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Delta-Shock Solution to the Eulerian Droplet Model by Variable Substitution Method

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Abstract: By introducing a special kind of variable substitution, we skillfully solve the delta-shock and vacuum solutions to the one-dimensional Eulerian droplet model. The position, propagation speed, and strength of the delta shock wave are derived under the generalised Rankine–Hugoniot relation and entropy condition. Moreover, we show that the Riemann solution of the Eulerian droplet model converges to the corresponding the pressureless Euler system solution as the drag coefficient goes to zero.

Keywords: Delta-Shock Wave; Eulerian Droplet Model; Generalised Rankine–Hugoniot Condition; Vacuum.

1 Introduction

We are concerned with the following one-dimensional Eulerian droplet model:

$$\begin{cases} \alpha_t + (\alpha u)_x = 0, \\ (\alpha u)_t + (\alpha u^2)_x = \mu\alpha(u_a - u), \end{cases} \tag{1}$$

where α and u are the volume fraction and velocity of the particles (droplets), respectively, u_a is the velocity of the carrier fluid (air), and μ is the drag coefficient between the carrier fluid and the particles.

This model was proposed by Bourgault et al. [1] to compute the impingement of droplets on airfoils. In (1), the virtual mass force is neglected since the density of particles exceeds the air density by orders of magnitude. Other forces, such as lift force, gravity, and other interfacial effects, are also negligible when compared to the viscous drag force, though they may be important in some applications [1]. The Eulerian droplet model (1) corresponds to a dispersed phase subsystem in its simplest form: for instance, a multi-phase system for particles suspended in a carrier fluid. Currently, it is not only used to predict deposition patterns for high-speed external gas-particle flows [1, 2] but also works for low-speed gas-particle internal flows [3, 4].

It is clear that, as $\mu = 0$, the system (1) is nothing but the pressureless Euler system, or the so-called the zero-pressure flow model, which can be used to describe the motion of free particles sticking together under collision [5] or to explain the formation of large-scale structures in the Universe [6, 7]. Since the delta-shock wave and vacuum are found in the solutions, the pressureless Euler system has been widely studied by a large number of scholars with strong interest since 1994 (see [8–11] and the references therein). If $u_a = 0$ and $\mu = 1$, the system (1) becomes a non-homogeneous, pressureless Euler system, which is derived from the Cucker-Smale model and can be used to describe the flocking phenomenon. The well posedness of entropy solution to this pressureless Euler system with flocking dissipation was systemically studied in [12–14], etc.

To our knowledge, investigations on the Eulerian droplet model (1) have mostly focused on the numerical level [1] and the practical level [3, 4, 15–17]. Recently, the theoretical arguments for (1) were completed by Keita and Bourgault [18]. They solved the Riemann problem for the Eulerian droplet model by going through the solution of the Riemann problems for the inviscid Burgers equation with a source term and the subsystem, respectively. Particularly, for the delta-shock solution, the generalised Rankine–Hugoniot condition, which is in the form of ordinary differential equations, was proposed. Nevertheless, as was pointed out in [18], "In general, it might be hard to find the analytical solution of this ordinary differential equations". Thus, for the delta-shock solution of the Eulerian droplet model (1), with the help of the Cauchy-Peáno theorem, the Cauchy–Lipschitz existence theorem, and the Arzla–Ascoli theorem, they obtained the existence of a solution to the generalised Rankine–Hugoniot condition satisfying the Lax entropy condition. Finally, for the Riemann problem for (1), they were lucky to find an analytical solution for the generalised Rankine–Hugoniot condition. However, hydrodynamicists and engineers find it difficult to apply the theory of delta-shock waves conveniently in practice because of the lack of a

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recondite mathematical foundation. Therefore, one of the main objectives of this paper is to propose another effective and workable method to solve the Riemann problem for the Eulerian droplet model (1).

For this purpose, we consider the Riemann problem of (1) with the following initial data:

$$(\alpha, u)(0, x) = (\alpha_\pm, u_\pm), \quad \pm x > 0 \tag{2}$$

and aim at putting forward a new method to construct the Riemann solutions. Here, $\alpha_\pm$ and $u_\pm$ are constants. As was done in [18], just for the convenience of the study, in the system (1), μ is assumed to be a positive constant, u_a is also supposed to be constant, and $\alpha > 0$ is required.

Recall that, in 2016, Shen [19] studied the Riemann problem for the pressureless Euler system with the following source term:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \beta\rho, \end{cases} \tag{3}$$

where ρ and u denote the density and velocity of fluids, respectively, and β is a constant. The source term $\beta\rho$ in (3) is known as the Coulomb-like friction term, which was introduced by Savage and Hutter [20] to describe the granular flow behaviour. By skillfully performing the variable substitution $v(x, t) = u(x, t) - \beta t$, which was earlier used by Faccanoni and Mangeney [21] for the shallow water equations, they rewrote (3) in a conservative form that is linearly degenerate and whose solution contains a delta-shock wave in certain situations. Then they successfully constructed the explicit Riemann solutions of (3) with contact discontinuity, vacuum state, and delta-shock wave. Variable substitution is the common method used to discuss the balance law with the source term. For example, see [22] for the generalised zero-pressure flow model, [23] for the Suliciu relaxation system, and [24] for the perfect fluid model. Besides, one can also refer to [19] and [25–30] for the Chaplygin and generalised Chaplygin gas equations with friction.

