H.Q. Kafri, S.A. Khuri* and Ali Sayfy

Convective Fins Problem with Variable Thermal Conductivity: An Approach Based on Embedding Green's Functions into Fixed Point Iterative Schemes

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Abstract: This article introduces a new numerical approach to solve the equation that models a rectangular purely convecting fin with temperature-dependent thermal conductivity. The algorithm embeds an integral operator, defined in terms of Green's function, into Krasnoselskii–Mann's fixed point iteration scheme. The validity of the method is demonstrated by a number of examples that consist of a range of values of the parameters that appear in the model. In addition, the evaluation of the fin efficiency is presented. The residual error computations show that the current method provides highly accurate approximations.

Keywords: Convective Fins; Fixed Point Iterative Schemes; Green's Function; Thermal Conductivity.

1 Introduction

Fins are the extensions of the surfaces from which the heat transfers to the ambient fluid. These extended surfaces have shown a significant improvement of convective heat transfers that are frequently encountered in various applications. A detailed review of the convective heat transfer with fins and their industrial applications is provided by Kern and Kraus [1].

The mathematical model of heat transfers with fins is extensively studied in the literature. Heat transfer with constant thermal conductivity results in a linear equation that could be solved analytically. Yet, it was studied numerically by a number of researchers (see [2, 3] and references therein). On the other hand, large temperature variations would cause changes in thermal conductivity, producing a nonlinear equation whose closed form solution is difficult to obtain [4]. Various numerical methods were employed to tackle the nonlinear equation. Ganji et al. [5] applied the perturbation method, homotopy perturbation method, and variational iteration method to solve the problem. Chiu and Chen [6] implemented the decomposition method. Joneidi et al. [7] used differential transform method to handle the problem. Homotopy analysis method was proposed by Domairry and Fazeli [8] and Khani et al. [9]. Adomian decomposition method (ADM) was implemented by Arslanturk [10]. For other related papers that apply Adomian decomposition approach to heat transfer problems, see [11–13]. The tanh method has also been utilised to tackle nonlinear heat conduction problems (see [14] and the references therein). In [15] the convective-radiative T-shape fin with variant thermal conductivity is analytically studied by Torabi and Aziz. A regular perturbation solution for a straight convecting fin with temperature-dependent thermal conductivity is presented by Aziz and Huq in [16]. The spectral collocation method is investigated for the heat transfer in the following various demonstrations: a continuously moving convective-radiative rod with variable thermal conductivity [17], a convective-radiative fin with temperature-dependent properties [18] and a radiative-conductive porous fin with temperature-dependent properties [19].

In this study, we present a strategy that is based on the implementation of fixed point iterative scheme, namely, Krasnoselskii–Mann's, to a designated integral operator that is expressed in terms of Green's function. Our approach is motivated and inspired by the work of the authors in [20–22] who introduced this novel iterative scheme for solving a wide spectrum of nonlinear initial and boundary value problems. The article is organised as follows. Section 2 presents the mathematical model describing heat transfer with fins. Section 3 briefly introduces Green's functions and derives the scheme that embeds them into fixed point iteration formula. Section 4 presents the numerical results and demonstrates their accuracy by considering the residual error.

***Corresponding author: S.A. Khuri,** Department of Mathematics and Statistics, American University of Sharjah, Sharjah, UAE, Tel.: +97165152918, Fax: +97165152950, E-mail: skhoury@aus.edu
H.Q. Kafri and Ali Sayfy: Department of Mathematics and Statistics, American University of Sharjah, Sharjah, UAE

Section 5 includes a conclusion that briefly summarises the results.

2 Description of the Problem

In this section, we present a brief physical interpretation of the problem (see [8] for further details). Consider a straight fin with perimeter P, length b, and cross-sectional area A_c. The thermal conductivity of the fin depends linearly on temperature T as follows:

$$k(T) = k_a[1+\lambda(T-T_a)], \tag{1}$$

where k_a denotes the thermal conductivity of the fin at the ambient fluid temperature T_a and λ represents the variation of thermal conductivity. The fin, which is attached to a base surface of temperature T_b, extends into the ambient fluid, and its tip is insulated. The one-dimensional energy balance equation is written as

$$A_c \frac{\mathrm{d}}{\mathrm{d}x}\left[k(T)\frac{\mathrm{d}T}{\mathrm{d}x}\right] - Ph(T_b - T_a) = 0, \tag{2}$$

where h is the heat transfer coefficient. The equation is then reduced by introducing the following parameters

$$u = \frac{T-T_a}{T_b-T_a},\ t = \frac{x}{b},\ \epsilon = \lambda(T_b-T_a),\ N = \left(\frac{hPb^2}{k_a A_c}\right)^{\frac{1}{2}}. \tag{3}$$

