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A Steady-state Trio for Bretherton Equation

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Abstract: To investigate if steady-state resonant solution exist for any system of weakly interacting waves in a dispersive medium, a trio is considered in the Bretherton equation based on the homotopy analysis method (HAM). Time-independent spectrum was found when all components were travelling in the same direction. Within the trio, the amplitude of longer component is larger than that of shorter one. As the difference of wave number between components in trio increases or the nonlinearity of whole system increases, the amplitudes of all components tends to increase simultaneously. These findings are helpful to enrich and deepen our understanding about resonant solutions in any dispersive medium, especially for a two-dimensional scenario.

Keywords: Bertherton Equation; Homotopy Analysis Method; Steady-State Resonant Solutions.

1 Introduction

The Bretherton equation

$$\frac{\partial^2\psi}{\partial t^2}+\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^2\psi}{\partial x^2}+\psi=\psi^2, \tag{1}$$

was introduced by Bretherton [1] as a model of a dispersive wave systems to investigate the resonant nonlinear interaction between three linear modes. Later, a modified Bretherton equation

$$\frac{\partial^2\psi}{\partial t^2}+\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^2\psi}{\partial x^2}+\psi=\psi^3, \tag{2}$$

was studied by Kudryashov [2] and Berloff and Howard [3] through truncated Painleve expansion to search periodic solution, and Darwish and Ramady [4] through algebraic method to obtain travelling wave solutions. Recently, a generalised Bretherton equation

$$\frac{\partial^2\psi}{\partial t^2}+\alpha\frac{\partial^4\psi}{\partial x^4}+\beta\frac{\partial^2\psi}{\partial x^2}+\delta\psi^m=\gamma\psi^n, \tag{3}$$

where $\alpha, \beta, \delta, \gamma, m$, and n are nonzero constants, was proposed. Romeiras [5] used both truncated Painleve expansion method and an algebraic method to investigate the existence of periodic and solitary travelling waves solutions, Kudryashov et al. [6] applied Backlund transformation to search several new families of simply periodic and elliptic solutions, Esfahani [7] used trigonometric function method and He's semi-inverse method to find traveling wave solutions, and Akbar et al. [8] applied improved (G'/G)-expansion method to construct travelling wave solutions.

Note the original Bretherton equation (1) was proposed to investigate the mathematical basis for resonance: to clarify some aspects of the mechanism whereby energy is transferred between different wave numbers [1]. When resonance condition is satisfied, the perturbation results contain secular terms and the perturbation theory breaks down due to singularities in the transfer functions [9], so most researchers considered the resonance from the point of view of initial/boundary-value problems (the spectrum changes with time during the wave evolution).

By means of the homotopy analysis method (HAM) [10–12], an analytic approximation method for highly nonlinear problems, the multiple solutions of steady-state resonant waves were obtained by Liao [13] from the point of view of boundary-value problems (the spectrum keeps unchanged with time during the wave evolution) in deep water. The same approach was extended further by Xu et al. [14] to special quartet in finite water depth, by Liu and Liao [15] to general resonant sets in deep water, and by Liao et al. [16] to nearly resonant waves. Especially, the multiple steady-state resonant waves were observed experimentally (Liu et al. [17]) in a basin at State Key Laboratory of Ocean Engineering, with excellent agreement to their theoretical predictions given by means of the HAM.

In this article, the object is to investigate the existence of steady-state resonant solutions for the Bretherton equation (1). The phenomenon of resonance happens for any system with weakly interacting waves in a dispersive medium, whereas steady-state resonant solutions

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has only been considered for water wave system up to now. So taking a trio as an example, we further applied the HAM to the Bretherton equation (1) to check if the steady-state resonant sets could be find for other nonlinear system.

