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Cosmological solutions in Hořava-Lifshitz scalar field theory

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Abstract: We perform a detailed study of the integrability of the Hořava-Lifshitz scalar field cosmology in a Friedmann-Lemaître-Robertson-Walker background space-time. The approach we follow to determine the integrability is that of singularity analysis. More specifically, we test whether the gravitational field equations possess the Painlevé property. For the exponential potential of the scalar field, we are able to perform an analytic explicit integration of the field equations and write the solution in terms of a Laurent expansion and more specifically write the solution in terms of right Painlevé series.

Keywords: cosmology; Hořava-Lifshitz; integrability; Painlevé analysis; scalar field.

1 Introduction

Modified theories of gravity have been a subject of special interest over recent years because they provide a geometric approach to the description of observable phenomena. In modified theories of gravity, new geometric quantities are introduced into the Einstein-Hilbert action of general relativity (GR). The new terms in the gravitational action provide new components of geometric origin in the field equations which change the dynamics such as the solutions of the latter to describe the phenomena observed. In this work, we are interested specifically in the Hořava-Lifshitz (HL) gravity [1]. HL gravity is a power-counting renormalization theory with consistent ultraviolet behaviour exhibiting an anisotropic Lifshitz scaling between time and space at the ultraviolet limit. HL theory can provide Einstein's GR as a critical point.

The Einstein-Aether theory [2–4] is related with HL theory in the classical limit [5]. In Einstein-Aether theory, the kinematic quantities of a time-like vector field (the Aether) are introduced into the gravitational action integral. The theory preserves locality and covariance while it contains GR. Now when the Aether field is hypersurface-orthogonal, then the classical limit of HL is recovered. Hence, every hypersurface-orthogonal Einstein-Aether solution is a solution of HL gravity [6].

Lorentz violation theories have been applied in various models of gravitational physics [7–9]. Applications of HL theory cover a wide range of subjects, from compact stars to cosmological studies [10–13]. In the studies by Calgani [14] and Kiritsis and Kofinas [15], it was found that HL cosmology provides an alternative to inflation and that the universe can be singularity free. For reviews of HL cosmology, we refer the reader to the studies by Mukohyama [16] and Saridakis [17]. Very few closed-form solutions exist in HL cosmology with or without any matter source [18–22]. However, as per our knowledge, there are no known analytical solutions in HL cosmology when a minimally coupled scalar field contributes to the total evolution of the universe.

Recently, a detailed study on the dynamics of the HL scalar field cosmology was performed by Leon and Paliathanasis [23]. More specifically, the phase space of HL cosmology was examined for a wide range of scalar field potentials by means of the powerful method of *f*-devisors. Applications were presented for the exponential, power law and other potentials. Singular power-law solutions or de Sitter solutions were found to be described at the critical points. However, these are peculiar solutions as they do not exist for any initial conditions and they describe only an approximation of the generic solution for arbitrary initial conditions.

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any initial conditions and they describe only an approximation of the generic solution for arbitrary initial conditions.

In general, physical theories provide a large number of free parameters or boundary conditions to be involved so that numerical solutions, even if practicable, give no real idea for the properties of the differential equations, which are defined by the theory. This means that analytic techniques are essential for the study of real-world problems. Hence, the knowledge for the existence and the determination of analytical or exact solutions for a given dynamical system is important for the detailed study and understanding of the dynamical system. More specifically, when a dynamical system is integrable, then know that trajectories of (numerical) solutions correspond to 'real' solutions for the given dynamical system, or when it is feasible, we can write the analytic solution by using closed-form functions. There are many methods to study the integrability of a dynamical system. In terms of Hamiltonian system usually in physics, we refer to the Liouville integrability, where the Hamiltonian system admits sufficient number of conservation laws in order to be solved by quadratures [24]. There is a plethora of cosmological models which are Liouville integrable, for instance, see [25–30] and references therein. The importance of the Liouville-integrable cosmological models, apart from the fact that the analytical solution can be determined, is that conservation laws can be applied to perform a canonical quantization which is a specific approach of quantum cosmology [31–34].

Although, there are dynamical systems that are not Liouville integrable, there is an analytical way to perform an explicit integration of the dynamical system. One of the alternative methods is the singularity analysis which is the main mathematical tool that we apply in this work. The modern treatment of singularity analysis is summarized in the Ablowitz, Ramani and Segur (ARS) algorithm [35–37] which provide us with the information if a given differential equation passes the Painlevé test and consequently if the solution of the differential equation can be written as a Laurent expansion around a movable singularity. Singularity analysis is a powerful method which has led to the determination of analytic solutions of various cosmological models in GR [38–44] or in modified theories of gravity [45–51]. The plan of the article is as follows.