Motivated by the works in [19] and [22–30], here we introduce a new state variable $v(x, t)$ and perform the following variable substitution for (1):

$$v(x, t) = u_a - (u_a - u(x, t))e^{\mu t}. \tag{4}$$

Then, system (1) is reduced to a modified system of conservation laws (see Section 2) and all the desired results on the Riemann problem (1) and (2) are obtained by the mathematical theory of hyperbolic conservation laws. Concretely, both the vacuum and delta-shock solutions for the modified system of conservation laws and (1) are obtained. For the delta-shock solution, we investigate and derive its position, propagation speed, and strength in detail under the generalised Rankine–Hugoniot relation and entropy condition. Interestingly, it is found that the generalised Rankine–Hugoniot relation proposed here can be reduced to a function equation. More precisely, it is a quadratic equation of one variable. Then, the existence and uniqueness of the delta-shock solution are solved completely under the entropy condition by studying the function equation, which is one of the outstanding advantages of our approach. Compared to the discussions in [18], this approach avoids using some analytical theorems that are complicated and not easy to understand for hydrodynamicists and engineers. Although we take a different method from [18], the results are just the same.

The novelty of this article comes from the following four aspects. First, compared with the method presented in [18], our method seems simpler and has the advantage of being easily workable. Second, the variable substitution (4) introduced in this paper is obviously different from that in previous works [19, 22–30]. Third, influenced by the source term, the Riemann solutions for (1) are not self-similar any more. All the characteristic curves, namely the curves of contact discontinuities and delta-shock waves, are bent into parabolic shapes. Fourth, we show that, as the drag coefficient $\mu \to 0^+$, the Riemann solution of the Eulerian droplet model (1) tends to the corresponding zero-pressure flow solution.

The rest of the paper is organised as follows. In Section 2, by the change of the state variable, we obtain a modified system of conservation laws with some new initial data, and constructively get its Riemann solutions containing delta-shock waves and vacuum states. In Section 3, we study the Riemann problem (1) and (2) and establish the existence and uniqueness of solutions under a suitable generalised Rankine–Hugoniot relation and entropy condition. Then, we rigorously prove that the delta-shock solution satisfies (1) in the sense of distributions. In addition, we show that the solutions of (1) and (2) converge to those of the pressureless Euler system with the same initial data when $\mu \to 0^+$. Finally, a brief conclusion is drawn in Section 4.

2 Riemann Problem to a Modified System of Conservation Laws

Under the transformation (4), the system (1) and initial data (2) are reduced into the following system of conservation laws:

$$\begin{cases} \alpha_t + (\alpha(u_a + (v - u_a)e^{-\mu t}))_x = 0, \\ (\alpha(v - u_a))_t + (\alpha(v - u_a)(u_a + (v - u_a)e^{-\mu t}))_x = 0 \end{cases} \tag{5}$$

with the new initial data

$$(\alpha, v)(0, x) = (\alpha_\pm, u_\pm), \quad \pm x > 0. \tag{6}$$

The system (5) can be rewritten in the quasi-linear form

$$\tilde{A}U_t + \tilde{B}U_x = 0,$$

where $U = (\alpha, v)^{\mathrm{T}}$, and

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ v - u_a & \alpha \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} u_a + (v - u_a)e^{-\mu t} & \alpha e^{-\mu t} \\ (v - u_a)(u_a + (v - u_a)e^{-\mu t}) & \alpha u_a + 2\alpha(v - u_a)e^{-\mu t} \end{pmatrix}.$$

From this form, we can calculate the eigenvalue of system (5) as

$$\lambda = u_a + (v - u_a)e^{-\mu t}$$

and the right eigenvector as $\vec{r} = (1, 0)^{\mathrm{T}}$. Since $\nabla\lambda \cdot \vec{r} = 0$, the system (5) is linear degenerate, and the elementary waves involve only contact discontinuities.

Noting that the parameter t appears in the flux functions of (5), it is quite different from the classical hyperbolic systems of conservation laws. However, the Rankine–Hugoniot conditions can also be derived via a standard technique as usual. For the bounded discontinuity $x = x(t)$, we have the following Rankine–Hugoniot condition:

$$\begin{cases} \sigma(t)[\alpha] = [\alpha(u_a + (v - u_a)e^{-\mu t})], \\ \sigma(t)[\alpha(v - u_a)] = [\alpha(v - u_a)(u_a + (v - u_a)e^{-\mu t})], \end{cases} \tag{7}$$

where $[p] = p_+ - p_-$ denotes the jump of p across the discontinuity, and $\sigma(t) = x'(t)$.

If $\sigma(t) \neq 0$, from (7), one has

$$\begin{aligned} &[\alpha(u_a + (v - u_a)e^{-\mu t})] \cdot [\alpha(v - u_a)] \\ &\quad = [\alpha] \cdot [\alpha(v - u_a)(u_a + (v - u_a)e^{-\mu t})]. \end{aligned} \tag{8}$$

Simplifying (8) yields

$$\alpha_-\alpha_+ e^{-\mu t}(v_+ - v_-)^2 = 0. \tag{9}$$

Therefore, we can connect the two non-vacuum constant states (α_-, v_-) and (α_+, v_+) by a contact discontinuity J if and only if $v_- = v_+$. At this moment, the propagation speed of J is

$$\sigma(t) = u_a + (v_- - u_a)e^{-\mu t} = u_a + (v_+ - u_a)e^{-\mu t}.$$

The solutions to (5) and (6) can be constructed in two cases.

