

Sanjay Kumar Pandey* and Jagdish Prasad Maurya

A General Viscous Model for Some Aspects of Tropical Cyclonic Winds

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Abstract: A previous investigation found that the existence of double exponential terms is a reason for rapid intensification of cyclonic winds with the assumption of a linearised form of viscosity. Here, we consider viscosity of general type and still get similar terms. A perturbation technique is applied to the solution. The domain of analysis is split into two regions: an inner one that experiences updraft, and an outer one that possesses no vertical component of velocity but does have azimuthal and radial components. It is observed that the radial pressure difference between an arbitrary radial distance and the point of the maximum wind diminishes with height, time, and Reynolds number. The azimuthal velocity, close to the ground, in region 1 increases fast with time, but its dependence diminishes at a height a little above the ground. At a considerable height, time ceases to be a factor, and further, above that, trends reverse. Perturbation terms behave almost identically with the terms without perturbation. The significance of their contribution depends on the magnitude of the Reynolds number and, hence, the viscosity. Trends for region 2 are qualitatively similar to those in region 1 but differ quantitatively. It is also observed that the central pressure drop decreases with time.

Keywords: Exact Solution; Hurricane; Intensification; Tropical Cyclone; Viscous Model.

1 Introduction

Cyclone is a weather-related phenomenon. It develops on the surface of tropical oceans due to atmospheric pressure disturbances. It is of huge size and is known by various

names in different geographical zones. In the Atlantic and the Eastern North Pacific, it is known as hurricane with wind speeds exceeding 33 ms^{-1} , whereas it is referred to as typhoon in the Western North Pacific and tropical cyclone in the Central and Eastern Pacific basins. It has enormous impacts on the society. Further details are given in subsection 2.1 under the subheading “The Physical Model.”

The scientific community is looking for answers to many fundamental questions related to cyclones. Unexpected variations in the wind direction from the bottom to the top of the hurricane, radial growth in the wind angular momentum in the boundary layer, unpredicted effect of ocean spray, enormous rise in the upper boundary layer temperature, etc., are a few of them. Outflow is a more recent aspect that is under study and worth mentioning [1–3]. Genesis and maturing are still underinvestigated.

Analytical solutions of the equations governing various atmospheric vortices such as dust devils, tornadoes, cyclones, etc., are always a challenging task due to the complex formulations of the atmospheric vortices. However, in this paper, we intend to focus only on cyclonic vortex. Here, we seek to answer some of the questions related to cyclones by providing an exact analytical solution of the equations governing the vortex motion of cyclones, of course idealized in some or the other way.

One of the most important idealized considerations for seeking a solution of a cyclone model is related to Coriolis force. The radial pressure gradient per unit mass balances approximately the sum of the centrifugal force and the Coriolis force [4, 5]. This balance refers to gradient wind balance. Aircraft measurements also endorse the validity of gradient wind balance in the lower to middle troposphere in tropical cyclones [6, 7]. Observations and modelling, however, revealed that at the upper level, such storms can deviate from balances but are valid everywhere else in the hurricane vortex [1, 5, 8]. Cohen et al. [1, 9] also refer to recent analyses of weather research and forecasting simulations to proclaim that the flow around high-pressure regions above the top of hurricanes at 15-km altitude violates the gradient wind balance. Bryan and Rotunno [10] also, while analyzing and discussing the maximum intensity of a tropical cyclone, insisted on more complete models to account for gradient wind imbalance.

*Corresponding author: Sanjay Kumar Pandey, Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi 221005, India, E-mail: skpandey.apm@itbhu.ac.in

Jagdish Prasad Maurya: Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi 221005, India; and Department of Mathematics, Jorhat Engineering College, Jorhat 785007, India

Analytical solutions for a full time-dependent cyclonic vortex have not yet been achieved. The current analytical solutions extend the Rankine combined vortex model, which is the first and foremost popular model for the azimuthal velocity [11]. In this model, the azimuthal velocity depends only on the radial coordinate, while the radial and vertical velocity components are considered zero. This was used to explain observed tangential wind and to deduce pressure distribution in whirlwinds [12–20]. The experimental observations match this formulation to a large extent. The main problem with the Rankine vortex model is that it has a sharp peak at the wall of the core. Later, Burgers [21] and Rott [22] independently obtained a solution for the viscous vortex motion of the steady incompressible flow embedded in a radially inward stagnation point flow over a plane boundary with all non-zero velocity components, which was an improvement over the Rankine model. Both the models are applicable for single-celled vortex flow. Some unrealistic aspects of the Burgers–Rott vortex, however, are that the radial and vertical velocity components increase linearly to infinity.

Kieu and Zhang [23] presented an analytical model of tropical cyclones to investigate rapid intensification from the perspective of rotational growth and central pressure falls. They considered a simplified version of the primitive equations with a linear first-order frictional term and

separated the entire domain into two regions, viz., region 1, is of a fixed radius, which is referred to as the radius of maximum wind, and contains the maximum exponential wind growth, and region 2, which lies outside region 1, has no vertical component of velocity. The velocities for the two regions were derived separately before stepping into further discussion. If we compare their division of regions with the existing nomenclature used for hurricane, we find that region 1 comprises the eye and the eye wall, while region 2 lies outside the eye wall (Fig. 1). The governing equations were solved for the axisymmetric flow by prescribing a time-dependent vertical velocity with exponential growth in region 1 but without growth in region 2.

For rapid intensification of tropical cyclones, they eventually held the double exponential term responsible, which exists in the azimuthal velocity of region 1 but not in the azimuthal velocity of region 2. They also added that most of the existing theoretical idealized models are based on the balanced vortex model in association with the Sawyer–Eliassen transverse circulation equation [24–32]. Models, based on balanced vortex, combine the hydrostatic and gradient wind balances with radial momentum and thermodynamic equations. The authors concluded in their study that the relative difference between the exact solution and gradient wind approximation at the initial

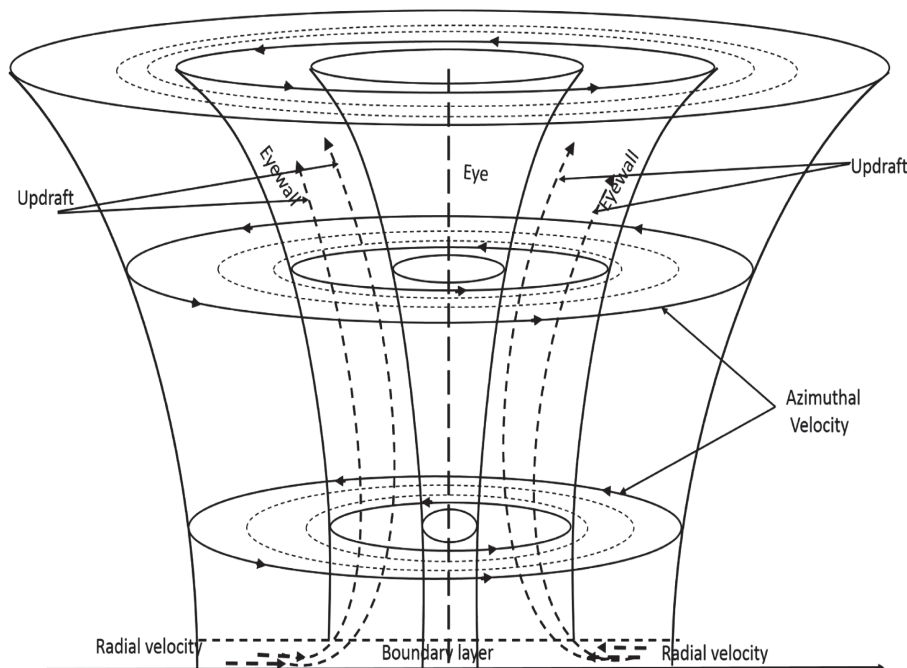


Figure 1: Schematic diagram of the physical model of hurricane. The narrow columnar geometry is the relatively motionless warm region called eye placed symmetrically with respect to the vertical axis. The eye is surrounded by the pure updraft zone, within the vertical layer called the eye wall, which witnesses a heavy downpour. The outermost zone contains violently rotating wind with extremely moist air at the lower altitudes, and its radial inflow downdraft supplies moisture for updraft through the boundary layer.

time approaches zero irrespective of the magnitude of the difference. They further added that the gradient balance relation is more easily satisfied in region 2.

Our curiosity about whether a double exponential term is responsible for intensification of tropical cyclones even when the viscosity is of the general type led us, unlike them, to consider a much more general viscous term. However, in order to get reliable inferences, we are required to compromise with several realities of cyclones such as gradient wind imbalance at high altitudes [1, 9], nor we claim to achieve an analytical solution for a full time-dependent cyclonic vortex. Hence, this makes it an idealized cyclone but helps to investigate particular aspects. Further, a perturbation technique is required to solve because of the additional consideration of general viscous effects.