The resulting equation is

$$[1+\epsilon u]u'' - N^2 u + \epsilon(u')^2 = 0, \tag{4}$$

subject to the boundary conditions

$$u'(0) = 0, \quad \text{and} \quad u(1) = 1. \tag{5}$$

2.1 Fin Efficiency

The efficiency of the fin is determined by taking the ratio of the actual heat transfer from the fin surface to that of the ideal. The actual heat transfer is obtained using Newton's law of cooling as

$$Q = \int_0^b P(T-T_a)\mathrm{d}x, \tag{6}$$

while the ideal transfer Q_{ideal} is the one that happens when the whole fin surface and the base share the same temperature. The fin efficiency η becomes

$$\eta = \frac{Q}{Q_{\text{ideal}}} = \frac{\int_0^b P(T-T_a)\mathrm{d}x}{Pb(T_b-T_a)} = \int_0^1 u(t)\mathrm{d}t. \tag{7}$$

3 Description of the Iterative Method

In this section, we start with an overview of the Green's function of a class of second order nonlinear boundary value problems (BVP), followed by the derivation of the algorithm that involves Green's functions and fixed point iteration schemes.

3.1 Overview of Green's Functions

Consider the following second order differential equation decomposed into a linear term, $L[u]$, and a nonlinear term, $f(t, u, u', u'')$, as follows

$$L[u] \equiv u'' - A^2 u = f(t, u, u', u''), \tag{8}$$

subject to the boundary conditions

$$\begin{cases} B_a[u] \equiv a_0 u(a) + a_1 u'(a) = \alpha, \\ B_b[u] \equiv b_0 u(b) + b_1 u'(b) = \beta, \end{cases} \tag{9}$$

where $a \le t \le b$ and A is a constant. The Green's function $G(t, s)$, corresponding to the linear term $L[u]$, is defined as the solution of the following BVP:

$$L[G(t,s)] = \delta(t-s),\ B_a[G(t,s)] = B_b[G(t,s)] = 0, \tag{10}$$

and takes on the following piecewise form

$$G(t,s) = \begin{cases} c_1 u_1 + c_2 u_2, & a \le t < s \\ d_1 u_1 + d_2 u_2, & s < t \le b \end{cases}, \tag{11}$$

where u_1 and u_2 form a fundamental set of solutions for $L[u] = 0$. The unknowns could be found using the corresponding homogenous conditions given in (9) and the fact that the Green's function is continuous and its derivative has a unit jump discontinuity (see [23]). More precisely, the constants are determined using the following properties:

A. G satisfies the corresponding homogeneous BCs:

$$B_a[G(t\,|\,s)] = B_b[G(t\,|\,s)] = 0, \tag{12}$$

B. G is continuous at $t = s$:

$$c_1 u_1(s) + c_2 u_2(s) = d_1 u_1(s) + d_2 u_2(s), \tag{13}$$

C. G' has a unit jump discontinuity at $t=s$:

$$d_1u_1'(s)+d_2u_2'(s)-c_1u_1'(s)-c_2u_2'(s)=1. \tag{14}$$

The Green's function serves as a particular solution to $u''-A^2u=f(t, u, u', u'')$ which satisfies the homogeneous boundary conditions and takes on the following structure

$$u_p=\int_a^b G(t, s)f(s, u_p, u_p', u_p'')\mathrm{d}s. \tag{15}$$

3.2 Green's Function and Fixed Point Iteration Scheme

To derive the Green's function embedded iterative scheme, we first define the following linear integral operator in terms of the Green's function and the particular solution, u_p. For convenience we set $u_p\equiv v$.

$$K[v]=\int_a^b G(t, s)[v''-A^2v]\mathrm{d}s. \tag{16}$$

Adding and subtracting $f(s, v, v', v'')$ from within the integral yields

$$\begin{aligned}K[v]&=\int_a^b G(t, s)(v''-A^2v-f(s, v, v', v''))\\&\quad+\int_a^b G(t, s)f(s, v, v', v'')\mathrm{d}s,\\&=v+\int_a^b G(t, s)(v''-A^2v-f(s, v, v', v''))\mathrm{d}s.\end{aligned} \tag{17}$$

