = 0.7$ and the other parameters are given in equation (2.11).

same as equation (2.11). In Figure 9, for the sub-figure (9a) we choose $m = 2$, where the green region denotes the set of initial conditions for which the positivity of both prey and predator populations are satisfied and this region is ecologically meaningful. The yellow and red regions denote the set of initial conditions for which the Z-control approach works, but the positivity condition for the prey/predator populations is violated. In the yellow region, predator population density becomes negative and in the red region, prey population density becomes negative in the temporal dynamics. Both yellow and red regions are not ecologically meaningful. Next, we choose $m = 2.5$ and investigate the same phenomena. We see that the ecologically feasible region (green shaded region) increases with an increase in the value of the design parameter m.

For a better interpretation of these phenomena, we select different initial conditions $(0.5, 1.2)$, $(0.4, 4)$, and $(1.5, 3.8)$ from the green, yellow, and red regions respectively and plot the time-series evaluations of prey and predator populations. In Figure 10, from the first row, it is seen that both prey and predator populations satisfy positivity conditions, whereas from the second row we observe that although the prey population is at positive level with desired prey population density, but the predator population has lost positivity condition. Lastly, from the third row, it is seen that both prey and predator populations have lost positivity condition. So it is important to note that, the selection of initial conditions is very significant while applying the Z-control mechanism to a system for attaining the desired population density.

3.3.1 Application of indirect Z-controller for a periodic desired state

The controller (3.9) is also capable to drive the prey population to a periodic desired state whereas the uncontrolled system shows chaotic, periodic, and stable dynamics respectively. For the successful execution of these phenomena, we consider the parameter values and the initial state same as subsection 2.4. We also choose the periodic desired states of the prey population as $p_i = 0.6$, $p_{i+1} = 1$

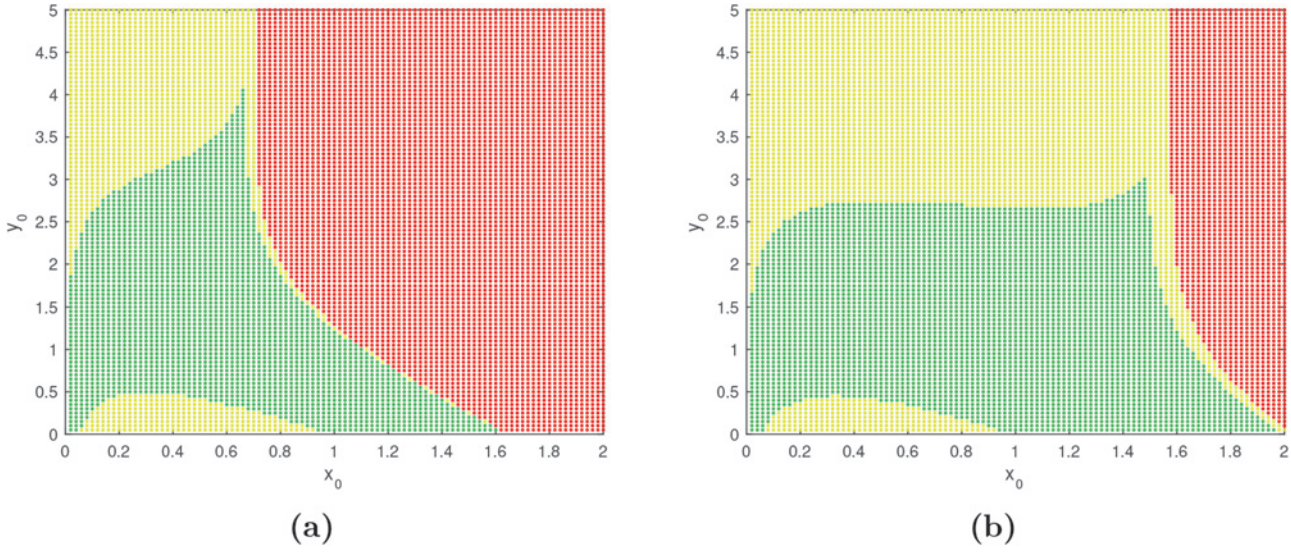

Figure 9: Figure shows the basin of attraction of the Z-controlled system (3.1) for two different values of the design parameter m. Here, in the left figure $m = 2$ and in the right figure $m = 2.5$ with $r = 3$ and the other parameter values are same as in equation (2.11). The value of the desired state is $p_n = 0.7$. For larger values of m, ecologically feasible region (green shaded region) increases.