2 Mathematical Formulation of the Problem

2.1 The Physical Model

A cyclone is a three-dimensional atmospheric phenomenon that combines a primary (i.e. horizontal) circulation with a secondary (i.e. in, up and out) circulation (see Fig. 1). In the inner cyclone, there is a region referred to as the eye, which is a vertical column of radius 20 km and is wrapped within another region known as the eye wall with an external radius 30–50 km. Above the boundary layer of thickness 2–3 km, the radius of the external eye wall changes with height [32]. The eye wall is made up of strongly revolving winds together with radial inflows, which are maximum at the bottom. The vertical velocity, which is ideally contained within the eye wall, is weaker than the radial and azimuthal winds and further weakens outside the eye wall. In the outermost part of radius 400–600 km, surrounding the eye wall, the relative rotation of the cyclone declines to zero. The wind along the radial direction points inward at the bottom and outward at the top of the hurricane. The entire vortex is vertically layered into the bottom hurricane boundary layer and the upper adiabatic layer with the total cyclonic height 20–30 kms [32].

The radial inflow in the hurricane boundary layer is the essential wind for the genesis of hurricane vortices, and when a hurricane matures, the azimuthal component of the wind velocity is much stronger than the radial and vertical components. These cyclones have been extensively investigated during the last seven decades. However, an exact analytical model is not available for

the circulation of the real cyclonic vortex and nor even for pressure distribution within and outside the vortex. Most studies considered either the linear form of the inflow radial component of velocity or neglected it in comparison with the azimuthal component.

As an attempt to investigate some aspects of the dynamics of an ideal atmospheric vortex, we try to present new analytical solutions of general viscous incompressible equations governing cyclones. As discussed in the Introduction section that gradient wind imbalance at high altitudes is an established fact, this needs to be taken care of while trying to model a fully developed cyclone. This will be unfair to hide that despite this fact, due to several complications, we shall consider it a balance for the entire height of cyclone in the model.

2.2 Mathematical Model of Cyclonic Vortex

A mathematical formulation of the dynamics of cyclone, whose structure was described above, is given below.

We opt for the cylindrical polar coordinates (r, θ, z) for the analysis, r, θ, z , respectively, being the radial, angular, and vertical coordinates. The vertical coordinate is a log-pressure coordinate defined as $z = -H \ln(p/p_s)$ with respect to the reference pressure p_s with p being the pressure and H the scale height of the cyclone. It is observed that a rotating fluid mass in the form of a mature vortex does not seem to differ much practically at different angles during its rotation about the vertical axis. Thus, it is reasonable to consider the flow as symmetric about the axis of rotation. This removes all terms where the angular coordinate θ is involved. Consequently, the three-dimensional model of atmospheric flows under the elastic and axisymmetric approximation [2, 33] of an incompressible Newtonian viscous fluid may be given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} - fv = -\frac{1}{\rho} \frac{\partial p}{\partial r} + F_u, \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} + fu = F_v, \quad (2)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + b + F_w, \quad (3)$$

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0, \quad (4)$$

$$\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial r} + N^2 w = Q, \quad (5)$$

where u, v , and w are the three wind components, respectively, in the r, θ, z directions, p is the pressure, $b = g(T - T_{\text{ref}})/T_{\text{ref}}$ is the buoyancy with $T_{\text{ref}}(z)$ being the

reference temperature of the undisturbed atmosphere, F_u , F_v , F_w are frictional forces, f is the Coriolis parameter, and N^2 is the Brunt–Vaisala frequency.

Characteristic quantities are required to make various parameters dimensionless. The radius, symbolized as a , of region 1 consists of the eye and the eye wall, and could be considered an appropriate characteristic length. The other candidate is the radius of the core region, at the periphery of which the azimuthal wind velocity is maximum. However, it is relatively more flexible. Moreover, the azimuthal wind velocity v_c at the core is very likely a characteristic velocity for non-dimensionalisation. Therefore, the system of equations (1)–(4), along with the boundary conditions are non-dimensionalised in terms of the following dimensionless parameters superscripted with $*$:

$$\begin{aligned} t^* &= \frac{t}{T}, \quad r^* = \frac{r}{a}, \quad z^* = \frac{z}{a}, \quad u^* = \frac{u}{v_c}, \\ v^* &= \frac{v}{v_c}, \quad w^* = \frac{w}{v_c}, \quad p^* = \frac{p}{P}, \end{aligned} \quad (6)$$

where T and P will be defined below. Further, we consider b constant for analytical solutions.

The dimensionless form of equations (1)–(4), by dropping asterisks, are transformed to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} - Sv = -\frac{\partial p}{\partial r} + F_u, \quad (7)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} + Su = F_v, \quad (8)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + b + F_w, \quad (9)$$

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{\partial w}{\partial z} - \frac{w}{H_1} = 0. \quad (10)$$

Here,

$$S = \frac{af}{v_c}, \quad H_1 = \frac{H}{a}, \quad P = \rho v_c^2, \quad T = \frac{a}{v_c}.$$

The classical Rankine combined vortex is the solution of the steady two-dimensional Euler equation governing an ideal inviscid fluid. The velocity field in this solution is purely azimuthal and is given, in cylindrical polar coordinates, by $q = [0, v(r), 0]$ where $v(r) = \zeta r/2$, $r \leq a$, and $v(r) = \zeta a^2/2r$, $r > a$. This flow consists of the circular inner region ($r \leq a$) of radius a moving with constant vorticity ζ surrounded by irrotational flow everywhere outside the inner region.

We seek to apply, here, the method of separation of variables and, hence, assume the azimuthal velocity as the product of $g(r)$, a function exclusive of r , and

$F(z, t)$, a function only of z and t , but not of r , i.e. $v(r, z, t) = g(r) \cdot F(z, t)$. Following the Rankine combined vortex model, we further assume the radial variation $g(r)$ of the azimuthal velocity in the non-dimensional form $g(r) = \begin{cases} \zeta r/2 & \text{for } 0 \leq r \leq a \\ \zeta a^2/2r & \text{for } r > a \end{cases}$, where a is the radius of the maximum wind. Further, in order to obtain a more general z -dependent solution, we consider a piecewise solution in r with the azimuthal velocity taken as

$$v(r, z, t) = \begin{cases} rF_1(z, t) & \text{for } 0 \leq r \leq 1 \\ r^{-1}F_2(z, t) & \text{for } r > 1 \end{cases}, \quad \text{and } 0 \leq z \leq H. \quad (11)$$

Kieu and Zhang [23] followed the theoretical framework established by Charney and Eliassen [25], Yanai [34], and Ooyama [26] for the secondary circulation growth in terms of an instability mode.

We consider a similar diabatically induced ascending velocity

$$\begin{aligned} w(r, z, t) &= W_0 \sin(\lambda z) e^{\beta t}, \\ \text{for } r \leq 1 \text{ and } w(r, z, t) &= 0 \text{ for } r > 1, \end{aligned} \quad (12)$$

where W_0 , β , λ , are constants non-dimensionalised, respectively, by v_c , a/v_c , a . This is to be noted that there cannot be any sort of discontinuity at $r = 1$, as $w(r, z, t) = 0$ at $r \geq 1$, z being an integral multiple of π/λ at $r = 1$. Kieu and Zhang [23] argue that β , the growth rate of the vertical flow, is affected by friction and surface heat fluxes. Hence, it is a function of the axial coordinate z and the buoyancy frequency. However, it may be approximated to a constant as it is dimensionally of the order of 10^{-6} – 10^{-5} s^{-1} (Ooyama [26]).

The dimensionless governing equations (7)–(9) are now constrained by the following boundary conditions:

$$u(r, z, t)|_{r=0} = 0, \quad w(r, z, t)|_{z=0} = 0, \quad v(r, z, t)|_{r=0} = 0. \quad (13)$$

Our purpose is to examine how the primary circulations evolve with time if the secondary circulations grow exponentially as often assumed in previous investigations. Of course, such an exponential growth will no longer be valid as tropical cyclones reach their maximum intensity, so the vertical profile given in equation (12) should be limited to the rapid intensifying period only [23].