The last equality in (17) follows from (15). The latter operator will be embedded in Krasnoselskii–Mann's fixed point iterative formula given in [24] by

$$v_{n+1}=(1-\alpha_n)v_n+\alpha_nK[v_n],$$

where $\{\alpha_n\}\subset(0, 1)$ is a real sequence satisfying appropriate conditions. Thus, we get Mann's Green's iterative scheme:

$$v_{n+1}=v_n+\alpha_n\left\{\int_a^b G(t, s)(v_n''-A^2v_n-f(s, v_n, v_n', v_n''))\mathrm{d}s\right\}, \tag{18}$$

for all $n\geq 0$. It is worth noting that the special case when $\alpha_n=1$ results in the well-known Picard's fixed point scheme. As for the initial iterate v_0, it is chosen such that it is a solution to $L[v]=0$ subject to the nonhomogeneous conditions, given the fact that Green's function satisfies the homogeneous boundary conditions.

The optimal values of the sequence $\{\alpha_n\}$ could be found either using trial and error strategy or by minimizing the L^2 norm of the residual error, $R_n(t; \alpha_n)$, of the nth iteration, u_n, given by

$$\|R_n(t; \alpha_n)\|_{L^2}^2=\frac{1}{b-a}\int_a^b R_n^2(t; \alpha_n)\mathrm{d}t, \tag{19}$$

where $R_n(t; \alpha_n)$ is the residual error

$$R_n(t; \alpha_n)\equiv u_n''(t)-A^2u_n(t)-f(t, u_n(t), u_n'(t), u_n''(t)). \tag{20}$$

4 Numerical Results

In this section, we will present our numerical results for the equation of rectangular purely convecting fin with temperature-dependent thermal conductivity. To apply the iterative algorithm (18) for our model problem given in (4) and (5), we rewrite (4) as follows:

$$u''-N^2u=-\epsilon(u')^2-\epsilon uu'', \tag{21}$$

where for this case $f(t, u, u', u'')=-\epsilon(u')^2-\epsilon uu''$. Thus, the iterative scheme (18) for our problem becomes

$$u_{n+1}=u_n+\alpha_n\left\{\int_a^b G(t, s)(u_n''-N^2u_n+\epsilon(u')^2+\epsilon u_nu_n'')\mathrm{d}s\right\}. \tag{22}$$

As for the initial iterate u_0, and as mentioned earlier, it is obtained by solving $u''-N^2u=0$ subject to the boundary conditions in (5) which yields $u_0=\mathrm{sech}(N)\cosh(Nt)$.

First, the Green's function, corresponding to the linear differential operator $u''-N^2u=0$, is to be constructed. Its two linearly independent solutions are $u_1(t)=e^{Nt}$ and $u_2(t)=e^{-Nt}$. Thus, upon using the properties of the Green's function, given in (11) and (12)–(14), it will take the following form:

$$G(t, s)=\begin{cases}\dfrac{(e^{N(s-1)}+e^{-N(s-1)})(e^{Nt}+e^{-Nt})}{2N(e^N+e^{-N})}, & 0\leq t<s\\[2ex]\dfrac{(e^{N(t-1)}-e^{-N(t-1)})(e^{Ns}+e^{-Ns})}{2N(e^N+e^{-N})}, & s<t\leq 1\end{cases}. \tag{23}$$

Substituting the latter Green's function into algorithm (22) yields the iterative scheme

$$\begin{aligned}u_{n+1}&=u_n+\alpha_n\frac{e^{N(t-1)}-e^{-N(t-1)}}{2N(e^N+e^{-N})}\int_0^t(e^{Ns}+e^{-Ns})([1+\epsilon u_n]u_n''-N^2u_n\\&\quad+\epsilon(u_n')^2)\mathrm{d}s+\alpha_n\frac{e^{Nt}+e^{-Nt}}{2N(e^N+e^{-N})}\int_t^1(e^{N(s-1)}+e^{-N(s-1)})\\&\quad([1+\epsilon u_n]u_n''-N^2u_n+\epsilon(u_n')^2)\mathrm{d}s,\end{aligned} \tag{24}$$

for $n=0, 1, 2, \ldots$, where the initial iterate u_0 is chosen as described above and is given by

$$u_0=\mathrm{sech}(N)\cosh(Nt)=\frac{e^{Nt}+e^{-Nt}}{e^N+e^{-N}}. \tag{25}$$

To find the best choice of α_n, we minimise the L^2 norm of the residual error, $R_n(t;\alpha_n)$, of the nth iteration, u_n. The residual error for the first iteration is given by

$$R_1(t;\alpha_1) \equiv u_1'' - N^2 u_1 + \epsilon u_1 u_1'' + \epsilon (u_1')^2. \tag{26}$$

The corresponding L^2 norm is given by

$$\| R_1(t;\alpha_1) \|_{L^2}^2 = \int_0^1 R_1^2(t;\alpha_1)\mathrm{d}t \tag{27}$$

which is to be minimised for α_1. Taking, for example, the case $\epsilon = 0.3$ and $N = 0.5$, we get $\alpha_1 = 0.901975$. In Figure 1 we plot the α_1 curve that confirms this optimal value.