In Section 2, the formal theory of HL cosmology under the detailed balance condition is presented, while the field equations of HL scalar field cosmology are given. Moreover, we derive the equivalent field equations in the dimensionless variables by using the H-normalization approach [23, 52]. By using the H-normalization approach, we present the dynamical systems of our consideration which we study in terms of integrability. In Section 3, we briefly discuss the ARS algorithm. The main results of our analysis and the new analytic solutions in HL cosmology are presented in Section 4. In Section 5, we discuss our results and draw our

2 Hořava-Lifshitz scalar field cosmology

The gravitational action integral in Hořava-Lifshitz gravity under the detailed balance condition is given by the following expression [15]:

$$\begin{aligned} S_g = \int dt d^3x \sqrt{g} N \Big\{ & \frac{2}{\kappa^2}\left(K_{ij}K^{ij} - \lambda K^2\right) \\ & + \frac{\kappa^2}{2w^4} C_{ij}C^{ij} - \frac{\kappa^2\mu}{2w^2}\frac{\varepsilon^{ijk}}{\sqrt{g}} R_{il}\nabla_j R^l_k + \frac{\kappa^2\mu^2}{8} R_{ij}R^{ij} \\ & - \frac{\kappa^2\mu^2}{8(3\lambda-1)}\left[\frac{1-4\lambda}{4}R^2 + \Lambda R - 3\Lambda^2\right]\Big\} \end{aligned} \tag{1}$$

where the underlying geometry is written as

$$ds^2 = -N^2dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{2}$$

in which the lapse and shift functions are, respectively, N and N_i. The spatial metric is given by g_{ij}, and roman letters indicate spatial indices. Tensor $K_{ij}$is the extrinsic curvature defined as

$$K_{ij} = \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)\Big/2N \tag{3}$$

and C^{ij} denotes the Cotton tensor

$$C^{ij} = \varepsilon^{ijk}\nabla_k\left(R^j_i - R\delta^j_i/4\right)\Big/\sqrt{g}. \tag{4}$$

The covariant derivatives are defined with respect to the spatial metric g_{ij}. Finally, ϵ^{ijk} is the totally antisymmetric unit tensor, λ is a dimensionless constant and the quantities w and μ are constants. For simplicity, we select to work with units where $\kappa^2 = 8\pi G = 1$.

In accordance with the cosmological principle, the universe in large scales is homogeneous and isotropic, consequently the line element (2) reduces to the Friedmann-Lemaître-Robertson-Walker (FLRW) space-time where $N^i = 0$ and $g_{ij} = a^2(t)\gamma_{ij}$ where now γ_{ij} denotes the maximally symmetric space of constant curvature k, i. e.,

$$\gamma_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2, \tag{5}$$

where $k = -1, 0, +1$ and $d\Omega_2$ is the two sphere.

In addition, we assume the contribution of a scalar field in the universe with action integral

$$S_\phi = \int dt d^3x \sqrt{g} N\left[\frac{3\lambda-1}{4}\frac{\dot{\phi}^2}{N^2} - V(\phi)\right], \tag{6}$$

where for the scalar field, it has been considered that ϕ inherits the symmetries of the FLRW space-time.

Variation with respect to the metric in the action integrals provides the second-order differential equations

$$\begin{aligned}3H^2 = &\frac{1}{2(3\lambda-1)}\left[\frac{3\lambda-1}{4}\dot{\phi}^2 + V(\phi)\right] \\ &+\frac{3}{16(3\lambda-1)^2}\left[-\frac{\mu^2k^2}{a^4} - \mu^2\Lambda^2 + \frac{2\mu^2\Lambda k}{a^2}\right]\end{aligned} \tag{7}$$

$$\begin{aligned}2\dot{H} + 3H^2 = &-\frac{1}{2(3\lambda-1)}\left[\frac{3\lambda-1}{4}\dot{\phi}^2 - V(\phi)\right] \\ &-\frac{1}{16(3\lambda-1)^2}\left[-\frac{\mu^2k^2}{a^4} + 3\mu^2\Lambda^2 - \frac{2\mu^2\Lambda k}{a^2}\right],\end{aligned} \tag{8}$$

where $H = \frac{\dot{a}}{a}$.