2.3 Analytical Solutions

In this section, we present an analytical solution for time-dependent viscous incompressible flows in the cyclonic

vortex governed by the azimuthal momentum equation (8). In equation (8), $\bar{F}_v$ represents the nondimensional viscous term in the azimuthal direction. Most of the earlier researchers considered $\bar{F}_v$ either negligibly small or of linear form [23]. However, we assume the general form

$$\bar{F}_v = \frac{1}{Re} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right).$$

Now, we solve the following equation for azimuthal velocity separately for the two regions (1) $0 \leq r \leq 1$ and (2) $r > 1$ by supplying radial and vertical velocities

$$\begin{aligned} \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} + Su \\ = \frac{1}{Re} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right). \end{aligned} \quad (14)$$

2.3.1 Solution for Region 1

Assuming $w_1(r, z, t) = W(z)e^{\beta t}$, where $W(z) = W_0 \sin(\lambda z)$ as in equation (12), the radial wind in region 1 with vanishing radial velocity at $r = 0$ may be obtained, from the continuity equation (10), as

$$u_1(r, z, t) = \frac{rW_0}{2} \left[\frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right] e^{\beta t}. \quad (15)$$

Using equations (12) and (15) into equation (14), we obtain the following equation for the tangential wind in region 1:

$$\begin{aligned} \frac{\partial v_1}{\partial t} + W_0 \sin(\lambda z) \frac{\partial v_1}{\partial z} e^{\beta t} \\ + \frac{rW_0}{2} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} \left(\frac{\partial v_1}{\partial r} + \frac{v_1}{r} + S \right) e^{\beta t} \\ = \frac{1}{Re} \left(\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r} \frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} + \frac{\partial^2 v_1}{\partial z^2} \right). \end{aligned} \quad (16)$$

The only solution separable in the radial and axial-temporal coordinates of equation (16) can be of the form $v_1(r, z, t) = rF_1(z, t)$. Hence, using this form in equation (16), we get

$$\begin{aligned} \frac{\partial F_1}{\partial t} + \left[W_0 \sin(\lambda z) \frac{\partial F_1}{\partial z} \right. \\ \left. + \frac{W_0}{2} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} (2F_1 + S) \right] e^{\beta t} \\ = \frac{1}{Re} \frac{\partial^2 F_1}{\partial z^2}. \end{aligned} \quad (17)$$

In terms of $G(z, t) (= F_1(z, t) + S/2)$, equation (17) is transformed to

$$\begin{aligned} \frac{\partial G}{\partial t} + W_0 \left[\sin(\lambda z) \frac{\partial G}{\partial z} \right. \\ \left. + \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} G \right] e^{\beta t} = \epsilon \frac{\partial^2 G}{\partial z^2}. \end{aligned} \quad (18)$$

We suppose that the vortex Reynolds number Re is very large or equivalently that $\epsilon = Re^{-1} \ll 1$. In view of this, we seek an asymptotic solution of equation (18) of the form

$$G(z, t) = G_0(z, t) + G_1(z, t) + \epsilon^2 G_2(z, t) + \dots \quad (19)$$

Assuming that this series expansion converges for higher orders, we can obtain a solution for $G(z, t)$ for various orders of ϵ by substituting equation (19) into equation (18). However, in order to avoid unnecessary derivations in view of $\epsilon \ll 1$, we present the solution only up to the first order of ϵ . Thus, equations of the zeroth and the first order of ϵ are

$$\begin{aligned} \epsilon^0 : \frac{\partial G_0}{\partial t} + W_0 \left[\sin(\lambda z) \frac{\partial G_0}{\partial z} \right. \\ \left. + \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} G_0 \right] e^{\beta t} = 0, \end{aligned} \quad (20)$$

$$\begin{aligned} \epsilon^1 : \frac{\partial G_1}{\partial t} + W_0 \left[\sin(\lambda z) \frac{\partial G_1}{\partial z} \right. \\ \left. + \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} G_1 \right] e^{\beta t} = \frac{\partial^2 G_0}{\partial z^2}. \end{aligned} \quad (21)$$

A possible solution of equation (20) is of the form

$$G_0(z, t) = G'_0(z) \exp(\mu e^{\beta t}), \quad (22)$$

where μ is an arbitrary positive dimensionless number. Substituting equation (22) into equation (20), followed by integration, the explicit form of $G_0(z, t)$ may be given by

$$\frac{dG'_0}{dz} = - \left[\left\{ \frac{1}{H_1} - \lambda \cot(\lambda z) \right\} + \frac{\beta \mu}{W_0} \operatorname{cosec}(\lambda z) \right] G'_0, \quad (23)$$

$$\begin{aligned} G_0(z, t) \\ = K e^{-(z/H_1)} \{ \sin(\lambda z) \}^{(1-\beta\mu/\lambda W_0)} \left\{ 2 \cos\left(\frac{\lambda z}{2}\right) \right\}^{2(\beta\mu/\lambda W_0)} \\ \exp(\mu e^{\beta t}), \end{aligned} \quad (24)$$

where K is an integration constant that determines the initial strength of the vortex.

The zeroth-order tangential wind velocity is given by

$$v_{10}(r, z, t) = r \left[K e^{-(z/H_1)} \{ \sin(\lambda z) \}^{(1-\beta\mu/\lambda W_0)} \left\{ 2 \cos\left(\frac{\lambda z}{2}\right) \right\}^{2(\beta\mu/\lambda W_0)} \exp(\mu e^{\beta t}) - \frac{S}{2} \right], \quad (25)$$

which has an infinite number of possible solutions depending on the values of μ . However, the requirements for the regularity of (25) at $z = 0$ impose a strong restriction in the range of μ . Thus, for a regular solution of equation (25), we have $\beta\mu/\lambda W_0 \leq 1$. Following Keiu and Zhang [23], we take $\beta\mu/\lambda W_0 = 1 - \delta$, where $0 \leq \delta \leq 1$. This substitution transforms equation (25) to

$$G_0(z, t) = 2 K e^{-(z/H_1)} \left\{ \sin\left(\frac{\lambda z}{2}\right) \right\}^\delta \left\{ \cos\left(\frac{\lambda z}{2}\right) \right\}^{2-\delta} \exp\left\{ \frac{\lambda W_0}{\beta} (1 - \delta) e^{\beta t} \right\}, \quad (26)$$

and the corresponding zeroth-order azimuthal wind velocity, in terms of δ , may be given by

$$v_{10}(r, z, t) = r \left[2 K e^{-(z/H_1)} \left\{ \sin\left(\frac{\lambda z}{2}\right) \right\}^\delta \left\{ \cos\left(\frac{\lambda z}{2}\right) \right\}^{2-\delta} \exp\left\{ \frac{\lambda W_0}{\beta} (1 - \delta) e^{\beta t} \right\} - \frac{S}{2} \right]. \quad (27)$$

Now, we seek to solve equation (21) by presenting it in the form

$$e^{-\beta t} \frac{\partial G_1}{\partial t} + W_0 \sin(\lambda z) \frac{\partial G_1}{\partial z} + W_0 \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} G_1 = e^{-\beta t} \frac{\partial^2 G_0}{\partial z^2}. \quad (28)$$

Assuming $G_1(z, t) = \Gamma(z, t) G_0(z, t)$ in equation (28) and using equation (20), we obtain

$$e^{-\beta t} \frac{\partial \Gamma}{\partial t} + W_0 \sin(\lambda z) \frac{\partial \Gamma}{\partial z} = N(z, t), \quad (29)$$

where

$$N(z, t) = \frac{e^{-\beta t} f(z, t)}{G_0(z, t)} = e^{-\beta t} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \cot^2\left(\frac{\lambda z}{2}\right) + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \tan^2\left(\frac{\lambda z}{2}\right) - \frac{\delta \lambda}{H_1} \cot\left(\frac{\lambda z}{2}\right) + \frac{(2 - \delta)\lambda}{H_1} \tan\left(\frac{\lambda z}{2}\right) + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \right]. \quad (30)$$

The solution of equation (29) is given by (the detailed analysis is given in Appendix B),

$$\begin{aligned} \Gamma(z, t) = & \frac{1}{\beta} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \exp\left(2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot^2\left(\frac{\lambda z}{2}\right) I_{11} \right. \\ & + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \exp\left(-2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan^2\left(\frac{\lambda z}{2}\right) I_{22} \\ & - \frac{\delta \lambda}{H_1} \exp\left(\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot\left(\frac{\lambda z}{2}\right) I_{33} \\ & + \frac{(2 - \delta)\lambda}{H_1} \exp\left(-\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan\left(\frac{\lambda z}{2}\right) I_{44} \\ & \left. + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \ln\left(\frac{e^{\beta t}}{\beta}\right) \right] + \emptyset(A), \end{aligned} \quad (31)$$

where $\emptyset(A)$ is an arbitrary function of $A = \frac{e^{\beta t}}{\beta} - \frac{1}{\lambda W_0} \ln\left\{ \tan\left(\frac{\lambda z}{2}\right) \right\}$, which is, itself, a constant and is given by equation (B4) in Appendix B. This may be noted that A has a singularity at $z = 0$, and hence, $\emptyset(A)$ also has a singularity at $z = 0$, which may be eliminated by considering $\emptyset(A) = \gamma$ a constant. Thus, the first-order solution of $G_1(z, t)$ may be given by

$$\begin{aligned} G_1(z, t) = & 2 K e^{-(z/H_1)} \left\{ \sin\left(\frac{\lambda z}{2}\right) \right\}^\delta \left\{ \cos\left(\frac{\lambda z}{2}\right) \right\}^{2-\delta} \\ & \left[\frac{1}{\beta} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \exp\left(2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot^2\left(\frac{\lambda z}{2}\right) I_{11} \right. \right. \\ & + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \exp\left(-2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan^2\left(\frac{\lambda z}{2}\right) I_{22} \\ & - \frac{\delta \lambda}{H_1} \exp\left(\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot\left(\frac{\lambda z}{2}\right) I_{33} \\ & + \frac{(2 - \delta)\lambda}{H_1} \exp\left(-\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan\left(\frac{\lambda z}{2}\right) I_{44} \\ & \left. + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \ln\left(\frac{e^{\beta t}}{\beta}\right) \right] + \gamma \\ & \exp\left\{ \frac{\lambda W_0}{\beta} (1 - \delta) e^{\beta t} \right\}. \end{aligned} \quad (32)$$