2 Mathematical Formulae

2.1 Solution Procedure

Consider the nonlinear interaction of a steady-state periodic oscillation. Write

$$\xi_i = k_i x - \sigma_i t, \quad i = 1,\ 2, \tag{4}$$

where k_i is the wave number and σ_i denotes actual wave frequency. Due to the nonlinearity, the actual wave frequency $\sigma_i(k_i,\ A_{m,n})$, which is larger than the linear frequency $\omega_i(k_i)$, depends on both the wave number k_i and amplitude $A_{m,n}$. And for Bretherton equation (1), the linear frequency $\omega_i = \sqrt{1-k_i^2+k_i^4}$. The "steady-state resonant solutions" means amplitude $A_{m,n}$ and actual wave frequency σ_i are constants (time independent) when resonance condition is satisfied. So the solution can be expressed in the form

$$\phi(\xi_1,\ \xi_2) = \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A_{m,n}\cos(m\xi_1 + n\xi_2). \tag{5}$$

The original initial/boundary-value problem governed by (1) can now be transformed into a boundary-value one by means of the new variables ξ_1 and ξ_2. In the new coordinate system (ξ_1, ξ_2), the governing equation is defined in (6), where $\mathcal{N}$ is a nonlinear operator defined on ψ.

$$\begin{aligned}\mathcal{N}[\psi] = {} & \sigma_1^2\frac{\partial^2\psi}{\partial\xi_1^2} + 2\sigma_1\sigma_2\frac{\partial^2\psi}{\partial\xi_1\partial\xi_2} + \sigma_2^2\frac{\partial^2\psi}{\partial\xi_2^2} + k_1^4\frac{\partial^4\psi}{\partial\xi_1^4} \\ & + 4k_1^3k_2\frac{\partial^4\psi}{\partial\xi_1^3\partial\xi_2} + 6k_1^2k_2^2\frac{\partial^4\psi}{\partial\xi_1^2\partial\xi_2^2} + 4k_1k_2^3\frac{\partial^4\psi}{\partial\xi_1\partial\xi_2^3} + k_2^4\frac{\partial^4\psi}{\partial\xi_2^4} \\ & + k_1^2\frac{\partial^2\psi}{\partial\xi_1^2} + 2k_1k_2\frac{\partial^2\psi}{\partial\xi_1\partial\xi_2} + k_2^2\frac{\partial^2\psi}{\partial\xi_2^2} + \psi - \psi^2 = 0,\end{aligned} \tag{6}$$

The steady-state resonant solutions for periodic oscillations in (1) can be obtained in a similar way as Liao [13] did, so the approach is briefly described below.

Based on the linear part of the governing equation (6), we choose the following auxiliary linear operator (7), which has the property

$$\begin{aligned}\mathcal{L}[\psi] = {} & \omega_1^2\frac{\partial^2\psi}{\partial\xi_1^2} + 2\omega_1\omega_2\frac{\partial^2\psi}{\partial\xi_1\partial\xi_2} + \omega_2^2\frac{\partial^2\psi}{\partial\xi_2^2} + k_1^4\frac{\partial^4\psi}{\partial\xi_1^4} \\ & + 4k_1^3k_2\frac{\partial^4\psi}{\partial\xi_1^3\partial\xi_2} + 6k_1^2k_2^2\frac{\partial^4\psi}{\partial\xi_1^2\partial\xi_2^2} + 4k_1k_2^3\frac{\partial^4\psi}{\partial\xi_1\partial\xi_2^3} + k_2^4\frac{\partial^4\psi}{\partial\xi_2^4} \\ & + k_1^2\frac{\partial^2\psi}{\partial\xi_1^2} + 2k_1k_2\frac{\partial^2\psi}{\partial\xi_1\partial\xi_2} + k_2^2\frac{\partial^2\psi}{\partial\xi_2^2} + \psi,\end{aligned} \tag{7}$$

$$\mathcal{L}[\cos(m\xi_1 + n\xi_2)] = \lambda_{m,n}\cos(m\xi_1 + n\xi_2), \tag{8}$$

where

$$\begin{aligned}\lambda_{m,n} &= 1-(mk_1+nk_2)^2 + (mk_1+nk_2)^4 - (m\omega_1+n\omega_2)^2 \\ &= \omega_{m,n}^2 - (m\omega_1+n\omega_2)^2\end{aligned} \tag{9}$$

is an eigenvalue of $\mathcal{L}$ with the property $\lambda_{1,0} \equiv \lambda_{0,1} \equiv 0$. When the linear resonance condition