On the other hand, variation with respect to the scalar field provides the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{2V'(\phi)}{3\lambda-1} = 0. \tag{9}$$

The latter field equations can be written equivalently as follows

$$3H^2 = \rho_\phi + \rho_{HL}, \tag{10}$$

$$-2\dot{H} - 3H^2 = p_\phi + p_{HL}, \tag{11}$$

and

$$\dot{\rho}_\phi + 3H\left(\rho_\phi + p_\phi\right) = 0, \tag{12}$$

where we have defined

$$\begin{aligned}\rho_\phi &= \frac{1}{2(3\lambda-1)}\left[\frac{3\lambda-1}{4}\dot{\phi}^2 + V(\phi)\right], \\ p_\phi &= \frac{1}{2(3\lambda-1)}\left[\frac{3\lambda-1}{4}\dot{\phi}^2 - V(\phi)\right],\end{aligned} \tag{13}$$

and

$$\rho_{HL} = \frac{3}{16(3\lambda-1)^2}\left[-\frac{\mu^2k^2}{a^4} - \mu^2\Lambda^2 + \frac{2\mu^2\Lambda k}{a^2}\right], \tag{14}$$

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In this work, we focus on the integrability of the dynamical system (7)–(8). However, to follow a similar singularity analysis as the one performed for the Einstein-Aether theory in the study by Latta et al [53], we prefer to work with H-normalized variables.

H-normalized variables defined as x, y, z, u in (16) were used in the study by Leon and Saridakis [52] to analyse the dynamics of HL cosmology in the presence of a scalar field with exponential potential. That analysis can be extended with the variable s, and the function $f(s)$ to analyse potentials beyond the exponential potential like in the study by Leon and Paliathanasis [23] by introducing the quantities s and f defined in (16). Hence, the dimensionless dynamical system [23] is defined by the new variables $\{x, y, z, u, s\}$

$$\begin{aligned}&x = \frac{\dot{\phi}}{2\sqrt{6}H}, \quad y = \frac{\sqrt{V(\phi)}}{\sqrt{6}H\sqrt{3\lambda-1}}, \\ &z = \frac{\mu}{4(3\lambda-1)a^2H}, \quad u = \frac{\Lambda\mu}{4(3\lambda-1)H}, \\ &s = -\frac{V'(\phi)}{V(\phi)}, \quad f(s) \equiv \frac{V''(\phi)}{V(\phi)} - \frac{V'^2(\phi)}{V(\phi)^2},\end{aligned} \tag{16}$$

where function $f(s)$ is defined by the specific form of the potential $V(\phi)$. As a new independent variable, it is assumed the number of e-fold $\tau = \ln\left(\frac{a}{a_0}\right)$ allows to recast the cosmological equations for arbitrary potentials in a closed form. We remark that for the exponential potential $V(\phi) = V_0e^{\sigma\phi}$, function $f(s)$ vanishes and $s = \sigma$, while for the power-law potential $V(\phi) = V_0\phi^{2n}$, function $f(s)$ is defined as $f(s) = -\frac{s^2}{2n}$.

At this point, it is important to mention that while the analysis of critical points in the study by Shababi and Pedram [23] for the HL scalar field cosmology is for arbitrary potential, to study the integrability of the field equations by applying the ARS algorithm, the potential function has to be specified. The exponential and power-law potentials are two well-known and well-studied potentials in GR, and they can approximate the functional behaviour of other nonlinear potentials in their limits; for instance, the hyperbolic potential $V(\phi) = V_0\sinh(\sigma\phi)^{2n}$ is approximated for small values of ϕ by the power-law potential $V(\phi) \simeq \phi^{2n}$, and for large values of ϕ, the hyperbolic potential has an exponential behaviour.

3 Singularity analysis

The integration of a differential equation is performed by the global knowledge of the general solution independently from the local solutions provided by Cauchy's existence theorem. A differential equation that passes the Painlevé test can be said to be integrable. However, because not all the integrable differential equations have the Painlevé property, when the latter property exists, it is better to be referred as the uniformizability of the general equation for the given differential equation [54].

To study if a given differential equation admits the Painlevé property, we apply the ARS algorithm which is summarized in three main steps: (a) determine the leading-order term which describes a movable singularity; (b) determine the resonances which give the position of the integration constants and (c) write a Painlevé series with exponent and step as given in steps (a) and (b) and test if it solves the differential equation; the latter is called the consistency test.

There are various criteria and conditions that should be satisfied in the ARS algorithm in order for a differential equation to have the Painlevé property; for instance, the number of the resonances should be equal to the number of the degrees of freedom of the differential equation, while one of the resonances should be $r = -1$, otherwise the singularity determined at step (a) is not movable. Moreover, if the resonances are positive, the Painlevé series is written by a right Laurent expansion, and when the resonances are negative, the Painlevé series is given by a left Laurent expansion; otherwise, the Painlevé series is given by a mixed Laurent expansion. For more details on the criteria of the ARS algorithm and the interpretation of the algorithm in the complex plane, we refer the reader to the studies by Conte [54] and Ramani et al [55].