The subsequent tables depict our numerical results for different selections of the parameters ϵ, variation of the thermal conductivity, and N, the thermo-geometric fin parameter. In addition, the evaluation of fin efficiency, η, is displayed and compared with the results of the ADM. Tables 1–3 display the approximate values of $u(t)$ obtained by our proposed Mann–Green Method (MGM) for a specified value of ϵ and various values of N. Corresponding to each N and ϵ, the nth iteration is carried out until the residual error reaches a value of at most 10^{-20}, which is the value of tolerance or the stopping criteria. Tables 4–6 report a comparison of the fin efficiency η, for different values of ϵ, between MGM and ADM. The results imply that the convergence of the method is slower for larger values of N. Yet, the CPU time remains small and negligible. It is also observed that when the thermal conductivity of the fins material, ϵ, increases the temperature $u(t)$ increases as well.

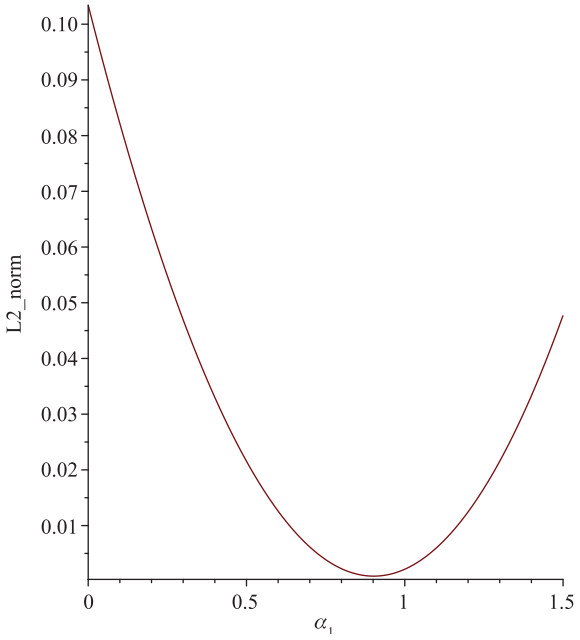

Figure 1: L^2 norm of the residual error, $R_1(t;\alpha_1)$, that depicts best choice of α_1.

5 Conclusion

In this paper, we introduced a novel technique to solve second order nonlinear heat transfer equation associated with variable thermal conductivity condition. The method incorporates Green's function and Mann's fixed point scheme. The solutions for selected values of the parameters in the equation and the corresponding fin efficiency were presented. In the absence of the exact solution, the high accuracy of the results was confirmed using residual error computations. In regard to the benefits and/or advantages of the proposed approach, unlike other existing methods, our proposed scheme provides highly accurate approximations with reasonable CPU time. Moreover, for the majority of other methods, such as differential transform method, spectral method, perturbation method, variation iteration method, and decomposition method, the approximations deteriorate when moving away from the initial endpoint, while when employing our strategy the errors are almost uniformly distributed over the domain. The Green's function is a piecewise-defined function that

Table 1: Approximate values for $u(t)$ with residual error 10^{-20}, for $\epsilon = 0.3$ and varying N.

t	$N=0.5, n=12$	$N=1, n=21$	$N=1.5, n=23$	$N=2, n=24$
0.1	0.9110145273890672968	0.7040965705774815940	0.4861507047068731107	0.3156044441702385142
0.2	0.9136962642767195999	0.7128008273880801020	0.5004200744620025031	0.3328107805818554538
0.3	0.9181679074166727508	0.7273458662436105667	0.5243934984808817427	0.3620007425482870426
0.4	0.9244325698015633969	0.7477880759963502394	0.5583541186946966696	0.4039372836413650518
0.5	0.9324945906975297821	0.7742053555938835350	0.6026911712757053398	0.4596750230602873599
0.6	0.9423595168898118528	0.8066960905751961337	0.6578927945782180437	0.5305441358351295200
0.7	0.9540340788099348710	0.8453778686714262781	0.7245367985350117152	0.6181265741595798546
0.8	0.9675261617215535391	0.8903859583583428447	0.8032794987386301972	0.7242234056812071001
0.9	0.9828447721781538506	0.9418715794780042253	0.8948428205482412094	0.8508128724994223146