$$mk_1 + nk_2 = k_0, \quad m\omega_1 + n\omega_2 = \omega_0 \tag{10}$$

is satisfied by a free component with frequency $\omega_0 = \sqrt{1-k_0^2+k_0^4}$, it holds $\omega_{m,n} = \omega_0 = m\omega_1 + n\omega_2$, so $\lambda_{m,n} \equiv 0$. For simplicity, a trio is considered in this article and the linear resonance condition reads

$$k_1 - k_2 = k_0, \quad \omega_1 - \omega_2 = \omega_0, \tag{11}$$

so the eigenvalue holds $\lambda_{1,0} \equiv \lambda_{0,1} \equiv \lambda_{1,-1} \equiv 0$. Based on the property of auxiliary linear operator $\mathcal{L}$, we define the initial guess as shown in (a11). The constants $A_{m,n}$ in (5) can now be obtained order by order in the framework of HAM. For details, refer to Appendix.

2.2 Results Verification

Let us consider the case of $\epsilon = \sigma_i/\omega_i = 1.01$ when $k_1 = 1.5$. The corresponding values of $k_2 = 1.03764$ and $k_0 = 0.462365$[1] are determined by the resonant condition (11). Figure 1 shows the averaged residual squares at different orders of approximation defined in (12),

$$\Pi_m = \frac{\pi^2}{25}\sum_{m=0}^{5}\sum_{n=0}^{5}\left[\sum_{i=0}^{m-1} R_i\left(\frac{\pi}{5}m,\ \frac{\pi}{5}n\right)\right]^2, \tag{12}$$

which suggest us the optimal convergence-control parameter $c_0 = -1.4$. Using this optimal convergence-control parameter, the residual squares of the governing equation indeed decreases quickly as the order of approximation increases, as shown in Figure 2.

1 It is assumed that $k_2 > k_0$.

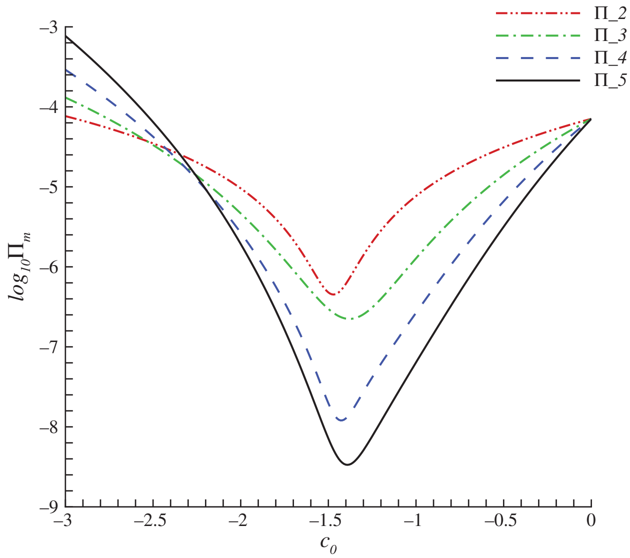

Figure 1: $\text{Log}_{10}\Pi_m$ versus the convergence-control parameter c_0 with $k_1 = 1.5$ and $\epsilon = 1.01$. Dash-dot-dot line: 2nd-order approximation; Dash-dot line: 3rd-order approximation; Dashed line: 4th-order approximation; and Solid line: 5th-order approximation.

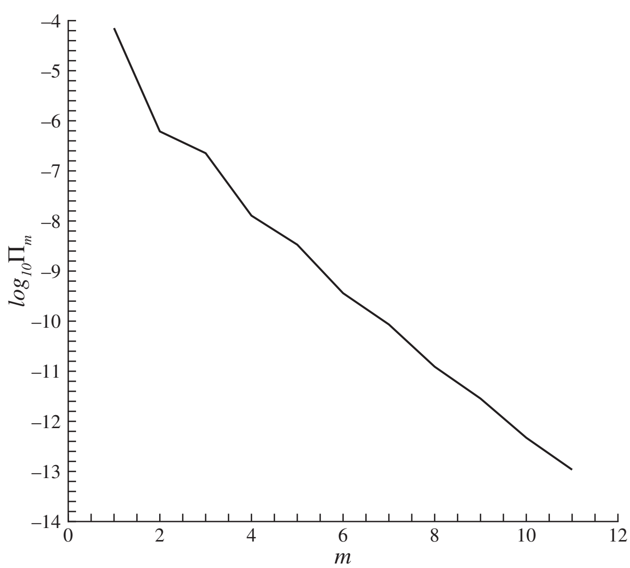

Figure 2: $\text{Log}_{10}\Pi_m$ versus the approximation order m with $k_1 = 1.5$ and $\epsilon = 1.01$.