When $u_- < u_+$, one can construct the Riemann solution of (5) and (6) by constant, vacuum, and contact discontinuity, which is

$$(\alpha, v)(t, x) = \begin{cases} (\alpha_-, u_-), & -\infty < x < x_1(t), \\ \left(0, u_a + \dfrac{\mu(x - u_a t)}{1 - e^{-\mu t}}\right), & x_1(t) \le x \le x_2(t), \\ (\alpha_+, u_+), & x_2(t) < x < +\infty, \end{cases} \tag{10}$$

in which

$$x_1(t) = u_a t + \frac{(u_- - u_a)(1 - e^{-\mu t})}{\mu},$$

$$x_2(t) = u_a t + \frac{(u_+ - u_a)(1 - e^{-\mu t})}{\mu}.$$

When $u_- > u_+$, the characteristic lines from the x-axis will overlap in the domain $\Omega = \{(t, x) | u_a t + \frac{(u_+ - u_a)(1 - e^{-\mu t})}{\mu} \le x \le u_a t + \frac{(u_- - u_a)(1 - e^{-\mu t})}{\mu}, t > 0\}$, so the singularity of solutions must occur. We affirm that there exist no solutions in the bounded variation space. Actually, for a smooth solution, (5) is equivalent to

$$\begin{cases} \alpha_t + (\alpha(u_a + (v - u_a)e^{-\mu t}))_x = 0, \\ v_t + (u_a + (v - u_a)e^{-\mu t})v_x = 0. \end{cases} \tag{11}$$

Then we consider (11) with sufficiently smooth initial data $(\alpha, v)(0, x) = (\alpha_0(x), v_0(x))$, in which $v_0'(x) < 0$.

The characteristic equations of the system (5) are

$$\frac{dx}{dt} = u_a + (v - u_a)e^{-\mu t},$$

$$\frac{dv}{dt} = 0, \quad \frac{d\alpha}{dt} = -\alpha e^{-\mu t} v_x. \tag{12}$$

For any given point $(0, b)$ on the x-axis, the characteristic curve passing through this point is

$$x(t) = b + u_a t - \frac{1}{\mu}(v_0(b) - u_a)(e^{-\mu t} - 1), \tag{13}$$

on which v takes the constant value $v = v_0(b)$.

Differentiating the second equation of (11) with respect to x gives

$$v_{tx} + (u_a + (v - u_a)\mathrm{e}^{-\mu t})v_{xx} + v_x^2 \mathrm{e}^{-\mu t} = 0, \tag{14}$$

that is

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -v_x^2 \mathrm{e}^{-\mu t}, \tag{15}$$

which is a standard type of the Riccati equation. As a result, along the characteristic curve (13), we can obtain

$$v_x = \frac{\mu v_0'(b)}{\mu + (1 - \mathrm{e}^{-\mu t})v_0'(b)}, \tag{16}$$

which when combined with the third equation of (12) yields

$$\alpha = \frac{\mu \alpha_0(b)}{\mu + (1 - \mathrm{e}^{-\mu t})v_0'(b)}. \tag{17}$$

Noting that $\mu > 0$ and $v_0'(b) < 0$, there exists b such that $\mu < -v_0'(b)$. Let $T^* = \inf\limits_{\mu < -v_0'(b)} \left\{ -\frac{\ln(1 + \frac{\mu}{v_0'(b)})}{\mu} \right\}$, so along the characteristic curve (13), one can easily observe that

$$\lim_{t \to T^*} (\alpha, v_x) \to (\infty, -\infty). \tag{18}$$

This implies that α and v_x must blow up simultaneously at a finite time, which leads to unboundedness and discontinuities in the solution.

We will construct the solution using a delta-shock wave for this case. In order to define the delta-shock solution, we give the following two definitions:

Definition 1: A two-dimensional weighted delta function $w(s)\delta_S$ supported on a smooth curve S parameterised as $t = t(s)$, $x = x(s) (a \le s \le b)$ can be defined by

$$\langle w(t(s))\delta_S, \varphi(t, x)\rangle = \int_a^b w(t(s))\varphi(t(s), x(s))\mathrm{d}s \tag{19}$$

for all test functions $\varphi \in C_0^\infty([0, +\infty) \times (-\infty, +\infty))$.

Definition 2: A pair (α, v) is called a delta-shock solution of (5) in the sense of distributions if there exist a smooth curve S and a function $w(t)$ such that α and v are represented in the following form:

$$\alpha = \bar{\alpha}(t, x) + w(t)\delta_S, \qquad v = \bar{v}(t, x), \tag{20}$$

$\bar{\alpha}, \bar{v} \in L^\infty(R \times [0, +\infty); R)$, $w(t) \in C^1(S)$, $v|_S = v_\delta$, and they satisfy

$$\begin{aligned} &\langle \alpha, \varphi_t\rangle + \langle \alpha(u_a + (v - u_a)\mathrm{e}^{-\mu t}),\ \varphi_x\rangle = 0, \\ &\langle \alpha(v - u_a),\ \ \varphi_t\rangle \\ &\quad + \langle \alpha(v - u_a)(u_a + (v - u_a)\mathrm{e}^{-\mu t}),\ \varphi_x\rangle = 0 \end{aligned} \tag{21}$$

for all test functions $\varphi \in C_0^\infty([0, +\infty) \times (-\infty, +\infty))$, where

$$\langle \alpha, \varphi\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \bar{\alpha}\varphi \mathrm{d}x\mathrm{d}t + \langle w(t)\delta_S, \varphi\rangle,$$

$$\langle \alpha(v - u_a), \varphi\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \bar{\alpha}(\bar{v} - u_a)\varphi \mathrm{d}x\mathrm{d}t + \langle w(t)(v_\delta - u_a)\delta_S, \varphi\rangle.$$

With these two definitions, for the Riemann problem (5) and (6) we seek the solution in the form

$$(\alpha, v)(t, x) = \begin{cases} (\alpha_-, v_-), & x < x(t), \\ (w(t)\delta(x - x(t)), v_\delta), & x = x(t), \\ (\alpha_+, v_+), & x > x(t), \end{cases} \tag{22}$$

where $x(t) \in C^1$ is the position of the discontinuity, $w(t)$ denotes the strength of the delta-shock wave, and v_δ is the corresponding value of v on the delta-shock wave curve.