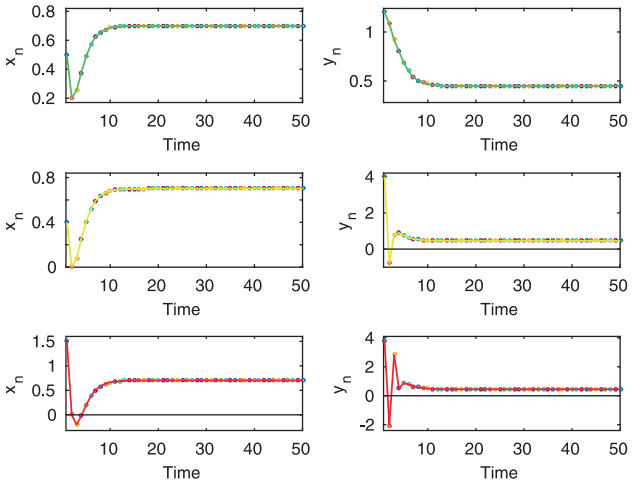

Figure 10: Here, from the first row it is seen that both prey and predator populations remain at positive level for the initial state $(0.5, 1.2)$. Second row shows that the prey population remains at positive level but predator population has lost positivity condition, where the initial state is $(0.4, 4)$. The third row shows that both prey and predator populations have lost positivity condition for the initial state $(1.5, 3.8)$. Here, $m = 2$, $r = 3$ and the other parameter values are same as in equation (2.11).

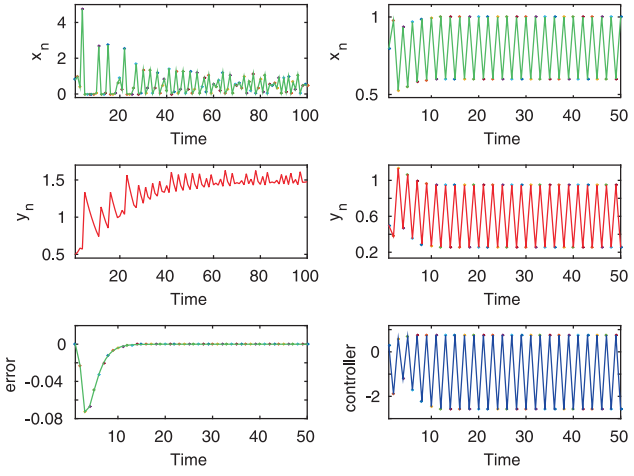

Figure 11: In this figure, the first two rows of left column show the chaotic dynamics for the uncontrolled system (2.8) for $r = 6$. On the other hand, the first two rows of right column exhibit periodic behavior of the controlled system (3.1). Here $m = 2$ and the other parameter values are taken as in equation (2.11). The last row displays the time-series evaluations of tracking error and update parameter, which execute successful implementation of indirect Z-type control.

and $p_{i+2} = p_i$, where i is a positive integer. The value of the design parameter is chosen as $m = 2$. From Figure 11, we observe that the controller drives the prey population to a periodic desired state after eliminating irregular oscillation

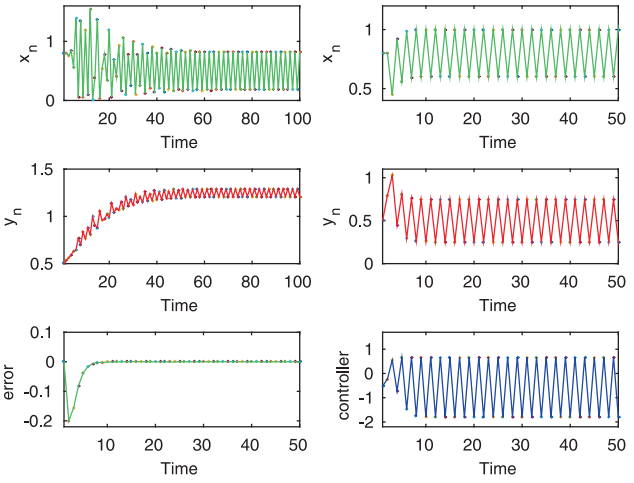

Figure 12: Here, from the first two rows, it is seen that both uncontrolled and controlled systems show periodic oscillations for $r = 5$, $m = 2$ and the other parameter values are same as in equation (2.11).

from the uncontrolled system (2.8). From Figure 12, it is seen that both the uncontrolled and the controlled systems show periodic oscillation. Here our aim is to show that the controller can drive the prey population density periodically in a desired state, which is away from zero. Lastly, from Figure 13 it is observed that the uncontrolled system shows stable dynamics, whereas, in the controlled system, the prey population converges to a periodic desired state due to the controller (3.9). In all these cases, we see that the tracking error converges to zero, which confirms the success of indirect Z-controller (3.9).