4 Explicit integration in Hořava-Lifshitz cosmology

In this section, we study if the field equations of HL scalar field cosmology (7)–(9) possess the Painlevé property; when the latter is true, we perform an explicit integration around the movable singularity and we present the generic solution in terms of Laurent expansion. We divide our analysis into four different cases as studied by Leon and Paliyathanasis [23] for arbitrary potentials, and previously by Leon and Saridakis [52] for exponential potential.

In case A, we assume that the underlying geometry is spatially flat ($k = 0$) and there is no cosmological constant term ($\Lambda = 0$), case B is with $k \neq 0$ and $\Lambda = 0$. Cases C and D are with $\Lambda \neq 0$, while for the spatial curvature, it holds $k = 0$ and $k \neq 0$, respectively. Last but not least, for the scalar field potential, we consider the exponential potential where $f(s) \equiv 0$and the power-law potential with $f(s) = -\frac{s^2}{2n}$.

4.1 Case A: $k = 0, \Lambda = 0$

For the first case of our analysis where the spatial curvature of the FLRW space-time vanishes and there is no cosmological constant term, the gravitational field equations in the dimensionless variables (16) form the following three-dimensional first-order differential equations [24]:

$$\frac{dx}{d\tau} = \left(3x - \sqrt{6}\,s\right)(x^2 - 1), \tag{17}$$

$$\frac{dz}{d\tau} = (3x^2 - 2)z, \tag{18}$$

$$\frac{ds}{d\tau} = -2\sqrt{6}xf(s), \tag{19}$$

defined on the phase space $\{(x, z, s) \in \mathbb{R}^3 : -1 \le x \le 1\}$.

We continue our analysis by presenting the application of the ARS algorithm for the dynamical system.

4.1.1 Exponential potential

For the exponential potential, because the RHS of equation (19) is identically zero, the dynamical system is reduced to a two-dimensional first-order differential equations, where $s(\tau) = const$.

We observe that the dynamical system (17), (18) for $s = const$ can be easily integrated by quadratures. Indeed, from equation (17), it follows that

$$\begin{gathered}\ln\left((x(\tau)+1)^{\frac{1}{6+2\sqrt{6}\,s}} (x(\tau)+1)^{\frac{1}{6-2\sqrt{6}\,s}} \right) + \\ \frac{3}{9-6s^2}\ln\left(\sqrt{6}\,s - 3x(\tau)\right) = \tau - \tau_0 \,, s \neq \pm\frac{\sqrt{6}}{2},\end{gathered} \tag{20}$$

or

$$\frac{1}{12}\ln\left(\frac{x+1}{x-1}\right)-\frac{1}{6(x+1)}=\tau-\tau_0\,,s=-\frac{\sqrt{6}}{2}, \tag{21}$$

or

$$\frac{1}{12}\ln\left(\frac{x-1}{x+1}\right)+\frac{1}{6(x+1)}=\tau-\tau_0\,,s=-\frac{\sqrt{6}}{2}, \tag{22}$$

while equation (18) gives

$$\ln z(\tau)=3\int x(t)^2 d\tau-2(\tau-\tau_1). \tag{23}$$

Note that this system was already integrated at [52], in a rather different way:

$$\begin{aligned} z(x) &= z_0\left(\frac{3x-\sqrt{6}s}{3x_0-\sqrt{6}s}\right)^{\frac{2(s^2-1)}{2s^2-3}}\left(\frac{x^2-1}{x_0^2-1}\right)^{\frac{1}{6-4s^2}}\cdot \\ &\cdot\exp\left\{\frac{\sqrt{6}s\left[\tanh^{-1}(x)-\tanh^{-1}(x_0)\right]}{6s^2-9}\right\}, \end{aligned} \tag{24}$$

$$\begin{aligned} \tau-\tau_0 &= \ln\left(\frac{\sqrt{6}-3x_0}{\sqrt{6}-3x}\sqrt{\frac{1-x^2}{1-x_0^2}}\right) \\ &-\sqrt{\frac{2}{3}}(\tanh^{-1}(x)-\tanh^{-1}(x_0)). \end{aligned} \tag{25}$$

for $x_0-\sqrt{6}s\neq 0$, $x_0^2\neq 1$, $s\neq\pm\frac{\sqrt{6}}{2}$.

Let us now apply the ARS algorithm in the system (17), (18). For the first step of the algorithm, we

$$x(\tau)=x_0\tau^p\,,z(\tau)=z_0\tau^q \tag{26}$$

and we find

$$px_0\tau^{-1+p}-3x_0^3\tau^{3p}+3x_0\tau^p+\sqrt{6}sx_0^2\tau^{2p}-\sqrt{6}s=0, \tag{27}$$

$$(q\tau^{-1+q}+2t^q-3x_0\tau^{2p+q})z_0=0. \tag{28}$$

To find the leading-order term, we should balance at least two of the exponents of powers of τ. Note that if a singularity is to occur, parameters p, q should be negative[1], while in general, $\tau\to\tau-\tau_0$ where τ_0 denotes the position of the singularity, and without loss of generality, we consider $\tau_0=0$.