We conclude that if (22) satisfies the generalised Rankine–Hugoniot condition

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \sigma(t) = u_a + (v_\delta - u_a)\mathrm{e}^{-\mu t}, \\ \dfrac{\mathrm{d}w(t)}{\mathrm{d}t} = \sigma(t)[\alpha] - [\alpha(u_a + (v - u_a)\mathrm{e}^{-\mu t})], \\ \dfrac{\mathrm{d}w(t)(v_\delta - u_a)}{\mathrm{d}t} = \sigma(t)[\alpha(v - u_a)] \\ \qquad\qquad - [\alpha(v - u_a)(u_a + (v - u_a)\mathrm{e}^{-\mu t})], \end{cases} \tag{23}$$

then the solution $(\alpha, v)(t, x)$ defined in (22) satisfies (5) in the sense of distributions.

Now we check that the solution obtained from solving (23) satisfies Definition 2 in the sense of distributions. The proof is similar to that of Theorem 4.2 in [18], so we only deliver the second equality in (21) for completeness. Actually, for any test function $\varphi \in C_0^\infty([0,+\infty)\times(-\infty,+\infty))$, if (23) holds, we have

$$\begin{aligned}
I &= \langle \alpha(v-u_a), \varphi_t\rangle \\
&\quad + \langle \alpha(v-u_a)(u_a+(v-u_a)\mathrm{e}^{-\mu t}, \varphi_x\rangle \\
&= \int_0^{+\infty}\int_{-\infty}^{x(t)} (\alpha_-(u_--u_a)\varphi)_t \\
&\quad + (\alpha_-(u_--u_a)(u_a+(u_--u_a)\mathrm{e}^{-\mu t}\varphi)_x \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^{+\infty}\int_{x(t)}^{+\infty} (\alpha_+(u_+-u_a)\varphi)_t \\
&\quad + (\alpha_+(u_+-u_a)(u_a+(u_+-u_a)\mathrm{e}^{-\mu t}\varphi)_x \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^{+\infty} w(t)(v_\delta-u_a)(\varphi_t+(u_a+(v_\delta-u_a)\mathrm{e}^{-\mu t})\varphi_x)\mathrm{d}t.
\end{aligned}$$

By using Green's formula and integrating by parts, one has

$$\begin{aligned}
I &= -\oint -\alpha_-(u_--u_a)(u_a+(u_--u_a)\mathrm{e}^{-\mu t})\varphi\mathrm{d}t \\
&\qquad + \alpha_-(u_--u_a)\varphi\mathrm{d}x \\
&\qquad + \oint -\alpha_+(u_+-u_a)(u_a+(u_+-u_a)\mathrm{e}^{-\mu t})\varphi\mathrm{d}t \\
&\qquad + \alpha_+(u_+-u_a)\varphi\mathrm{d}x \\
&\qquad - \int_0^{+\infty} \varphi \frac{\mathrm{d}w(t)(v_\delta-u_a)}{\mathrm{d}t}\mathrm{d}t \\
&= \int_0^{+\infty} \varphi\Big(\sigma(t)[\alpha(v-u_a)] - [\alpha(v-u_a)(u_a \\
&\qquad +(v-u_a)\mathrm{e}^{-\mu t})] - \frac{\mathrm{d}w(t)(v_\delta-u_a)}{\mathrm{d}t}\Big)\mathrm{d}t \\
&= 0.
\end{aligned}$$

It means that the second equation of (21) holds. The first one of (21) can be proved similarly.

Moreover, the entropy condition

$$\lambda(\alpha_+, v_+) < \sigma(t) < \lambda(\alpha_-, v_-), \tag{24}$$

that is

$$u_+ < v_\delta < u_- \tag{25}$$

should be assumed to guarantee the uniqueness.

Now we solve the Riemann problem (5) and (6) when $u_- > u_+$. Under the entropy condition (24), the Riemann problem is reduced to solving the ordinary differential equations (23) with the initial data $x(0)=0$, $w(0)=0$.

From (23), we can compute that

$$\begin{aligned}
\frac{\mathrm{d}w(t)}{\mathrm{d}t} &= (v_\delta-u_a)(\alpha_+-\alpha_-)\mathrm{e}^{-\mu t} \\
&\quad - (\alpha_+(u_+-u_a)\mathrm{e}^{-\mu t} - \alpha_-(u_--u_a)\mathrm{e}^{-\mu t})
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&(v_\delta-u_a)\frac{\mathrm{d}w(t)}{\mathrm{d}t} \\
&\quad = (v_\delta-u_a)\mathrm{e}^{-\mu t}(\alpha_+(u_+-u_a)-\alpha_-(u_--u_a)) \\
&\qquad - (\alpha_+(u_+-u_a)^2\mathrm{e}^{-\mu t} - \alpha_-(u_--u_a)^2\mathrm{e}^{-\mu t}).
\end{aligned} \tag{27}$$

Then, we immediately obtain from (26) and (27)

$$\begin{aligned}
&(v_\delta-u_a)^2(\alpha_+-\alpha_-) \\
&\quad - 2(v_\delta-u_a)(\alpha_+(u_+-u_a)-\alpha_-(u_--u_a)) \\
&\quad + \alpha_+(u_+-u_a)^2 - \alpha_-(u_--u_a)^2 = 0.
\end{aligned} \tag{28}$$

If $\alpha_+-\alpha_-=0$, from (28) we get

$$\begin{cases}
v_\delta = \dfrac{u_++u_-}{2}, \\
x(t) = u_a t + \dfrac{1}{\mu}\Big(\dfrac{u_++u_-}{2}-u_a\Big)(1-\mathrm{e}^{-\mu t}), \\
w(t) = \dfrac{1}{\mu}\alpha_+(u_--u_+)(1-\mathrm{e}^{-\mu t}).
\end{cases} \tag{29}$$