With the first-order viscous correction, the tangential wind velocity, in region 1, is now given by

$$v_1(r, z, t) = r[G_0(z, t)\{1 + \Gamma(z, t)\} - S/2] = r\Psi(z, t), \quad (33)$$

where

$$\begin{aligned} \Psi(z, t) = & 2 K e^{-(z/H_1)} \left\{ \sin\left(\frac{\lambda z}{2}\right) \right\}^{\delta} \left\{ \cos\left(\frac{\lambda z}{2}\right) \right\}^{2-\delta} \\ & \left[1 + \frac{1}{Re} \left\{ \frac{1}{\beta} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \exp\left(2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot^2\left(\frac{\lambda z}{2}\right) I_{11} \right. \right. \right. \\ & + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \exp\left(-2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan^2\left(\frac{\lambda z}{2}\right) I_{22} \\ & - \frac{\delta \lambda}{H_1} \exp\left(\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot\left(\frac{\lambda z}{2}\right) I_{33} \\ & + \frac{(2 - \delta)\lambda}{H_1} \exp\left(-\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan\left(\frac{\lambda z}{2}\right) I_{44} \\ & \left. \left. + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \ln\left(\frac{e^{\beta t}}{\beta}\right) + \theta(A) \right] \right\} - S/2. \end{aligned} \quad (34)$$

The radial pressure gradient is obtained using equations (12), (15), and (33) into equation (7), as

$$\begin{aligned} \frac{\partial p_1}{\partial r} = & -r \left[\frac{W_0 e^{\beta t}}{4} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} \right. \\ & \left\{ 2\beta + W_0 \left(\frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right) e^{\beta t} + \frac{2\lambda^2}{Re} \right\} \\ & + \frac{W_0^2 \lambda}{2} \sin(\lambda z) \left\{ \frac{1}{H_1} \cos(\lambda z) + \lambda \sin(\lambda z) \right\} e^{2\beta t} \\ & \left. - \Psi^2 - S\Psi \right]. \end{aligned} \quad (35)$$

Integration of equation (35), from r to 1, gives

$$\begin{aligned} p_1(1, z, t) - p_1(r, z, t) = & \frac{(1 - r^2)}{2} \left[-\frac{W_0 e^{\beta t}}{4} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} \right. \\ & \left\{ 2\beta + W_0 \left(\frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right) e^{\beta t} + \frac{2\lambda^2}{Re} \right\} \\ & - \frac{W_0^2 \lambda}{2} \sin(\lambda z) \left\{ \frac{1}{H_1} \cos(\lambda z) + \lambda \sin(\lambda z) \right\} e^{2\beta t} \\ & \left. + \Psi^2 + S\Psi \right]. \end{aligned} \quad (36)$$

Using equations (12) and (15) into equation (9), the axial pressure gradient is obtained, as

$$\begin{aligned} \frac{\partial p_1}{\partial z} = & -\frac{W_0 \lambda W_0}{2} \sin(2\lambda z) e^{2\beta t} \\ & + \bar{b} - W_0 \left(\beta + \frac{\lambda^2}{Re} \right) \sin(\lambda z) e^{\beta t}. \end{aligned} \quad (37)$$

Integrating equation (37) from the initial level z_0 to z , we get

$$\begin{aligned} p_1(r, z, t) = & p_1(r, z_0, t) + \frac{W_0^2}{4} \{ \cos(2\lambda z) - \cos(2\lambda z_0) \} e^{2\beta t} \\ & + \frac{W_0}{\lambda} \left(\beta + \frac{\lambda^2}{Re} \right) \{ \cos(\lambda z) - \cos(\lambda z_0) \} e^{\beta t} + \int_{z_0}^z \bar{b} dz. \end{aligned} \quad (38)$$

Evaluating $p_1(1, z, t)$ by substituting $r = 1$ into equation (38) and then substituting it further in equation (36), we get

$$\begin{aligned} p_1(r, z, t) - p_1(1, z_0, t) = & \frac{(1 - r^2)}{2} \left[\frac{W_0 e^{\beta t}}{4} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} \right. \\ & \left\{ 2\beta + W_0 \left(\frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right) e^{\beta t} + \frac{2\lambda^2}{Re} \right\} \\ & + \frac{W_0^2 \lambda}{2} \sin(\lambda z) \left\{ \frac{1}{H_1} \cos(\lambda z) + \lambda \sin(\lambda z) \right\} e^{2\beta t} \\ & \left. - \Psi^2 - S\Psi \right] + \frac{W_0^2}{4} \{ \cos(2\lambda z) - \cos(2\lambda z_0) \} e^{2\beta t} \\ & + \frac{W_0}{\lambda} \left(\beta + \frac{\lambda^2}{Re} \right) \{ \cos(\lambda z) - \cos(\lambda z_0) \} e^{\beta t} + \int_{z_0}^z \bar{b} dz. \end{aligned} \quad (39)$$

2.3.2 Solution for Region 2

From the continuity equation (10) and assuming $w_2(r, z, t) = 0$, the radial wind velocity in region 2 may be given by

$$u_2(r, z, t) = \frac{C_1(z, t)}{r}, \quad (40)$$

where $C_1(z, t)$, an integral function, is obtained by equating the radial velocities of the two regions at the outer boundary of the core, i.e. at $r = 1$. Thus, the radial velocity may be given by

$$u_2(r, z, t) = \frac{W_0}{2r} \left\{ \frac{\sin(\lambda z)}{H_1} - \lambda \cos(\lambda z) \right\} e^{\beta t}. \quad (41)$$

Substitution of u_2 into equation (15), followed by some manipulations, yields

$$\begin{aligned} \frac{\partial v_2}{\partial t} + \frac{W_0}{2r} \left(\frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right) e^{\beta t} \left(\frac{\partial v_2}{\partial r} + \frac{v_2}{r} + S \right) \\ = \epsilon \left(\frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r} \frac{\partial v_2}{\partial r} - \frac{v_2}{r^2} + \frac{\partial^2 v_2}{\partial z^2} \right). \end{aligned} \quad (42)$$

A possible separable solution of equation (42) is of the form $v_2(r, z, t) = F_2(z, t)/r$ so that we have

$$\frac{\partial F_2}{\partial t} - \epsilon \frac{\partial^2 F_2}{\partial z^2} = -\frac{W_0}{2} \left\{ \frac{\sin(\lambda z)}{H_1} - \lambda \cos(\lambda z) \right\} e^{\beta t} S, \quad (43)$$

which, itself, is separable in the form $F_2(z, t) = M(z)e^{\beta t}$, where $M(z)$ satisfies the following equation:

$$\frac{d^2 M}{dz^2} - \frac{\beta}{\epsilon} M = \frac{W_0}{2\epsilon} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} S,$$

whose solution is

$$M(z) = c_1 e^{-\sqrt{(\beta/\epsilon)z}} + c_2 e^{\sqrt{(\beta/\epsilon)z}} - \frac{W_0 S}{2(\lambda^2 + \beta)} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\}. \quad (44)$$

The variables $F_2(z, t)$ and, hence, $M(z)$ must be finite for a finite solution of $v_2(r, z, t)$, which is possible only when $c_2 = 0$, which increases $M(z)$ infinitely. This constraint reduces equation (44) to

$$M(z) = c_1 e^{-\sqrt{(\beta/\epsilon)z}} - \frac{W_0 S}{2(\epsilon\lambda^2 + \beta)} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\}. \quad (45)$$

Thus, the azimuthal velocity for this region may be given by

$$v_2(r, z, t) = \frac{e^{\beta t}}{r} M(z). \quad (46)$$

We obtain c_1 using the second condition, i.e. the azimuthal velocities v_1 and v_2 of the two regions are the same when $t = 0$, $z = 0$, and $r = 1$. Thus, we have

$$v_2(r, z, t) = -\frac{S}{2r} \left[\left\{ 1 + \frac{\lambda W_0}{(\epsilon\lambda^2 + \beta)} \right\} e^{-\sqrt{(\beta/\epsilon)z}} + \frac{W_0}{(\epsilon\lambda^2 + \beta)} \left(\frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right) \right] e^{\beta t}. \quad (47)$$

Corresponding radial pressure gradient is obtained by the application of equations (9), (38), and (47) into equation (7) and is given by

$$\begin{aligned} \frac{\partial p_2}{\partial r} = & -\frac{W_0}{2} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} e^{\beta t} \left[\frac{1}{r} \left(\beta + \frac{\lambda^2}{Re} \right) \right. \\ & \left. - \frac{W_0}{2r^3} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} e^{\beta t} \right] \\ & + \frac{M^2}{r^3} e^{2\beta t} + \frac{SM}{r} e^{\beta t}. \end{aligned} \quad (48)$$

Integrating equation (48) from 1 to r , we get

$$\begin{aligned} p_2(r, z, t) - p_2(1, z, t) = & -\frac{W_0}{2} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} e^{\beta t} \left[\left(\beta + \frac{\lambda^2}{Re} \right) \log_e r \right. \\ & \left. + \frac{W_0}{4} \left(\frac{1}{r^2} - 1 \right) \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} e^{\beta t} \right] \\ & - \frac{M^2}{2} \left(\frac{1}{r^2} - 1 \right) e^{2\beta t} + SM e^{\beta t} \log_e r. \end{aligned} \quad (49)$$

3 Results and Discussion

Presented here is an analytical model of an intense idealized hurricane vortex by considering a diabatically induced ascending motion proposed by Kieu and Zhang [23]. We further assumed that the vertical velocity is independent of the radial coordinate r .