To increase the convergent rate of the series solution, we apply the Homotopy-Pade approximation and the corresponding $[n, n]$ approximation of $A_{1,0}$, $A_{0,1}$, and $A_{1,-1}$ are shown in Table 1. It is found from Table 1 that as the approximation order n increases, the values of $[n, n]$ Homotopy-Pade approximation of $A_{1,0}$, $A_{0,1}$, and $A_{1,-1}$ gradually tend to constants. Hereinafter all the following calculations stop until the results of Homotopy-Pade approximation agree to four significant figures.

Table 1: $[m, m]$ Homotopy-Pade approximation of $A_{1,0}$, $A_{0,1}$, and $A_{1,-1}$ with $k_1 = 1.5$ and $\epsilon = 1.01$.

n	$\lvert A_{1,0}\rvert$	$\lvert A_{0,1}\rvert$	$\lvert A_{1,-1}\rvert$
1	0.026909	0.044490	0.049736
2	0.027388	0.045053	0.050313
3	0.027419	0.045093	0.050353
4	0.027421	0.045096	0.050356
5	0.027421	0.045096	0.050356

3 Results Analysis

First of all, the resonance condition (10) is analyzed for six resonant components, with the corresponding resonance curves shown in Figure 3. Note that the other five resonant curves overlap with the one we defined for the trio in (11). This means when the wave number k_1 changes along the resonance curve of trio (11), the resonance condition (10) may be exactly or nearly satisfied by other quartet, quintet, sextet, or octet. Remember Liu and Liao [15] found that the near-resonant components as a whole contain more and more wave energy as the nonlinearity of the resonant wave system increases. So for steady-state trio (11), other high-order exact or near resonant components may be involved into the resonance and the total energy would be distributed among more components.

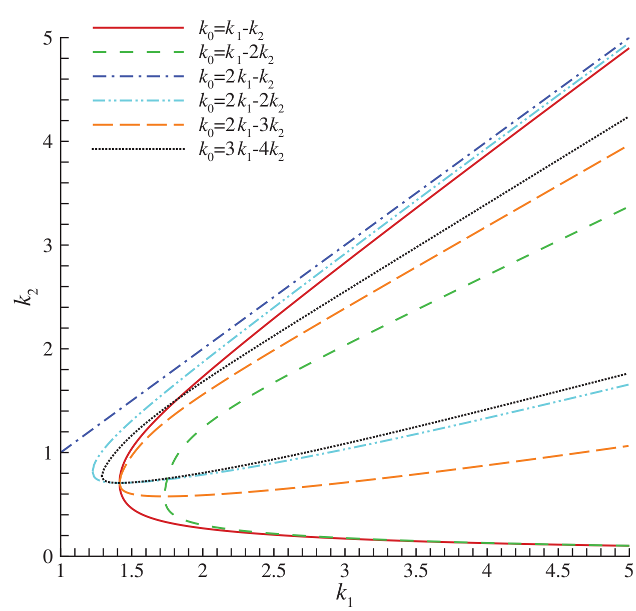

Figure 3: Linear resonance curves satisfying resonance condition (10).

Figure 4 shows the amplitude of two primary components $A_{1,0}$ and $A_{0,1}$ and the second-order resonant component $A_{1,-1}$ for $\sigma_i/\omega_i = 1.0001$, 1.001, 1.01, respectively. In all three cases, the amplitudes of three components keep increasing as k_1 changes along the resonance curve of trio (11). Besides, the amplitude of resonant component $A_{1,-1}$ is bigger than the amplitudes of second primary component $A_{0,1}$, which is then bigger than the amplitude of first primary component $A_{0,1}$. And the difference among amplitudes of three components is enlarged for bigger values of k_1. So, when components of a trio traveling in the same direction, the amplitude of longer component (with a smaller wave number) is larger than the amplitude of shorter component (with a bigger wave number).