If $\alpha_+-\alpha_-\neq 0$, by solving the quadratic equation of $v_\delta-u_a$ in (28), one can obtain

$$v_\delta = \frac{\alpha_+u_+-\alpha_-u_- \pm \sqrt{\alpha_-\alpha_+(u_--u_+)^2}}{\alpha_+-\alpha_-}. \tag{30}$$

Under the entropy condition (24), we have

$$\begin{aligned}
v_\delta &= \frac{\alpha_+u_+-\alpha_-u_- + \sqrt{\alpha_-\alpha_+(u_--u_+)^2}}{\alpha_+-\alpha_-} \\
&= \frac{\sqrt{\alpha_+}u_+ + \sqrt{\alpha_-}u_-}{\sqrt{\alpha_+}+\sqrt{\alpha_-}}.
\end{aligned} \tag{31}$$

Owing to (23), one has

$$\begin{cases}
x(t) = u_a t + \dfrac{1}{\mu}\Big(\dfrac{\sqrt{\alpha_+}u_+ + \sqrt{\alpha_-}u_-}{\sqrt{\alpha_+}+\sqrt{\alpha_-}} - u_a\Big)(1-\mathrm{e}^{-\mu t}), \\
w(t) = \dfrac{1}{\mu}\sqrt{\alpha_-\alpha_+}(u_--u_+)(1-\mathrm{e}^{-\mu t}).
\end{cases} \tag{32}$$

So far, we have obtained the solution of the Riemann problem (6) for the system of conservation laws (5).

3 Riemann Solutions for the Eulerian Droplet Model (1)

In this section, we pay attention to the Riemann solutions for the original Eulerian droplet model (1). Based on the results in Section 2, when $u_- < u_+$, as shown in Figure 1, the Riemann solution of (1) and (2) can be expressed as

$$(\alpha, u)(t, x) = \begin{cases} (\alpha_-, u_a + (u_- - u_a)\mathrm{e}^{-\mu t}), & -\infty < x < x_1(t), \\ \left(0, u_a + \dfrac{\mu(x - u_a t)}{\mathrm{e}^{\mu t} - 1}\right), & x_1(t) \le x \le x_2(t), \\ (\alpha_+, u_a + (u_+ - u_a)\mathrm{e}^{-\mu t}), & x_2(t) < x < +\infty, \end{cases} \tag{33}$$

in which

$$x_1(t) = u_a t + \frac{(u_- - u_a)(1 - \mathrm{e}^{-\mu t})}{\mu},$$

$$x_2(t) = u_a t + \frac{(u_+ - u_a)(1 - \mathrm{e}^{-\mu t})}{\mu},$$

which is a vacuum-state solution.

For convenience, here we only give the structure of the Riemann solution in the (x, t)-plane for the cases where $u_-, u_+, u_a > 0$. At this moment, the two curves J_1 and J_2 are monotonously increasing.

When $u_- > u_+$, the delta-shock solution for (1) and (2) in the sense of distributions can be introduced as follows:

Definition 3: A pair (α, u) is called a delta-shock solution of (1) in the sense of distributions if there exist a smooth curve S and a function $w(t)$ such that α and u are represented in the following form:

$$\alpha = \bar{\alpha}(t, x) + w(t)\delta_S, \qquad u = \bar{u}(t, x), \tag{34}$$

$\bar{\alpha}, \bar{u} \in L^\infty(R \times (0, +\infty); R)$, $w(t) \in C^1(S)$, $u|_S = u_\delta(t)$ and they satisfy

$$\begin{aligned} &\langle \alpha, \ \varphi_t \rangle + \langle \alpha u, \ \varphi_x \rangle = 0, \\ &\langle \alpha u, \ \varphi_t \rangle + \langle \alpha u^2, \ \varphi_x \rangle = -\langle \mu\alpha(u_a - u), \ \varphi \rangle \end{aligned} \tag{35}$$

for all test functions $\varphi \in C_0^\infty([0, +\infty) \times (-\infty, +\infty))$, where

$$\langle \alpha, \varphi \rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \bar{\alpha}\varphi \mathrm{d}x\mathrm{d}t + \langle w(t)\delta_S, \varphi \rangle,$$

$$\langle \alpha u, \varphi \rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \bar{\alpha}\bar{u}\varphi \mathrm{d}x\mathrm{d}t + \langle w(t)u_\delta(t)\delta_S, \varphi \rangle.$$

Then, we give the solution to the Riemann problem (1) and (2) in the following form:

$$(\alpha, u)(t, x) = \begin{cases} (\alpha_-, u_a + (u_- - u_a)\mathrm{e}^{-\mu t}), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\alpha_+, u_a + (u_+ - u_a)\mathrm{e}^{-\mu t}), & x > x(t), \end{cases} \tag{36}$$

where $u_\delta(t)$ is the value of u on the delta-shock wave curve $x = x(t)$, and $(u_\delta(t) - u_a)\mathrm{e}^{\mu t}$ is assumed to be a constant based on the result in Section 2.