Hence, from (28), it follows that $-1+q=2p+q$ or $q=2p+q$. The second case $p=0$ provides $p=-\frac{1}{2}$ which we reject, while from the other case, we derive $\tau^{-\frac{3}{2}}$ where the leading-order terms have exponents which are cancelled when $x_0^2=\sqrt{\frac{q}{3}}$. Moreover, from expression (27), we find that the leading-order terms are those with power $\tau^{-\frac{3}{2}}$ and coefficient $-\frac{x_0}{2}(1+6x_0^2)$ where by demanding the latter to be zero, we find that $q=0$ or $q=-\frac{1}{2}$. We conclude that the leading-order behaviour is

$$x(\tau)=\pm\frac{i}{\sqrt{6}}\tau^{-\frac{1}{2}}\,,z(\tau)=z_0\tau^{-\frac{1}{2}}. \tag{29}$$

To find the resonance r, we substitute

$$x(\tau)=\pm\frac{i}{\sqrt{6}}\tau^{-\frac{1}{2}}+m\tau^{-\frac{1}{2}+r}\,,z(\tau)=z_0\tau^{-\frac{1}{2}}+v\tau^{-\frac{1}{2}+r} \tag{30}$$

in (17), (18) from where we get

$$\begin{aligned} 0 &= -\frac{x_0}{2}(1+6x_0^2)\tau^{-\frac{3}{2}}+\sqrt{6}sx_0^2\tau^{-1}+3x_0\tau^{-\frac{1}{2}}-\sqrt{6}s+ \\ &+\left(\frac{1}{2}(2r-1-18x_0^2)t^{-\frac{3}{2}+r}+2\sqrt{6}\,sx_0\tau^{-1+r}+3\tau^{-\frac{1}{2}+r}\right)m \\ &+O(m^2), \end{aligned} \tag{31}$$

$$\begin{aligned} 0 &= \left(-\frac{1+6x_0^2}{2}\tau^{-\frac{3}{2}}+2t^{-\frac{1}{2}}\right)+\left(-6x_0\tau^{-\frac{3}{2}+r}\right)m+ \\ &+\left(\left(r-\frac{1}{2}-3x_0^2\right)\tau^{-\frac{3}{2}+r}+2\tau^{-\frac{1}{2}+r}\right)n \\ &+O(m^2,n^2,mn), \end{aligned} \tag{32}$$

where $x_0=\pm\frac{i}{\sqrt{6}}$.

From the latter system, we define the two-dimensional matrix

$$A=\begin{pmatrix} \frac{1}{2}(2r-1-18x_0^2) & 0 \\ \left(-6x_0\tau^{-\frac{3}{2}+r}\right) & \left(r-\frac{1}{2}-3x_0^2\right) \end{pmatrix},\quad \det A=0 \tag{33}$$

whose requirement says that parameters m and n are arbitrary. We find $\det A=r(r+1)$ from where we calculate the two resonances to be

1 This is not absolute: in the modern treatment, the exponent p of the leading-order term can also be a positive fractional number.

$$r = -1 \quad \text{and} \quad r = 0. \tag{34}$$

The existence of $r = -1$ indicates that the singularity is movable while $r = 0$ indicates that the second integration constant is the coefficient parameter z_0 in (29). Since the two integration constants have been determined, we can say that the systems (17), (18) possess the Painlevé property. The algebraic solution is expressed by the Laurent expansions

$$\begin{aligned} x(\tau) &= \pm\frac{i}{\sqrt{6}}\tau^{-\frac{1}{2}} + \sum_{j=1} x_j \tau^{-\frac{1}{2}+j}, \\ z(\tau) &= z_0\tau^{-\frac{1}{2}} + \sum_{j=1} z_j \tau^{-\frac{1}{2}+j}, \end{aligned} \tag{35}$$

where the first coefficient terms for the $x_0 = +\frac{i}{\sqrt{6}}$ coefficient are determined to be

$$x_1 = \frac{1}{3}\sqrt{\frac{2}{3}}s \,,\; x_2 = -\frac{i}{6\sqrt{6}}\left(9 + 2s^2\right), \;\dots \tag{36}$$

$$z_1 = \frac{4}{3}sz_0 i \,,\; z_2 = -\frac{z_0}{6}\left(3 + 2s^2\right), \;\dots \tag{37}$$

Therefore, the Laurent expansions (35) pass the third step of the ARS algorithm and we conclude that the two-dimensional dynamical systems (17), (18) possess the Painlevé property. In the following section, we avoid the presentation of the calculations and we give directly the main results from our analysis.