The amplitudes of other high-order components are shown in Tables 2–4 for $\sigma_i/\omega_i = 1.0001$, 1.001, 1.01, respectively. Within each table, the values of $|A_{1,-2}|$, $|A_{2,-1}|$, $|A_{2,-2}|$, $|A_{2,-3}|$, and $|A_{3,-4}|$ increase for bigger value of k_1, which means that the amplitudes of all components increase simultaneously when the difference of wave number between long and short components in a resonant trio increases. Besides, as σ_i/ω_i increases from 1.0001 to 1.01, the amplitudes of all five high-order components keeps increasing, too. So for a steady-state trio, the near-resonant components do become increasingly important as the nonlinearity of whole system increases.

4 Conclusions and Discussions

In this article, we considered a trio in Bretherton equation to check if steady-state resonant solution exist not only for water wave system, but for any other system with weakly interacting waves in a dispersive medium. For all components travelling in the same direction, time-independent spectrum (with constant amplitude and actual wave frequency) is obtained for the first time in a resonant set. Within the trio, it is found that the amplitude of longer component is larger than the amplitude of shorter component. When the difference of wave number between long and short components in a trio increases, the amplitudes of all components increase simultaneously. Besides, the high-order near-resonant components become increasingly important as the nonlinearity of whole system increases.

Theoretically speaking, the number of singularities caused by linear resonance condition become infinity when components are travelling in one direction. Plenty of high-order resonant components would join in the

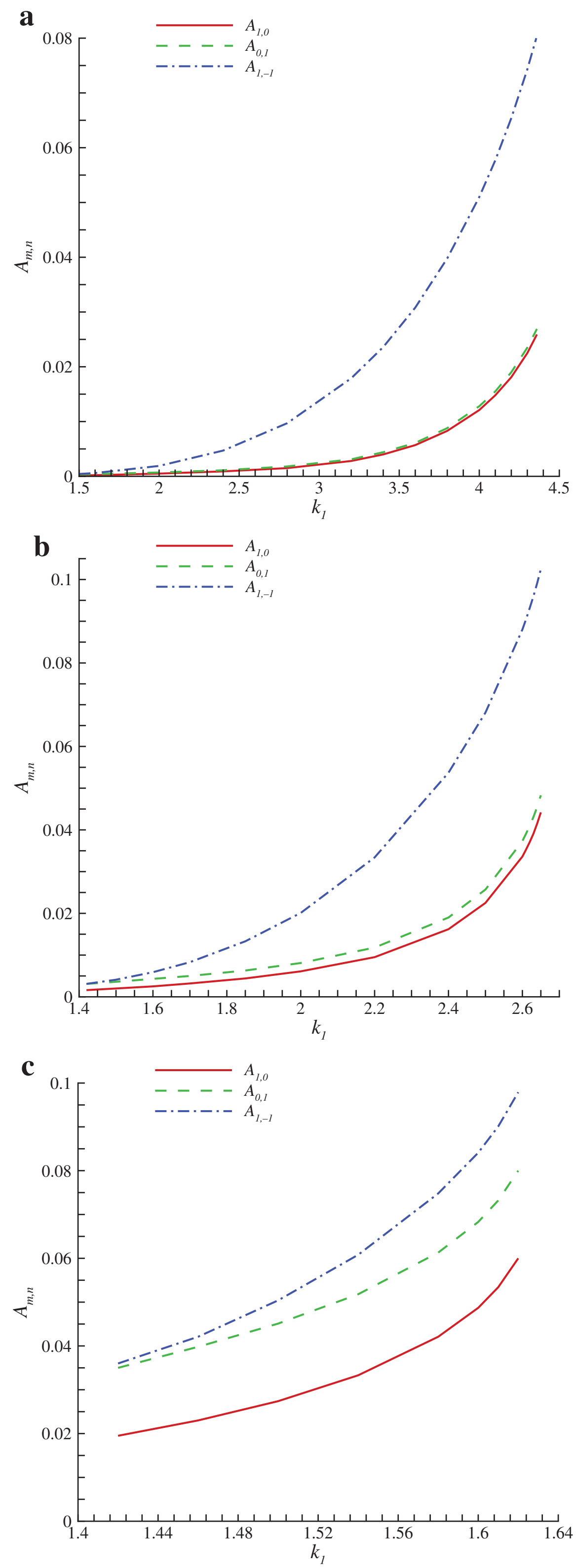

Figure 4: Wave amplitude in trio (11): (a) $\sigma_i/\omega_i = 1.0001$; (b) $\sigma_i/\omega_i = 1.001$; (c) $\sigma_i/\omega_i = 1.01$. Solid line: $|A_{1,0}|$ (first primary component); Dashed line: $|A_{0,1}|$ (second primary component); and Dash-dot line: $|A_{1,-1}|$ (2nd-order resonant component).