The delta-shock solution of the Riemann problem (1) and (2) defined above should satisfy the following generalised Rankine–Hugoniot relation:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \sigma(t) = u_\delta(t), \\ \dfrac{\mathrm{d}w(t)}{\mathrm{d}t} = \sigma(t)[\alpha] - [\alpha u], \\ \dfrac{\mathrm{d}w(t)u_\delta(t)}{\mathrm{d}t} = \sigma(t)[\alpha u] - [\alpha u^2] + \mu w(t)(u_a - u_\delta), \end{cases} \tag{37}$$

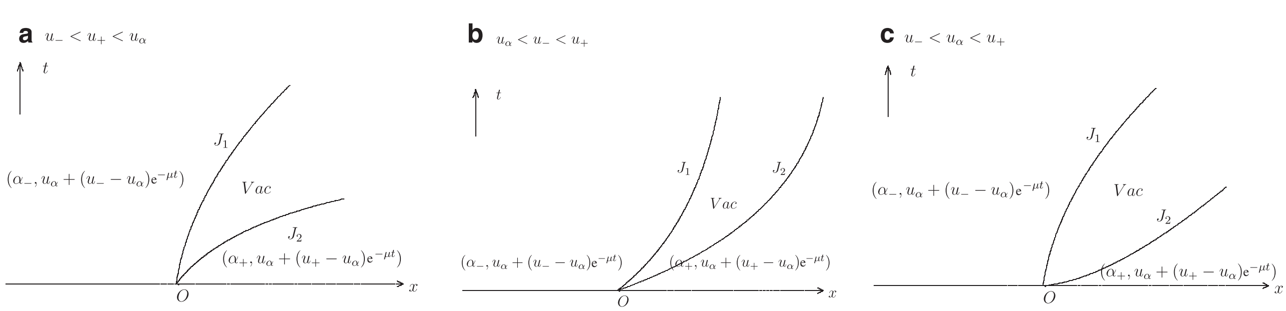

Figure 1: Riemann solution of (1) and (2) in the (x, t)-plane when $u_- < u_+$.

in which the discontinuity becomes

$$\begin{cases} [\alpha u] = \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t}) \\ \qquad - \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t}), \\ [\alpha u^2] = \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})^2 \\ \qquad - \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})^2. \end{cases} \quad (38)$$

To ensure the uniqueness of the solution, we add the entropy condition

$$u_a + (u_+ - u_a)\mathrm{e}^{-\mu t} < u_\delta(t) < u_a + (u_- - u_a)\mathrm{e}^{-\mu t}. \quad (39)$$

In this case, the solution can be described by the following theorem:

Theorem 1: *Suppose that $u_- > u_+$ and $\alpha_- \neq \alpha_+$; then the delta-shock solution of the Riemann problem (1) and (2) can be shown to be*

$$(\alpha, u)(t, x) = \begin{cases} (\alpha_-, u_a + (u_- - u_a)e^{-\mu t}), & x < x(t), \\ (w(t)\delta(x - x(t)), u_a + (v_\delta - u_a)e^{-\mu t}), & x = x(t), \\ (\alpha_+, u_a + (u_+ - u_a)e^{-\mu t}), & x > x(t), \end{cases} \quad (40)$$

where $x(t) = u_a t + \frac{1}{\mu}(v_\delta - u_a)(1 - e^{-\mu t})$ and $w(t) = A(1 - e^{-\mu t})$ denote the position and strength of the delta-shock wave, respectively, in which

$$v_\delta = \frac{\sqrt{\alpha_+}u_+ + \sqrt{\alpha_-}u_-}{\sqrt{\alpha_+} + \sqrt{\alpha_-}}, \quad A = \frac{1}{\mu}\sqrt{\alpha_-\alpha_+}(u_- - u_+). \quad (41)$$

Proof. From the second equation of (37), we have

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = (\sigma(t) - u_a)(\alpha_+ - \alpha_-) - (\alpha_+(u_+ - u_a)\mathrm{e}^{-\mu t} - \alpha_-(u_- - u_a)\mathrm{e}^{-\mu t}). \quad (42)$$

Noticing that $(u_\delta(t) - u_a)\mathrm{e}^{\mu t}$ is a constant, one can get

$$\frac{\mathrm{d}w(t)u_\delta(t)}{\mathrm{d}t} = u_\delta(t)w'(t) + \mu w(t)(u_a - u_\delta(t)). \quad (43)$$

Then, we have

$$\begin{aligned} u_\delta(t)w'(t) = {} & \sigma(t)(\alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t}) \\ & - \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})) \\ & - \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})^2 \\ & + \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})^2. \end{aligned} \quad (44)$$

Combining (42) and (44) leads to

$$\begin{aligned} & (\alpha_+ - \alpha_-)(u_\delta - u_a)^2 - 2(u_\delta - u_a)\mathrm{e}^{-\mu t} \\ & (\alpha_+(u_+ - u_a) - \alpha_-(u_- - u_a)) \\ & + \alpha_+(u_+ - u_a)^2\mathrm{e}^{-2\mu t} - \alpha_-(u_- - u_a)^2\mathrm{e}^{-2\mu t} = 0, \end{aligned} \quad (45)$$

from which one has

$$u_\delta(t) - u_a = \frac{\mathrm{e}^{-\mu t}(\sqrt{\alpha_+}v_+ \pm \sqrt{\alpha_-}v_-)}{\sqrt{\alpha_+} + \sqrt{\alpha_-}} - \mathrm{e}^{-\mu t}u_a. \quad (46)$$

By the entropy condition (39), we choose

$$u_\delta(t) = u_a + \mathrm{e}^{-\mu t}\left(\frac{\sqrt{\alpha_+}v_+ + \sqrt{\alpha_-}v_-}{\sqrt{\alpha_+} + \sqrt{\alpha_-}} - u_a\right) \quad (47)$$

as the admissible solution. Furthermore, we can compute that

$$x(t) = u_a t - \frac{1}{\mu}\left(\frac{\sqrt{\alpha_+}u_+ + \sqrt{\alpha_-}u_-}{\sqrt{\alpha_+} + \sqrt{\alpha_-}} - u_a\right)(\mathrm{e}^{-\mu t} - 1) \quad (48)$$

and

$$w(t) = \frac{1}{\mu}\sqrt{\alpha_-\alpha_+}(u_- - u_+)(1 - \mathrm{e}^{-\mu t}). \quad (49)$$