4.1.2 Power-law potential

We continue with the study of the three-dimensional dynamical system (17)–(19) where $f(s) = -\frac{s^2}{2n}$. The leading-order terms are found to be

$$x(\tau) = {}_0\tau^{-\frac{1}{2}},\; z(\tau) = z_0\tau^{-\frac{1+n}{2}},\; s(\tau) = s_0\tau^{-\frac{1}{2}} \tag{38}$$

in which

$$x_0^2 = -\frac{1+n}{6}\,,\; s_0 = -\frac{n}{2\sqrt{6}x_0} \quad \text{and} \quad z_0 = \text{arbitrary}. \tag{39}$$

where $n \neq 0, -1, -3$. The resonances are the zeros of the polynomial equation $(1+2r)(r+1)r = 0$ which are

$$r = -1\,,\; r = 0 \text{ and } r = -\frac{1}{2}. \tag{40}$$

At this case, the generic solution is given by the following mixed Laurent expansions

$$x(\tau) = \sum_{j=1}\bar{x}_j\tau^{-\frac{1}{2}-j} + x_0\tau^{-\frac{1}{2}} + \sum_{j=1} x_j\tau^{-\frac{1}{2}+j} \tag{41}$$

$$z(\tau) = \sum_{j=1}\bar{z}_j\tau^{-\frac{1+n}{2}-j} + z_0\tau^{-\frac{1+n}{2}} + \sum_{j=1} z_j\tau^{-\frac{1+n}{2}+j} \tag{42}$$

$$s(\tau) = \sum_{j=1}\bar{s}_j\tau^{-\frac{1}{2}-j} + s_0\tau^{-\frac{1}{2}} + \sum_{j=1} s_j\tau^{-\frac{1}{2}+j} \tag{43}$$

where x_0, z_0 and s_0 given in (41).

To perform the consistency test, we select $n = 1$, and for $x_0 = \frac{i}{\sqrt{3}}$, we find that $x_j = 0\,, s_j = 0$ for every value of j and the third integration constant is parameter $\bar{s}_1$; recall that the other two integration constants are the position of the movable singularity τ_0 and the coefficient z_0. For negative values of n, the consistency test fails. Hence, the system possesses the Painlevé property only for $n > 0$.

We proceed with the second case of our analysis.

4.2 Case B: $k \neq 0, \Lambda = 0$

In the presence of curvature, the dimensionless field equations form the following three-dimensional dynamical system [23]

$$\frac{dx}{d\tau} = x\left(3x^2 - 2z^2 - 3\right) + \sqrt{6}s\left(1 - x^2 + z^2\right), \tag{44}$$

$$\frac{dz}{d\tau} = z\left[3x^2 - 2\left(z^2 + 1\right)\right], \tag{45}$$

$$\frac{ds}{d\tau} = -2\sqrt{6}xf(s), \tag{46}$$

which is defined on the phase space $\{(x,z,s) \in \mathbb{R}^3 : x^2 - z^2 \leq 1\}$.

4.2.1 Exponential potential

For the exponential potential and for the two-dimensional system (44), (45), the leading-order behaviour is found to be

$$x(\tau) = \pm\sqrt{\frac{4z_0 - 1}{6}}\tau^{-\frac{1}{2}},\ z(\tau) = z_0\tau^{-\frac{1}{2}} \tag{47}$$

in which z_0 is an arbitrary constant. The resonances are determined to be again $r = -1$ and $r = 0$ as in the spatially flat case. The solution is expressed in the right Laurent expansions

$$\begin{aligned} x(\tau) &= \pm\frac{\sqrt{4z_0 - 1}}{\sqrt{6}}\tau^{-\frac{1}{2}} + \sum_{j=1} x_j\tau^{-\frac{1}{2}+j}, \\ z(\tau) &= z_0\tau^{-\frac{1}{2}} + \sum_{j=1} z_j\tau^{-\frac{1}{2}+j}, \end{aligned} \tag{48}$$

where now the first coefficients are defined as

$$\begin{aligned} x_1 &= \frac{1}{3}\sqrt{\frac{2}{3}}s\left(1 + 2z_0^2\right)\left(1 + 8z_0^2\right), \\ z_1 &= \frac{4}{3}sz_0\left(1 + 2z_0^2\right)\sqrt{4z_0^2 - 1}, \dots \end{aligned} \tag{49}$$

We continue with the three-dimensional system defined by the power-law potential.