Table 2: Amplitudes of four high-order components in trio (11) for $\sigma_i/\omega_i=1.0001$.

k_1	$\lvert A_{1,-2}\rvert$	$\lvert A_{2,-1}\rvert$	$\lvert A_{2,-2}\rvert$	$\lvert A_{2,-3}\rvert$
3.2	0.0001			
3.4	0.0001	0.0001		
3.6	0.0002	0.0002		
3.8	0.0004	0.0003		
4	0.0008	0.0006	0.0004	
4.1	0.0011	0.0009	0.0005	
4.2	0.0015	0.0012	0.0007	
4.3	0.0021	0.0018	0.0009	0.0001
4.36	0.0027	0.0022	0.0010	0.0001

Table 3: Amplitudes of four high-order components in trio (11) for $\sigma_i/\omega_i=1.001$.

k_1	$\lvert A_{1,-2}\rvert$	$\lvert A_{2,-1}\rvert$	$\lvert A_{2,-2}\rvert$	$\lvert A_{2,-3}\rvert$
1.7	0.0001			
1.85	0.0001			
2	0.0003	0.0001		
2.2	0.0006	0.0003	0.0002	
2.4	0.0016	0.0008	0.0005	
2.5	0.0026	0.0014	0.0007	0.0001
2.6	0.0050	0.0028	0.0012	0.0002
2.61	0.0053	0.0030	0.0013	0.0003
2.62	0.0058	0.0033	0.0013	0.0003
2.63	0.0063	0.0036	0.0014	0.0003
2.64	0.0068	0.0040	0.0015	0.0004
2.65	0.0075	0.0044	0.0016	0.0004

Table 4: Amplitudes of five high-order components in trio (11) for $\sigma_i/\omega_i=1.01$.

k_1	$\lvert A_{1,-2}\rvert$	$\lvert A_{2,-1}\rvert$	$\lvert A_{2,-2}\rvert$	$\lvert A_{2,-3}\rvert$	$\lvert A_{3,-4}\rvert$
1.42	0.0013	0.0001	0.0005	0.0004	
1.46	0.0020	0.0002	0.0004	0.0003	
1.5	0.0030	0.0004	0.0005	0.0004	
1.54	0.0046	0.0007	0.0006	0.0006	
1.58	0.0072	0.0012	0.0009	0.0010	
1.6	0.0094	0.0017	0.0011	0.0014	0.0001
1.61	0.0110	0.0021	0.0012	0.0017	0.0001
1.62	0.0133	0.0027	0.0014	0.0022	0.0002

resonant interaction. So it is hard to even image the existence of steady-state solutions in two-dimensional system when resonance mechanism is triggered. We acknowledge the Bretherton equation (1) is a simple system compared with fully nonlinear water wave equation when considering the resonant interaction, whereas we, too, believe the work in this article helps for further investigation in steady-state resonant water waves, especially when all wave components traveling in one direction.

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Appendix

Let $\psi_0(\xi_1, \xi_2)$ denote a initial guess solution of $\psi(\xi_1, \xi_2)$, $\mathcal{L}$ an auxiliary linear operator, and $q\in[0, 1]$ an embedding parameter with no physical meaning, respectively. Then construct a family of solution $\Psi(\xi_1, \xi_2; q)$ in q by the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\Psi(\xi_1, \xi_2; q)-\psi_0(\xi_1, \xi_2)]=c_0 q\mathcal{N}[\Psi(\xi_1, \xi_2; q)], \quad \text{(a1)}$$

where $c_0\neq 0$ is the "convergence-control parameter" with no physical meaning. When $q=0$, we have