In most of the investigations, the vortex motion was considered inviscid. However, Kieu and Zhang [23] considered viscous flow by taking a linear form of viscosity. Unlike them, we considered the general form and used a perturbation technique to analyze the contribution of viscosity to hurricane dynamics despite the fact that the Reynolds number is very large in such a rotational motion. Because of the large Reynolds number and highly complicated expressions, we confined the entire solution to the first order of $\epsilon = Re^{-1} (\ll 1)$. Besides the azimuthal velocity, pressure also is worth discussing.

3.1 Analysis of the Solution in Region 1 ($r < 1$)

As per our assumptions made in Section 2, we have only the azimuthal velocity derived for the two regions viz., the inner region and the outer region. In the eye wall, updraft and rotational wind motion about the vertical axis are witnessed. As the vertical velocity is the same as that assumed by Kieu and Zhang [23], we shall confine the discussion around the azimuthal velocity.

3.1.1 Azimuthal Velocity

The role of δ , which is a parameter in the formulation of the azimuthal velocity, the contribution due to the perturbation term and the edge of general viscosity consideration over the linear form assumed by Kieu and Zhang [23] is worth discussing.

In order to examine the impact of δ , where $\beta\mu/\lambda W_0 = 1 - \delta$, $0 \leq \delta \leq 1$ on the unperturbed azimuthal velocity v_{10} , we plot v_{10} versus z , displayed in Figure 2,

against a wide range of δ at $t = 0$ and for $\lambda = 2$, $\beta = 0.5$, $W_0 = 0.12$, and $r = 1$, which is the interface of the two regions. It is observed that the azimuthal velocity increases while ascending along the vertical axis up to a certain height and then begins to fall in magnitude. An interesting observation is that up to that height v_{10} , the zeroth-order azimuthal velocity increases with δ but coincides at non-dimensional $z = 1 \forall \delta$. Trends are exactly reverse above that. This is almost similar as that Kieu and Zhang [23] discovered. However, it is not clear how they non-dimensionalised.

The contribution of the first perturbation term is another important aspect of this investigation. Accordingly, we plot v_{10} and v_{11} versus z by varying t in the

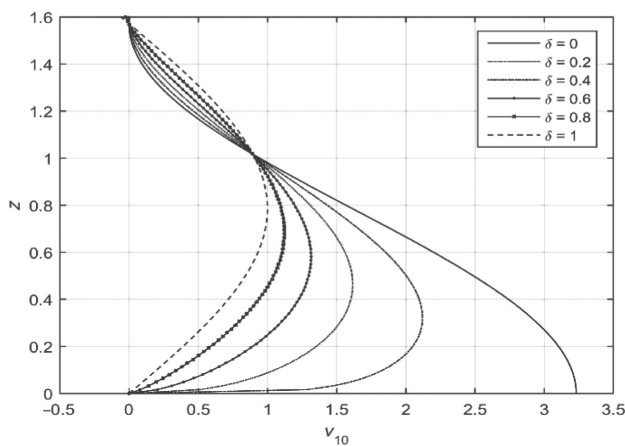


Figure 2: The diagram, based on (27), represents the vertical profile of v_{10} , the zeroth-order azimuthal velocity, for $t = 0$. $\lambda = 2$, $\beta = 0.5$, $W_0 = 0.12$ are the parameters used for the plot.

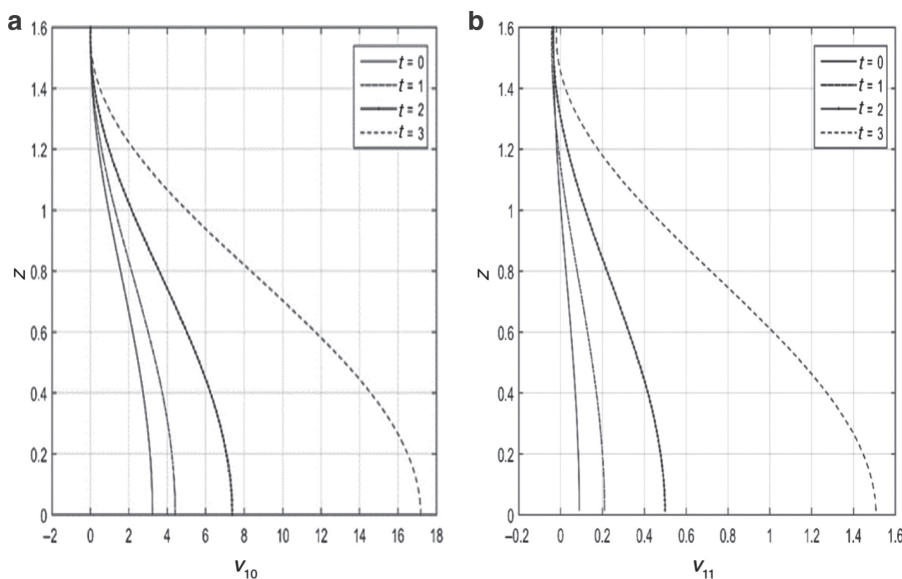


Figure 3: The diagrams display (a) the zeroth-order and (b) the first-order perturbed azimuthal velocity along the vertical axis, based on (27) and (33), for $\delta = 0$ at different instants mentioned in the legend.

range of 0–3 in Figure 3. The two have similar patterns with v_{11} exceeding a little bit in magnitude, but both of them increase with time t . However, the real contribution of the perturbation term will be much less as it is multiplied by $\epsilon = 1/Re$, which is of the order of 10^{-4} for a real hurricane. The combined effect is displayed in Figure 4a,b, respectively, for $Re = 10,000$ and 100 . For comparatively small Reynolds number, the contribution of the perturbation term, in terms of magnitude, is quite significant Figure 5. Moreover, the pattern we get here is quite similar to what Kieu and Zhang [23] observed. Scales are distinct for the reason that they used dimensional parameters; however, they claim the figures to use non-dimensional units. If so, then probably they used different characteristic parameters, which are nowhere mentioned in the article.

3.1.2 Vertical Pressure Distribution

The difference of pressures between the core and the inner part of region 1 is given by (39). Apart from the radial and axial coordinates, it depends also on time, viscosity, and the radial distance from the point of maximum wind. Therefore, in this subsection, we would study the temporal, viscous, and radius of maximum wind impacts on the vertical pressure distribution.

The vertical distribution of the pressure difference between the point of maximum wind and an arbitrary inner radial distance from the centre is displayed in Figure 6, in which the parameters viz., time, the Reynolds number and the radial distance are varied, respectively,

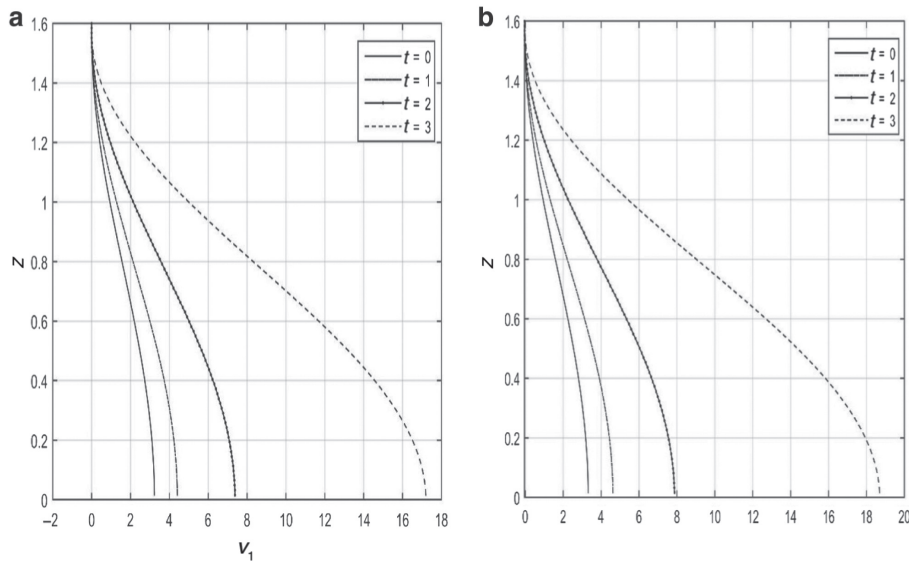


Figure 4: The diagrams display the azimuthal velocity, based on (33), (a) for Reynolds number $Re = 10,000$; (b) Reynolds number $Re = 100$, for $\delta = 0$ along the vertical axis at different instants.