4.2.2 Power-law potential

For the power-law potential, the first step of the ARS algorithm for the dynamical system (44)–(46) provides the leading-order behaviour

$$\begin{aligned} x(\tau) &= \pm\sqrt{\frac{4z_0^2 - 1}{6}}\tau^{-\frac{1}{2}},\ z(\tau) = z_0\tau^{-\frac{1}{2}}, \\ s(\tau) &= -\frac{(n+1)p}{\sqrt{6}x_0}\tau^{-\frac{1}{2}},\ z_0 = \pm\frac{i}{\sqrt{2}}, \end{aligned} \tag{50}$$

for $n \neq 0, -1$.

As far as the resonances are concerned, they are derived from

$$r = -1,\ r = -\sqrt{6},\ r = \frac{1}{2} + \frac{\sqrt{6}}{n} \tag{51}$$

where we conclude that the given dynamical system does not possess the Painlevé property.

4.3 Case C: $k = 0, \Lambda \neq 0$

For the spatially flat background space and in the presence of the cosmological constant Λ, the field equations reduce to the following dimensionless system [23]

$$\frac{dx}{d\tau} = \sqrt{6}s\left(u^2 - x^2 + 1\right) + 3x\left(x^2 - 1\right), \tag{52}$$

$$\frac{du}{d\tau} = 3ux^2, \tag{53}$$

$$\frac{ds}{d\tau} = -2\sqrt{6}xf(s). \tag{54}$$

defined on the phase space $\{(x, u, s) \in \mathbb{R}^3 : x^2 - u^2 \leq 1\}$.

4.3.1 Exponential potential

The two-dimensional dynamical system (52), (53) with $s = const.$ passes the Painlevé test. The solution is given by the right Laurent expansions

$$\begin{aligned} x(\tau) &= \pm\frac{i}{\sqrt{6}}\tau^{-\frac{1}{2}} + \sum_{j=1} x_j\tau^{-\frac{1}{2}+j}, \\ u(\tau) &= u_0\tau^{-\frac{1}{2}} + \sum_{j=1} u_j\tau^{-\frac{1}{2}+j} \end{aligned} \tag{55}$$

where the constants of integrations are u_0 and the position of the movable singularity is τ_0.

By replacing (55) in the dynamical system, the first coefficients are calculated (for $x_0 = \frac{i}{\sqrt{6}}$)

$$\begin{aligned} x_1 &= \frac{\sqrt{6}s\left(1 + 6u_0^2\right)}{9}, \\ x_2 &= \frac{i}{6\sqrt{6}}\left(s^2\left(360u_0^4 + 48u_0^2 - 2\right) - 9\right), \dots \end{aligned} \tag{56}$$

$$\begin{aligned} u_1 &= i\frac{4}{3}su_0\left(1 + 6u_0^2\right), \\ u_2 &= -\frac{u_0}{6}\left(s^2\left(504u_0^4 + 96u_0^2 + 2\right) - 9\right), \dots \end{aligned} \tag{57}$$

4.3.2 Power-law potential

In the case of the power-law potential and for the three-dynamical system (52)–(54) with $n \neq -1, 0$, from the first step of the ARS algorithm, we determine the leading-order behaviour

$$x(\tau) = \pm\frac{i}{\sqrt{6}}\tau^{-\frac{1}{2}},\ u(\tau) = \pm\frac{i}{\sqrt{6}}\tau^{-\frac{1}{2}},\ s(\tau) = \frac{i}{2}\tau^{-\frac{1}{2}}. \tag{58}$$

The resonances are found to be the solutions of the polynomial equation $(r+1)(2r+1)(r-n) = 0$, which gives

$$r=-1\,,r=-\frac{1}{2}\,,r=n.$$

To write the Laurent expansions and perform the consistency test, the power n should be defined. We have studied various values of n, positives and negatives, integers and fractional, and we found that in the possible cases, the Laurent expansions fail at the consistency test. Hence, we conclude that the dynamical system (52)–(54) does not pass the Painlevé test.

4.4 Case D: $k\neq 0,\Lambda\neq 0$

The dimensionless dynamical system that describe the field equations in the last case of our consideration consists of the following four first-order differential equations [23]

$$\frac{dx}{d\tau}=\sqrt{6}s\left[-x^2+(u-z)^2+1\right]+x\left[3x^2+2(u-z)z-3\right], \tag{59}$$

$$\frac{dz}{d\tau}=z\left[3x^2+2(u-z)z-2\right], \tag{60}$$

$$\frac{du}{d\tau}=u\left[3x^2+2(u-z)z\right], \tag{61}$$

$$\frac{ds}{d\tau}=-2\sqrt{6}xf(s), \tag{62}$$

defined on the phase space $\{(x,z,u,s)\in\mathbb{R}^3:x^2-(u-kz)^2\leq 1\}$.