As was done in [19] and [22], in what follows, we need to check that the delta-shock solution satisfies Definition 3 in the sense of distributions. We only prove the second equation in (35), because the proof for the other one is similar. In fact, one can deduce that

$$\begin{aligned} I = {} & \langle \alpha u, \ \varphi_t \rangle + \langle \alpha u^2, \ \varphi_x \rangle \\ = {} & \int_0^{+\infty}\int_{-\infty}^{x(t)} \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})\varphi_t(x, t) \\ & + \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})^2\varphi_x(x, t)\mathrm{d}x\mathrm{d}t \\ & + \int_0^{+\infty}\int_{x(t)}^{+\infty} \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})\varphi_t(x, t) \\ & + \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})^2\varphi_x(x, t)\mathrm{d}x\mathrm{d}t \\ & + \int_0^{+\infty} A(1 - \mathrm{e}^{-\mu t})(u_a + (v_\delta - u_a)\mathrm{e}^{-\mu t})(\varphi_t(x(t), t) \\ & + (u_a + (v_\delta - u_a)\mathrm{e}^{-\mu t})\varphi_x(x(t), t))\mathrm{d}t. \end{aligned}$$

When $u_a, v_\delta > 0$ (see Fig. 2a,b) or $u_a, v_\delta < 0$, the curve of the delta-shock wave is always monotonous with

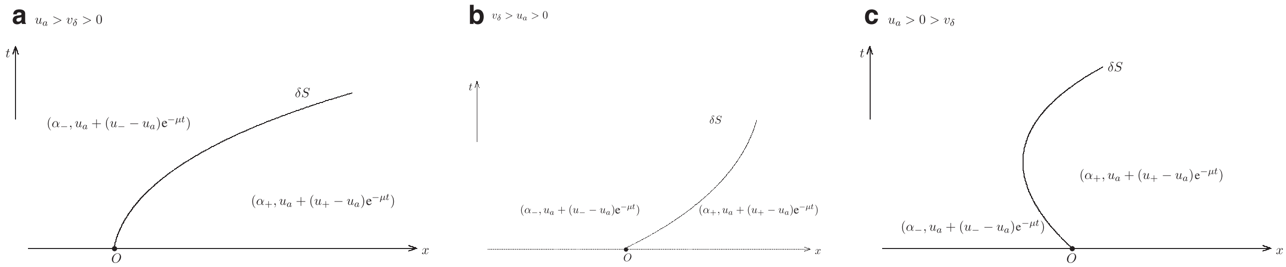

Figure 2: Delta-shock solution of (1) and (2) in the (x, t)-plane when $u_- > u_+$, where v_δ is determined by (29) if $\alpha_- = \alpha_+$ or (31) if $\alpha_- \neq \alpha_+$.

respect to the time t, so $x = x(t)$ exists as an inverse function for $t \geqslant 0$. However, when $u_a < 0 < v_\delta$ or $v_\delta < 0 < u_a$ (see Fig. 2c), there exists a point $t^\star = \frac{1}{\mu}\ln\frac{u_a - v_\delta}{u_a}$, such that the inverse function of $x(t)$ should be solved for $t \leq t^\star$ and $t > t^\star$, respectively. At this moment, the integral region $(t, x) \in [0, +\infty) \times (-\infty, +\infty)$ should be divided into two parts $(t, x) \in [0, t^\star] \times (-\infty, +\infty)$ and $(t, x) \in (t^\star, +\infty) \times (-\infty, +\infty)$. As a result, we need to check that (35) holds in the two parts.

Here, without loss of generality, we suppose that $u_a > 0$ and $v_\delta > 0$. Then, we can solve $t = t(x)$ from $x(t)$ uniquely. By exchanging the integral orders, one has

$$
\begin{aligned}
I &= \langle \alpha u, \ \varphi_t \rangle + \langle \alpha u^2, \ \varphi_x \rangle \\
&= \int_0^{+\infty}\int_{t(x)}^{+\infty} \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})\varphi_t(x, t)\,\mathrm{d}t\mathrm{d}x \\
&\quad + \int_0^{+\infty}\int_{-\infty}^{x(t)} \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})^2 \varphi_x(x, t)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^{+\infty}\int_0^{t(x)} \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})\varphi_t(x, t)\,\mathrm{d}t\mathrm{d}x \\
&\quad + \int_0^{+\infty}\int_{x(t)}^{+\infty} \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})^2 \varphi_x(x, t)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^{+\infty} A(1 - \mathrm{e}^{-\mu t})(u_a + (v_\delta - u_a)\mathrm{e}^{-\mu t})\,\mathrm{d}\varphi(x(t), t) \\
&= \int_0^{+\infty} (\alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t(x)}) \\
&\quad - \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t(x)}))\varphi(x, t(x))\,\mathrm{d}x \\
&\quad + \int_0^{+\infty} (-\alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})^2 \\
&\quad + \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})^2)\varphi(x(t), t)\,\mathrm{d}t \\
&\quad + \int_0^{+\infty}\int_{t(x)}^{+\infty} \alpha_-\mu(u_- - u_a)\mathrm{e}^{-\mu t}\varphi(x, t)\,\mathrm{d}t\mathrm{d}x \\
&\quad + \int_0^{+\infty}\int_0^{t(x)} \alpha_+\mu(u_+ - u_a)\mathrm{e}^{-\mu t}\varphi(x, t)\,\mathrm{d}t\mathrm{d}x \\
&\quad - \int_0^{+\infty} (\mu A\mathrm{e}^{-\mu t}(u_a + (v_\delta - u_a)\mathrm{e}^{-\mu t}) \\
&\quad - \mu A\mathrm{e}^{-\mu t}(1 - \mathrm{e}^{-\mu t})(v_\delta - u_a))\varphi(x(t), t))\,\mathrm{d}t.
\end{aligned}
$$