$$\Psi(\xi_1, \xi_2; 0)=\psi_0(\xi_1, \xi_2). \quad \text{(a2)}$$

When $q=1$, we have

$$\Psi(\xi_1, \xi_2; 1)=\psi(\xi_1, \xi_2). \quad \text{(a3)}$$

So, as q increases from 0 to 1, $\Psi(\xi_1, \xi_1; q)$ varies from the initial guess ψ_0 to the unknown $\psi(\xi_1, \xi_2)$. Assume the auxiliary linear operator $\mathcal{L}$, the initial guess $\psi_0(\xi_1, \xi_2)$ and the convergence-control parameter c_0 are properly chosen so that the Maclaurin series of $\Psi(\xi_1, \xi_2; q)$

$$\Psi(\xi_1, \xi_2; q)=\psi_0(\xi_1, \xi_2)+\sum_{m=1}^{+\infty}\psi_m(\xi_1, \xi_2)q^m, \quad \text{(a4)}$$

is convergent at $q=1$, where $\psi_m(\xi_1, \xi_2)=\frac{1}{m!}\left.\frac{\partial^m\Psi(\xi_1, \xi_2; q)}{\partial q^m}\right|_{q=0}$. Based on (a3) and (a4), we have

$$\psi(\xi_1, \xi_2)=\psi_0(\xi_1, \xi_2)+\sum_{m=1}^{+\infty}\psi_m(\xi_1, \xi_2). \quad \text{(a5)}$$

Substituting the Maclaurin series (a4) into the zeroth-order deformation equation (a1) and equating the like power of q, we can derive the high-order deformation equation

$$\mathcal{L}[\psi_m]=c_0 R_{m-1}+\chi_m\mathcal{L}[\psi_{m-1}], \quad \text{(a6)}$$

where $\chi_1=0$ and $\chi_m=1$ for $m>1$, and R_{m-1} is determined by the known approximations ψ_i $(i=0, 1, 2, \dots, m-1)$. The special solution of ψ_m is

$$\psi_m^* = \mathcal{L}^{-1}[c_0 R_{m-1}] + \chi_m \psi_{m-1}, \tag{a7}$$

where $\mathcal{L}^{-1}$ is the inverse operator of the auxiliary linear operator $\mathcal{L}$, which has the property

$$\mathcal{L}^{-1}[\cos(m\xi_1 + n\xi_2)] = \frac{\cos(m\xi_1 + n\xi_2)}{\lambda_{m,n}}. \tag{a8}$$

Note when the resonance condition (11) is satisfied, the auxiliary linear operator (7) has three eigenvalues being zero:

$$\lambda_{1,0} = \lambda_{0,1} = \lambda_{1,-1} = 0, \tag{a9}$$

which means $\cos(\xi_1)$, $\cos(\xi_2)$, and $\cos(\xi_1 - \xi_2)$ are all primary solutions. Thus, the common solution of ψ_m reads

$$\psi_m = \psi_m^* + A_m^{1,0}\cos(\xi_1) + A_m^{0,1}\cos(\xi_2) + A_m^{1,-1}\cos(\xi_1 - \xi_2), \tag{a10}$$

where ψ_m^* is a special solution defined in (a7), and $A_m^{1,0}$, $A_m^{0,1}$, and $A_m^{1,-1}$ are three constants to be determined.

To start the iteration for ψ_m, we define the initial guess

$$\psi_0 = A_0^{1,0}\cos(\xi_1) + A_0^{0,1}\cos(\xi_2) + A_0^{1,-1}\cos(\xi_1 - \xi_2), \tag{a11}$$

where $A_0^{1,0}$, $A_0^{0,1}$, and $A_0^{1,-1}$ are three constants, which are determined by enforcing the coefficients of terms $\cos(\xi_1)$, $\cos(\xi_2)$, and $\cos(\xi_1 - \xi_2)$ on the right-hand side of (a6) equal to 0, otherwise, secular terms will appear at the first-order approximation ψ_1. In this way, we solve three coupled, nonlinear algebraic equations about the unknown constants $A_0^{1,0}$, $A_0^{0,1}$, and $A_0^{1,-1}$. The detailed expression for ψ_0 can be obtained now and further calculation for ψ_i $(i>1)$ can be carried on in a similar way.

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