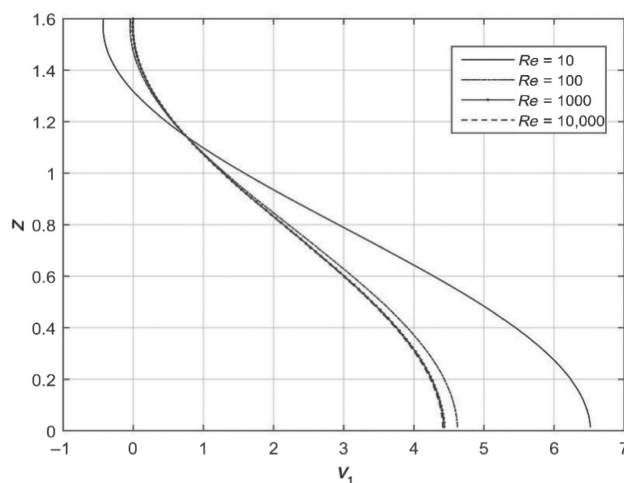


Figure 5: Diagram for azimuthal velocity v vs. z , based on (33), for different Reynolds number at time $t = 1$ and $\delta = 0$.

in Figure 6a–c in order to examine their roles when other parameters are kept constant. Keeping the reference pressure $p_1(1, 0, t)$, i.e. the pressure at the radius of maximum wind on the ground, we observe that pressure difference diminishes when we move from the ground to higher altitudes irrespective of the variations of the three parameters. This diminishing character of the pressure difference with altitude contributes to the possible funnel shape of the cyclone as shown in Figure 1.

Temporal variation reveals in Figure 6a for $Re = 5000$ that pressure difference grows with time. This growth in pressure difference enhances the centripetal force and, hence, leads to intensification of cyclone with time.

Radial dependence of the pressure difference is displayed in Figure 6b. $Re = 5000$ and $t = 1$ are fixed; r is varied in the range of 0.2–0.8. The difference diminishes obviously as we move towards the point of maximum wind, but this difference further decreases as we move along the axial distance from the ground.

Setting $r = 0.2$, $t = 1$, we observe in Figure 6c that the pressure difference between the point of maximum wind and $r = 0.2$ increases slightly when Re is varied from 100 to 3000. Further increase in Re brings about insignificant change.

3.2 Analysis of the Solution in Region 2 ($r > 1$)

Region 2 that contains fast rotating wind with nearly saturated air at the lower altitudes and has a radial inflow that supplies moisture for updraft through the boundary layer plays an important role for the updraft in the eye wall.

The vertical profile of the azimuthal velocity is plotted in Figure 7 in a temporal range of 0–3. Unlike within region 1 for which perturbation technique had to be used, an exact solution was obtained for region 2. Close to the ground, the azimuthal velocity is found to rise rapidly with time, but reverse is the trend at a little height and becomes independent of time at high altitudes. It drops further with time at even higher altitudes. The trends are qualitatively similar to that in region 1, but the quantitative difference is considerable. In fact, the two regions conform to the Rankine's model as assumed in the beginning.

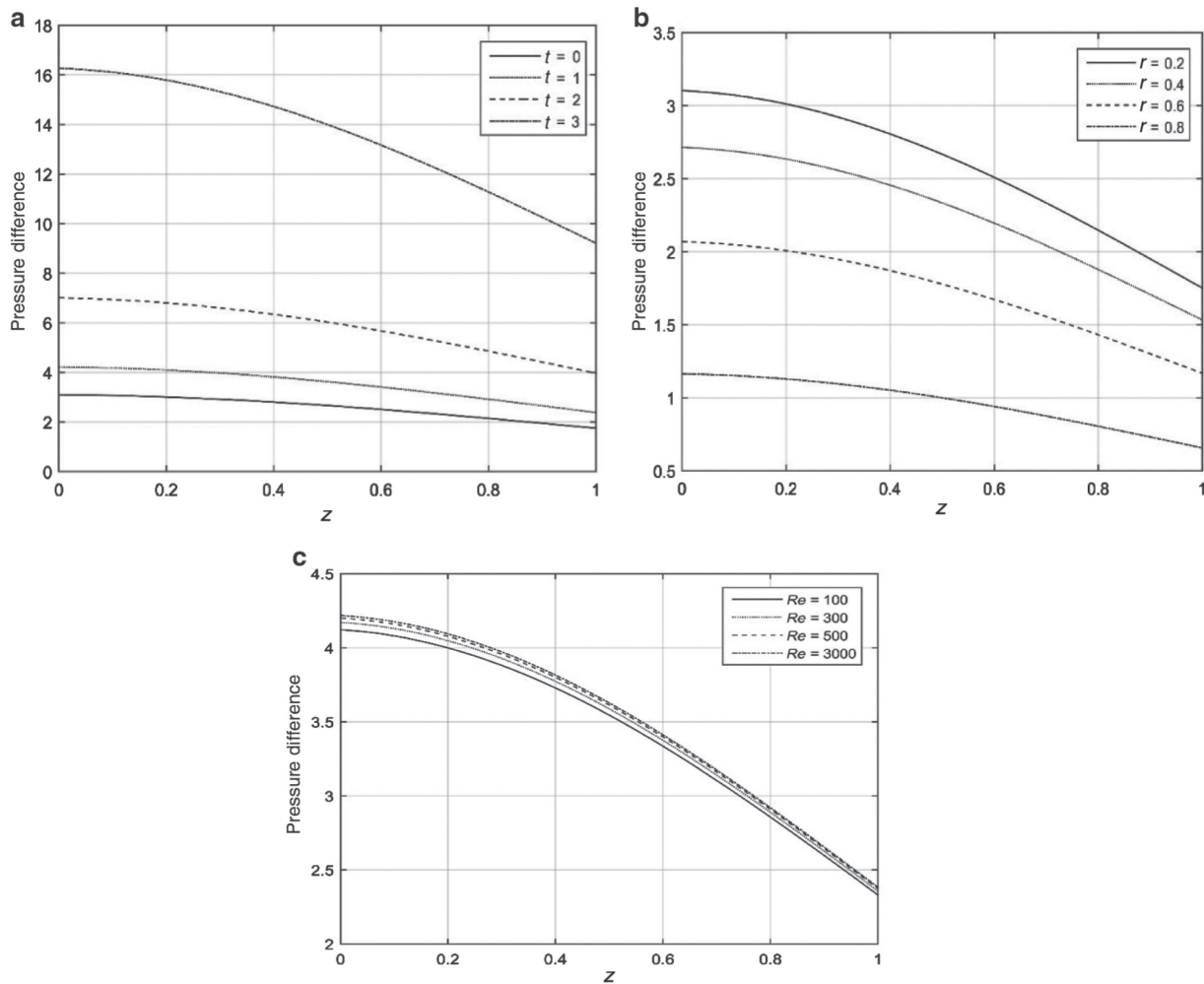


Figure 6: The diagrams display pressure difference vs. z , given by (39), for $\delta = 0$ and the impact of (a) time t ($Re = 5000$), (b) the radial distance ($Re = 5000$, $t = 1$), and (c) the Reynolds number Re ($r = 0.2$, $t = 1$).

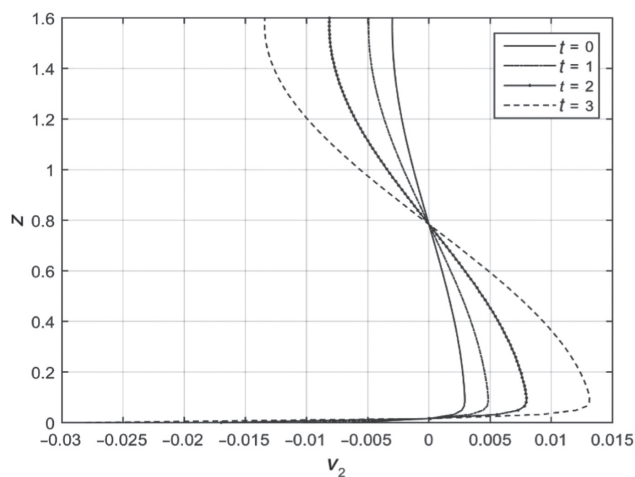


Figure 7: The diagram represents azimuthal velocity along the vertical axis for $\delta = 0$ at different instants mentioned in the legend for the second region.