By using the change of variables and exchanging the ordering of the integrals again, we have

$$
\begin{aligned}
I = &\int_0^{+\infty} B(t)\varphi(x(t), t)\,\mathrm{d}t \\
&+ \int_0^{+\infty}\int_{-\infty}^{x(t)} \alpha_-\mu(u_- - u_a)\mathrm{e}^{-\mu t}\varphi(x, t)\,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^{+\infty}\int_{x(t)}^{+\infty} \alpha_+\mu(u_+ - u_a)\mathrm{e}^{-\mu t}\varphi(x, t)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{50}
$$

in which

$$
\begin{aligned}
B(t) \ = \ &(\alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t}) \\
&- \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})) \\
&\cdot (u_a + (v_\delta - u_a)\mathrm{e}^{-\mu t}) \\
&- \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})^2 \\
&+ \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})^2 \\
&- \mu A\mathrm{e}^{-\mu t}(u_a + (v_\delta - u_a)\mathrm{e}^{-\mu t}) \\
&+ \mu A\mathrm{e}^{-\mu t}(1 - \mathrm{e}^{-\mu t})(v_\delta - u_a)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_+(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t})(v_\delta - u_+)\mathrm{e}^{-\mu t} \\
&\quad + \alpha_-(u_a + (u_- - u_a)\mathrm{e}^{-\mu t})(u_- - v_\delta)\mathrm{e}^{-\mu t} \\
&\quad - \mu A\mathrm{e}^{-\mu t}(u_a + (v_\delta - u_a)\mathrm{e}^{-\mu t}) \\
&\quad + \mu A\mathrm{e}^{-\mu t}(1 - \mathrm{e}^{-\mu t})(v_\delta - u_a).
\end{aligned}
$$

Substituting v_δ into $B(t)$ yields

$$
\begin{aligned}
B(t) &= \sqrt{\alpha_+}\sqrt{\alpha_-}\mathrm{e}^{-\mu t}(u_- - u_+) \\
&\quad \frac{\sqrt{\alpha_+}}{\sqrt{\alpha_+} + \sqrt{\alpha_-}}(u_a + (u_+ - u_a)\mathrm{e}^{-\mu t}) \\
&\quad + \sqrt{\alpha_+}\sqrt{\alpha_-}\mathrm{e}^{-\mu t}(u_- - u_+) \\
&\quad \frac{\sqrt{\alpha_-}}{\sqrt{\alpha_+} + \sqrt{\alpha_-}}(u_a + (u_- - u_a)\mathrm{e}^{-\mu t}) \\
&\quad - \sqrt{\alpha_+}\sqrt{\alpha_-}\mathrm{e}^{-\mu t}(u_- - u_+) \\
&\quad \left(u_a + \mathrm{e}^{-\mu t}\frac{\sqrt{\alpha_+}u_+ + \sqrt{\alpha_-}u_- - u_a(\sqrt{\alpha_+} + \sqrt{\alpha_-})}{\sqrt{\alpha_+} + \sqrt{\alpha_-}}\right) \\
&\quad + \mu A\mathrm{e}^{-\mu t}(1 - \mathrm{e}^{-\mu t})(v_\delta - u_a) \\
&= \mu A\mathrm{e}^{-\mu t}(1 - \mathrm{e}^{-\mu t})(v_\delta - u_a),
\end{aligned}
$$

from which we immediately get

$$
B(t) = \mu w(t)(v_\delta - u_a)\mathrm{e}^{-\mu t}. \tag{51}
$$

Combining (50) and (51), one can see that the second equation in (35) holds in the sense of distributions. The proof is complete. □

Remark 1: Particularly, when $u_- > u_+$ and $\alpha_- = \alpha_-$, the delta-shock solution of (1) and (2) can be shown to be

$$
\begin{cases}
u_\delta(t) = u_a + \mathrm{e}^{-\mu t}\Big(\dfrac{u_- + u_+}{2} - u_a\Big), \\
x(t) = u_a t + \dfrac{1}{\mu}\Big(\dfrac{u_- + u_+}{2} - u_a\Big)(1 - \mathrm{e}^{-\mu t}), \\
w(t) = \dfrac{1}{\mu}\alpha_-(u_- - u_+)(1 - \mathrm{e}^{-\mu t}).
\end{cases} \tag{52}
$$

Remark 2: It is easily observed that the solutions obtained here are completely coincident with those in [18]. In other words, the transformation (4) is an effective way to obtain the solutions of the Euler droplet model (1). It also shows that the method used in [18] is not the only one to solve the Riemann problem of (1).

Remark 3: In view of $\lim\limits_{\mu \to 0^+} \frac{1-\mathrm{e}^{-\mu t}}{\mu} = t$, from (47)–(49) and (52), one can find that the Riemann solutions of the Euler droplet model (1) converge to those of the pressureless Euler system studied in [10] and others. Nevertheless, it should be noticed that, under the influence of external forces, all the characteristic curves, namely curves of contact discontinuities and delta-shock waves of the Euler droplet model (1), are bent into parabolic shapes.

4 Conclusions

The Riemann problem for the one-dimensional Eulerian droplet model was solved. Compared to the discussions in [18], we introduced a special kind of non-linear variable substitution to rewrite the original Eulerian droplet model into a conservative one. Then, the delta-shock and vacuum solutions for both systems were constructively obtained. This method adopted here is easy to understand and workable. Furthermore, the variable substitution introduced here is different from that in [19, 22–30], etc. It is also shown that, because of the effect of the external force, the contact discontinuities and delta-shock waves are curved, and then the solutions are not self-similar. Finally, this work can provide a fundamental method of exploration for the study of the Riemann problem of the Eulerian droplet model with the initial data containing the Dirac measure, namely the Radon measure initial data problem. We leave it for a future study.

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