4.4.1 Exponential potential

For the three-dimensional dynamical system (59)–(61) where parameter s is constant, we find the leading-order behaviour

$$x(\tau)=x_0\tau^{-\frac{1}{2}}\,,z(\tau)=z_0\tau^{-\frac{1}{2}}\,,u(\tau)=u_0\tau^{-\frac{1}{2}}, \tag{63}$$

where x_0,z_0 are arbitrary constants and $u_0=\frac{4z_0^2-1-6x_0^2}{4z_0}$, with $4z_0^2-1-6x_0^2\neq 0$. Because there are already three arbitrary constants, including the position of the singularities, someone will expect to find two resonances with value zero.

Indeed, by replacing

$$\begin{aligned}&x(\tau)=x_0\tau^{-\frac{1}{2}}+m\tau^{-\frac{1}{2}+r}\,,z(\tau)=z_0\tau^{-\frac{1}{2}}+\nu\tau^{-\frac{1}{2}+r}\,,\\&u(\tau)=u_0\tau^{-\frac{1}{2}}+\kappa\tau^{-\frac{1}{2}+r},\end{aligned} \tag{64}$$

in the dynamical system (59)–(61), we derive that the resonances are the zeros of the polynomial equation $r^2(r+1)$. We conclude that the dynamical systems (59)–(61) with s constant possess the Painlevé property. The algebraic solution is given by the following Laurent expansions

$$\begin{aligned}&x(\tau)=x_0\tau^{-\frac{1}{2}}+\sum_{j=1}x_j\tau^{-\frac{1}{2}+j},\\&z(\tau)=z_0\tau^{-\frac{1}{2}}+\sum_{j=1}z_j\tau^{-\frac{1}{2}+j},\end{aligned} \tag{65}$$

$$u(\tau)=\frac{4z_0^2-1-6x_0^2}{4z_0}\tau^{-\frac{1}{2}}+\sum_{j=1}u_j\tau^{-\frac{1}{2}+j}. \tag{66}$$

4.4.2 Power-law potential

For the power-law potential, the leading-order behaviour is found to be

$$\begin{aligned}&x(\tau)=x_0\tau^{-\frac{1}{2}},\,z(\tau)=z_0\tau^{-\frac{1}{2}},\\&u(\tau)=u_0\tau^{-\frac{1}{2}},\,s(\tau)=s_0\tau^{-\frac{1}{2}},\end{aligned} \tag{67}$$

with

$$z_0=-\frac{6x_0^2+1}{4x_0},\,u_0=-\frac{2x_0^2+1}{4x_0},\,s_0=-\frac{n}{2\sqrt{6}x_0},$$

and x_0 arbitrary. The second step of the ARS algorithm provides the four resonances

$$r=-1\,,r=0,\,r=-\frac{1}{2}\text{ and }r=n. \tag{68}$$

In a similar way, with case C, we have to define power index n to perform the consistency test. We have performed the consistency test for various rational numbers of n, and we can conclude that the four-dimensional dynamical system (59)–(62) does not possess the Painlevé property.

5 Conclusion

In this work, we studied the integrability of the HL scalar field cosmology in an FLRW background space-time for the exponential and power-law potentials. We performed our study for the dimensionless dynamical system under H-normalization which is usually applied in the fixed-point analysis of gravitational dynamical systems. We categorized our study into cases of study based on the existence of the cosmological constant term Λ and whether the spatially curvature $k$vanishes or not. More specifically, the four cases of study are as follows: case A: $k = 0, \Lambda = 0$, case B: $k \neq 0, \Lambda = 0$, case C: $k = 0, \Lambda \neq 0$, and case D: $k \neq 0, \Lambda \neq 0$.

The main mathematical tool that we applied for the study of the integrability of the dimensionless field equations for the aforementioned cases of study is that of the singularity analysis. In particular, we examined if the given gravitational dynamical system possesses the Painlevé property which tell us that an explicit analytic integration can be performed where the solution is expressed in Laurent expansion.

As far as the power-law potential is concerned, we found that only case A provides an integrable system, which is in contrary to the exponential potential where the field equations always possess the Painlevé property. Our results are summarized in the following proposition: *"The gravitational field equations for the HL scalar field cosmology in a FLRW background (7)–(9) expressed in the dimensionless variables in the H –normalization (16) pass always the Painlevé test when $V(\phi) = V_0 e^{-\sigma\phi}$, and the equations can be explicitly integrated by Laurent expansions, where in all cases the resonances are $r = -1$ and $r = 0$, where the rank of $r = 0$ is greater of equal to one"*. At this point, we want to mention that by applying the same analysis for the field equations beyond the detailed balance condition [23], we found that the field equations do not possess the Painlevé property.

The results of this analysis complement the dynamical study of HL scalar field equations [23] and the analysis in the study by Leon and Saridakis [52]. Last but not the least, this work contributes to the subject of integrability of gravitational field equations in cosmological studies.

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