The presence of sine and cosine terms seems to periodically change the trends with height. However, this region has no vertical velocity.

3.3 Pressure Deficit

The relationship between the central pressure deficit and the peak wind speed near the ground surface in a tropical cyclone has important consequences in meteorology from a physical point of view. It is also related to risk of damage and loss of life [35]. The central pressure deficit in a tropical cyclone is defined as the difference in pressure between the centre of the storm and outside it. We denote it by $\Delta\tilde{p} = 1 - p_m/p_0$, where p_m is the minimum central pressure near the surface, and p_0 is the environmental pressure at the outer edge of the storm.

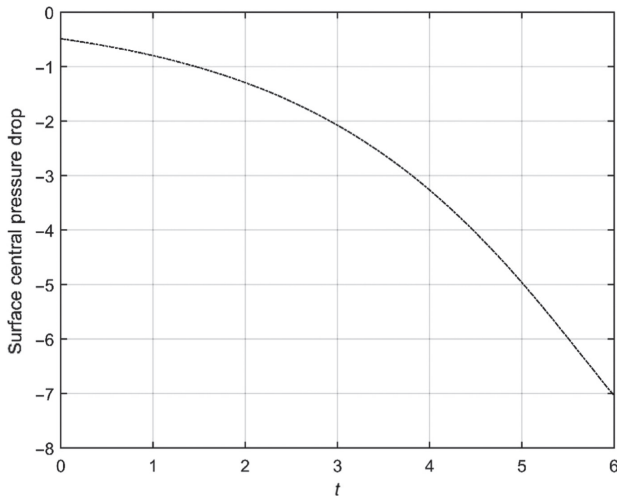


Figure 8: The diagram represents the variation of surface central pressure drop with time based on (50).

The minimum central pressure near the surface may be obtained by means of equations (39) and (49), which is

$$\begin{aligned}
 p_m(t) = & -\frac{\lambda W_0 e^{\beta t}}{2} \left[\left(\beta + \frac{\lambda^2}{Re} \right) \log R_m \right. \\
 & + \frac{\lambda W_0}{4} \left(1 - \frac{1}{R_m^2} \right) e^{\beta t} \left. \right] + \frac{S^2}{8} \left(1 - \frac{1}{R_m^2} \right) e^{2\beta t} \\
 & + \frac{S^2}{2} e^{\beta t} \log R_m - \frac{1}{8} \left[\lambda W_0 e^{\beta t} \left\{ 2 \left(\beta + \frac{\lambda^2}{Re} \right) \right. \right. \\
 & \left. \left. - \lambda W_0 e^{\beta t} \right\} - S^2 \right]. \quad (50)
 \end{aligned}$$

It is observed that the central pressure drop decreases with time (Fig. 8). The observation is similar to that of Kieu and Zhang [23] who found it to conform to the experimental data.

4 Conclusions

Unlike the linear approximated form of viscosity, we assumed a general type of viscosity for investigating the reason behind the rapid intensification of the cyclonic wind. The existence of the double exponential terms was discovered by Kieu and Zhang [23] as the reason for linear viscosity. Similar terms are observed even for the general form of viscosity. Hence, it is concluded that double exponential terms accelerate the rotational motion irrespective of the form of the viscosity of a tropical cyclone. This is to be noted that this inference is based on the fact that gradient wind imbalance for higher altitudes of cyclone was ignored.

The domain is split into two regions: one which contains all the updraft and lies entirely within the radius of maximum velocity and the other, which lies beyond it, contains no updraft but possesses azimuthal and radial velocities.

The radial pressure difference depends on time, viscosity, and height from the ground. The difference in pressures at the radius of maximum wind and an arbitrary radial point that falls with height increases with time and the Reynolds number. This gives the cyclone a funnel shape.

The azimuthal velocity rises fast with time close to the ground, but the dependence diminishes at a height a little above the ground. At a considerable height, time sieges to be a factor, and further, above that, trends reverse.

Using a perturbation analysis, we found that the perturbation terms behave almost identically with the terms without perturbation. The significance of their contribution definitely depends on the magnitude of the Reynolds number. Unlike that for region 1 where perturbation technique had to be used, exact solution is obtained for region 2. The trends are qualitatively similar to that in region 1, but quantitative difference is significant. It is also observed that the central pressure drop decreases with time.

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Appendix A

Analytical solution for the eye wall

Using equation (22) into equation (20), we obtain

$$\frac{dG'_0}{dz} = - \left[\left\{ \frac{1}{H_1} - \lambda \cot(\lambda z) \right\} + \frac{\beta \mu}{W_0} \operatorname{cosec}(\lambda z) \right] G'_0. \quad (A1)$$

On integration, we have

$$\begin{aligned}
 \log \left(\frac{G'_0}{K} \right) = & - \left[\frac{z}{H_1} - \log(\sin(\lambda z)) - \frac{\beta \mu}{\lambda W_0} \log(\cot(\lambda z)) \right. \\
 & \left. + \operatorname{cosec}(\lambda z) \right],
 \end{aligned}$$

or $G'_0(z)$

$$= Ke^{-(z/H_1)} \{\sin(\lambda z)\}^{(1-\beta\mu/\lambda W_0)} \left\{ 2 \cos\left(\frac{\lambda z}{2}\right) \right\}^{2(\beta\mu/\lambda W_0)}, \quad (A2)$$

where K is an integration constant (with unit per second) that determines the initial strength of the vortex.

Equation (A2) contains an infinite number of possible solutions depending on the values of μ . However, the requirements for the regularity of (A2) at $z = 0$ impose a strong restriction in the range of μ . Using the L'Hospital rule for a regular solution, we have $1 \geq \beta\mu/\lambda W_0$. Taking $\beta\mu/\lambda W_0 = 1 - \delta$, where $0 \leq \delta \leq 1$, thus, (A2) reduces to

$$G'_0(z) = Ke^{-(z/H_1)} \left\{ \sin\left(\frac{\lambda z}{2}\right) \right\}^\delta \left\{ \cos\left(\frac{\lambda z}{2}\right) \right\}^{2-\delta}. \quad (A3)$$

Appendix B

To solve equation (28), we let $G_1(z, t) = \Gamma(z, t)G_0(z, t)$ and use equation (20) to obtain

$$e^{-\beta t} \frac{\partial \Gamma}{\partial t} + W_0 \sin(\lambda z) \frac{\partial \Gamma}{\partial z} = N(z, t), \quad (B1)$$

where

$$N(z, t) = e^{-\beta t} \left[\frac{(2\delta - 1)\delta\lambda^2}{4} \cot^2\left(\frac{\lambda z}{2}\right) + \frac{(1 - 3\delta)\lambda}{2H_1} \cot\left(\frac{\lambda z}{2}\right) + \left\{ \frac{1}{H_1^2} - \frac{(5 - 2\delta)\delta\lambda^2}{4} \right\} + \frac{(3 - \delta)\lambda}{2H_1} \tan\left(\frac{\lambda z}{2}\right) - \frac{\delta\lambda^2}{4} \operatorname{cosec}^2\left(\frac{\lambda z}{2}\right) \right]. \quad (B2)$$

Now, we solve equation (B2) by applying the Lagrange subsidiary equation

$$\frac{dt}{e^{-\beta t}} = \frac{dz}{W_0 \sin(\lambda z)} = \frac{d\Gamma}{F(z, t)}. \quad (B3)$$

The solution of the first equality is

$$\frac{e^{\beta t}}{\beta} - \frac{1}{\lambda W_0} \ln \tan\left(\frac{\lambda z}{2}\right) = A, \quad (B4)$$

where A is an integration constant.