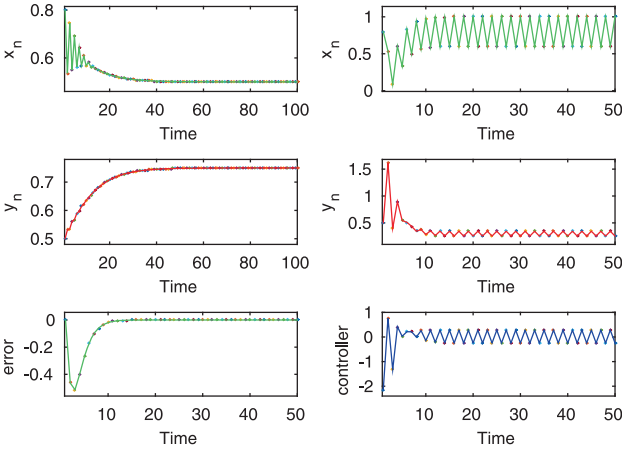

Figure 13: Here, the first two rows of left column show stable dynamics of the uncontrolled system (2.8), on the other hand, from the first two rows of right column it is observed that the desired state of prey population of the controlled system (3.1) is periodic for $r = 3$, $m = 2$ and the other parameter values are given in equation (2.11).

4 Conclusion

In the real-world, all species are important to maintain the ecological balance, and they interact with each other and also with their environment in an elegantly balanced cycle. But as time goes, ecological imbalance increases due to the excessive growth of some species, sudden death of some species, careless human activities, changing climate, habitat loss, degradation, etc., for which negative consequences occur in the ecosystem. But for the survival of all species in the world, preservation of ecological balance is very important and it will be maintained if all the species are in the desired states. So it is sometimes needed to control the population of a system to a reasonable level, because either this population drives others to become extinct or even itself becomes extinct. According to top-down control, predator plays an important role in controlling the prey population size and community structure in a food web ecology. However, a pre-defined prey population abundance may not be achieved only through top-down control. It can be done successfully by applying Z-type controller, which is a combination of immigration, emigration, culling, and harvesting.

To investigate the effectiveness of Z-type control mechanism in a discrete-time predator–prey system, we have considered a Lotka–Volterra type predator–prey model with direct and indirect Z-control laws in discrete-time set-up. We observed that in both direct and indirect cases the Z-type controller group (2.10) (for prey and predator species) and the controller (3.9) (for the prey species) drive the populations to the respective desired states. It is also investigated that the controller is able to keep species far from the risk of disappearance i.e. the species do not die out from the environment. Here, the time-series evaluation of the update parameters (or parameter) has great importance. We have observed that the update parameters can take both positive and negative values for achieving the desired population density. In both the direct and indirect Z-control mechanisms, the positive value of the update parameter implies the removal of a population from the system through emigration, harvesting, or culling, whereas the negative value of the update parameter implies the addition of population into the system through immigration. In the Z-type control mechanism, we can change the prey and predator population densities or predator population density through emigration, immigration, culling, or harvesting to obtain the desired dynamics. For direct control, addition or removal of a certain amount of prey and predator species (for indirect control, addition, or removal of a certain amount of predator species) stabilize the system. Now it is worthy to note that the rate at which the species should be added into or removed from the system to achieve the desired state and it is quantified by the magnitude of the update parameter. Also, we have observed that for any value of the design parameter $m\,(m > 1)$ the Z-type controller drives the population to the desired state. It is also interesting to note that the choice of initial conditions is very important from an ecological point of view. We have drawn basin of attractions and observed that depending on the initial condition, the solutions may converge to the desired population density via negative solution trajectories, which are not ecologically meaningful. The density of prey or predator population becomes negative in the transient dynamics means that the extinction of the respective populations. So, we have to careful while implementing the Z-control mechanism, otherwise one or both of the populations may go to extinction.

We can conclude that such an error-based dynamic method (Z-type control) plays an important role for maintaining the ecological balance. Riechert et al. [46] experimentally showed that when spiders are added in the vegetable system, then the number of pests significantly decline and the amount of average damage reduced to 31.8% from 93.3%. Therefore, predators limit associated prey populations. The above experimental results can be implemented in an ecosystem most effectively by using Z-control mechanism. This error based Z-control mechanism is very useful for serving ecosystems, via, stabilizing the unstable equilibrium point, originating new stable equilibrium point or shifting the population oscillation away from zero. Therefore, Z-type control mechanism can be considered as a control mechanism that can be applied for conservation biology and strategic management of pest control in an ecological system.

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