The second integral is obtained by means of last two equalities

$$\begin{aligned} \frac{d\Gamma}{dz} &= \frac{F(z, t)}{W_0 \sin(\lambda z)} = \frac{e^{-\beta t}}{W_0 \sin(\lambda z)} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \cot^2\left(\frac{\lambda z}{2}\right) + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \tan^2\left(\frac{\lambda z}{2}\right) - \frac{\delta \lambda}{H_1} \cot\left(\frac{\lambda z}{2}\right) + \frac{(2 - \delta)\lambda}{H_1} \tan\left(\frac{\lambda z}{2}\right) + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \right]. \end{aligned} \quad (B5)$$

Applying equation (B4), we have

$$\begin{aligned} \frac{d\Gamma}{dz} &= \frac{1}{W_0 \beta \sin(\lambda z) \left(A + \frac{1}{\lambda W_0} \ln \left\{ \tan\left(\frac{\lambda z}{2}\right) \right\} \right)} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \cot^2\left(\frac{\lambda z}{2}\right) + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \tan^2\left(\frac{\lambda z}{2}\right) - \frac{\delta \lambda}{H_1} \cot\left(\frac{\lambda z}{2}\right) + \frac{(2 - \delta)\lambda}{H_1} \tan\left(\frac{\lambda z}{2}\right) + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \right]. \end{aligned} \quad (B6)$$

Integrating equation (B6), with respect to z , we obtain

$$\begin{aligned} \Gamma(z, t) &= \frac{1}{W_0 \beta} \int \frac{1}{\sin(\lambda z) \left(A + \frac{1}{\lambda W_0} \ln \left\{ \tan\left(\frac{\lambda z}{2}\right) \right\} \right)} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \cot^2\left(\frac{\lambda z}{2}\right) + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \tan^2\left(\frac{\lambda z}{2}\right) - \frac{\delta \lambda}{H_1} \cot\left(\frac{\lambda z}{2}\right) + \frac{(2 - \delta)\lambda}{H_1} \tan\left(\frac{\lambda z}{2}\right) + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \right] dz + \phi(A) \end{aligned} \quad (B7)$$

or

$$\begin{aligned} \Gamma(z, t) &= \frac{1}{\beta} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} I_1 + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} I_2 - \frac{\delta \lambda}{H_1} I_3 + \frac{(2 - \delta)\lambda}{H_1} I_4 + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) I_5 \right] + \phi(A), \end{aligned} \quad (B8)$$

where

$$\begin{aligned} I_1 &= \int \frac{\cot^2\left(\frac{\lambda z}{2}\right)}{W_0 \sin(\lambda z) \left(A + \frac{1}{\lambda W_0} \ln \left\{ \tan\left(\frac{\lambda z}{2}\right) \right\} \right)} dz, \\ I_2 &= \int \frac{\tan^2\left(\frac{\lambda z}{2}\right)}{W_0 \sin(\lambda z) \left(A + \frac{1}{\lambda W_0} \ln \left\{ \tan\left(\frac{\lambda z}{2}\right) \right\} \right)} dz, \\ I_3 &= \int \frac{\cot\left(\frac{\lambda z}{2}\right)}{W_0 \sin(\lambda z) \left(A + \frac{1}{\lambda W_0} \ln \left\{ \tan\left(\frac{\lambda z}{2}\right) \right\} \right)} dz, \\ I_4 &= \int \frac{\tan\left(\frac{\lambda z}{2}\right)}{W_0 \sin(\lambda z) \left(A + \frac{1}{\lambda W_0} \ln \left\{ \tan\left(\frac{\lambda z}{2}\right) \right\} \right)} dz, \\ I_5 &= \int \frac{dz}{W_0 \sin(\lambda z) \left(A + \frac{1}{\lambda W_0} \ln \left\{ \tan\left(\frac{\lambda z}{2}\right) \right\} \right)}. \end{aligned} \quad (B9)$$

Substituting $A + \frac{1}{\lambda W_0} \ln \left\{ \tan\left(\frac{\lambda z}{2}\right) \right\} = P$, we get $dz/W_0 \sin(\lambda z) = dP$, $\tan\left(\frac{\lambda z}{2}\right) = e^{(P-A)\lambda W_0}$, and

$\cot\left(\frac{\lambda z}{2}\right) = e^{-(P-A)\lambda W_0}$. Thus, the integrals reduce

$$\begin{aligned} I_1 &= \int \frac{e^{-2(P-A)\lambda W_0}}{P} dP, I_2 = \int \frac{e^{2(P-A)\lambda W_0}}{P} dP, \\ I_3 &= \int \frac{e^{-(P-A)\lambda W_0}}{P} dP, I_4 = \int \frac{e^{(P-A)\lambda W_0}}{P} dP, \\ I_5 &= \int \frac{dP}{P} \end{aligned} \quad (B10)$$

or

$$\begin{aligned} I_1 &= e^{2\lambda W_0 A} \int \frac{e^{-2\lambda W_0 P}}{P} dP, I_2 = e^{-2\lambda W_0 A} \int \frac{e^{2\lambda W_0 P}}{P} dP, \\ I_3 &= e^{\lambda W_0 A} \int \frac{e^{-\lambda W_0 P}}{P} dP, I_4 = e^{-\lambda W_0 A} \int \frac{e^{\lambda W_0 P}}{P} dP, \\ I_5 &= \int \frac{dP}{P} = W_0 \ln\left(\frac{e^{\beta t}}{\beta}\right). \end{aligned} \quad (B11)$$

Further, we denote

$$\begin{aligned} I_{11} &= \int \frac{e^{-2\lambda W_0 P}}{P} dP, I_{22} = \int \frac{e^{2\lambda W_0 P}}{P} dP, \\ I_{33} &= \int \frac{e^{-\lambda W_0 P}}{P} dP, I_{44} = \int \frac{e^{\lambda W_0 P}}{P} dP, \\ I_{55} &= \lambda W_0 \ln\left(\frac{e^{\beta t}}{\beta}\right). \end{aligned} \quad (B12)$$

Thus, we have

$$\begin{aligned} \Gamma(z, t) &= \frac{1}{\beta} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} e^{2\lambda W_0 A} I_{11} \right. \\ &\quad + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} e^{-2\lambda W_0 A} I_{22} \\ &\quad - \frac{\delta \lambda}{H_1} e^{\lambda W_0 A} I_{33} + \frac{(2 - \delta)\lambda}{H_1} e^{-\lambda W_0 A} I_{44} \\ &\quad \left. + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \ln\left(\frac{e^{\beta t}}{\beta}\right) \right] + \emptyset(A), \end{aligned} \quad (B13)$$

or

$$\begin{aligned} \Gamma(z, t) &= \frac{1}{\beta} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \exp\left(2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot^2\left(\frac{\lambda z}{2}\right) I_{11} \right. \\ &\quad + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \exp\left(-2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan^2\left(\frac{\lambda z}{2}\right) I_{22} \\ &\quad - \frac{\delta \lambda}{H_1} \exp\left(\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot\left(\frac{\lambda z}{2}\right) I_{33} \\ &\quad + \frac{(2 - \delta)\lambda}{H_1} \exp\left(-\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan\left(\frac{\lambda z}{2}\right) I_{44} \\ &\quad \left. + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \ln\left(\frac{e^{\beta t}}{\beta}\right) \right] + \emptyset(A). \end{aligned} \quad (B14)$$

Thus, we get the first-order solution as

$$\begin{aligned} G_1(z, t) &= 2 K e^{-(z/H_1)} \left\{ \sin\left(\frac{\lambda z}{2}\right) \right\}^\delta \left\{ \cos\left(\frac{\lambda z}{2}\right) \right\}^{2-\delta} \\ &\quad \left[\frac{1}{\beta} \left[\frac{\lambda^2 \delta (\delta - 1)}{4} \exp\left(2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot^2\left(\frac{\lambda z}{2}\right) I_{11} \right. \right. \\ &\quad + \frac{\lambda^2 (2 - \delta)(1 - \delta)}{4} \exp\left(-2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan^2\left(\frac{\lambda z}{2}\right) I_{22} \\ &\quad - \frac{\delta \lambda}{H_1} \exp\left(\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \cot\left(\frac{\lambda z}{2}\right) I_{33} \\ &\quad + \frac{(2 - \delta)\lambda}{H_1} \exp\left(-\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan\left(\frac{\lambda z}{2}\right) I_{44} \\ &\quad \left. + \left(\frac{1}{H_1^2} - \frac{(2\delta - \delta^2 + 1)\lambda^2}{2} \right) \ln\left(\frac{e^{\beta t}}{\beta}\right) \right] \\ &\quad + \emptyset(A)] \exp\left\{ \frac{\lambda W_0}{\beta} (1 - \delta) e^{\beta t} \right\}. \end{aligned} \quad (B15)$$

For $\delta = 0$, we have

$$\begin{aligned} G_1(z, t) &= 2 K e^{-(z/H_1)} \left\{ \cos\left(\frac{\lambda z}{2}\right) \right\}^2 \\ &\quad \left[\frac{1}{\beta} \left\{ \frac{\lambda^2}{2} \exp\left(-2\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan^2\left(\frac{\lambda z}{2}\right) I_{22} \right. \right. \\ &\quad + \frac{2\lambda}{H_1} \exp\left(-\lambda W_0 \frac{e^{\beta t}}{\beta}\right) \tan\left(\frac{\lambda z}{2}\right) I_{44} \\ &\quad \left. + \left(\frac{1}{H_1^2} - \frac{\lambda^2}{2} \right) \ln\left(\frac{e^{\beta t}}{\beta}\right) \right\} + \emptyset(A)] \exp\left\{ \frac{\lambda W_0}{\beta} e^{\beta t} \right\}, \end{aligned} \quad (B16)$$

$$I_{22} = \int \frac{e^{2\lambda W_0 x}}{x} dx, I_{44} = \int \frac{e^{\lambda W_0 x}}{x} dx, I_{22} = \int \frac{e^{2\lambda W_0 x}}{x} dx. \quad (B17)$$

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