Table 2: Approximate values for $u(t)$ with residual error 10^{-20}, for $\epsilon = 0.6$ and varying N.

t	$N=0.5, n=13$	$N=1, n=23$	$N=1.5, n=32$	$N=2, n=28$
0.1	0.7504582319191158410	0.7447724769702965246	0.5358701232174225119	0.3588747955057333225
0.2	0.7573546973641643544	0.7524769778495454628	0.5494941563577533683	0.3764680715431864944
0.3	0.7689322704211684642	0.7653222411228093326	0.5722702521052938287	0.4060875828497119974
0.4	0.7853189407889304222	0.7833147947909563294	0.6042992837565059940	0.4481629080278328063
0.5	0.8066996704916730494	0.8064635545332225772	0.6457163643869606779	0.5032637010795651600
0.6	0.8333226341254120177	0.8347796134608539900	0.6966853249331083656	0.5720708229866122708
0.7	0.8655079676976486896	0.8682759895629383759	0.7573923770941915319	0.6553432613171433221
0.8	0.9036596941883805649	0.9069673422146975059	0.8280394100953244257	0.7538835605923969069
0.9	0.9482818112558209894	0.9508696697567548830	0.9088373557191893229	0.8685045688442466995

Table 3: Approximate values for $u(t)$ with residual error 10^{-20}, for $\epsilon = 0.3$ and varying N.

t	$N=0.5, n=24$	$N=1, n=29$	$N=1.5, n=30$	$N=2, n=32$
0.1	0.8503776177098373046	0.5828484899025314185	0.3680367836885145086	0.2266950579963812878
0.2	0.8546611949113598808	0.5934537326608102242	0.3820037626438002596	0.2412730171158224088
0.3	0.8618330419115957930	0.6113653967848411046	0.4058283740927973900	0.2664295640080135613
0.4	0.8719427003060758125	0.6369508694112494105	0.4403735433410974105	0.3035399831270695098
0.5	0.8850609933363754017	0.6707521875488762237	0.4869423865421726450	0.3547193620907040472
0.6	0.9012815802521769421	0.7135181149870534368	0.5473950623529988052	0.4230648153220207452
0.7	0.9207230918557521204	0.7662528842552628167	0.6243394387688524078	0.5130694820879195703
0.8	0.9435319680898089057	0.8302899676708631604	0.7214486125227140274	0.6313580896192450505
0.9	0.9698861689132278449	0.9074056768332206711	0.8440158081150480245	0.7880941544469556568

Table 4: Comparison of the fin efficiency η, for the case $\epsilon = 0.3$, between MGM and ADM.

MGM	0.9400124386925088509	0.7996042259270025418	0.6483917552012893881	0.5239264520762520187
ADM	0.9404680680535717202	0.7982728814406818913	0.6482934151933514615	0.5263499844586178403

Table 5: Comparison of the fin efficiency η, for the case $\epsilon = 0.6$, between MGM and ADM.

t	$N=0.5, n=13$	$N=1, n=23$	$N=1.5, n=32$	$N=2, n=28$
MGM	0.9504607918461572431	0.8280032546465179919	0.6856238188182954092	0.5607498285833497693
ADM	0.9404680680535717202	0.8284696415591576352	0.6728967892394042266	0.5503887686991067316

Table 6: Comparison of the fin efficiency η, for the case $\epsilon = 0.3$, between MGM and ADM.

t	$N=0.5, n=24$	$N=1, n=29$	$N=1.5, n=30$	$N=2, n=32$
MGM	0.8981102295635205016	0.7094065712679741977	0.5487424572444185055	0.4338598836963126839
ADM	0.8986675174475007375	0.7128624012602439825	0.5552763131563923018	0.4428305460729853247

satisfies the boundary conditions at both ends of the interval; thus, it is designed for boundary value problems and a domain decomposition will be unnecessary. This is dissimilar to other methods which rely solely on one endpoint, so that the accuracy worsens as we approach the other endpoint.

Nomenclature

A_c	cross-sectional area of the fin	m^2
h	heat transfer coefficient	$Wm^{-2}K^{-1}$
b	fin length	m
u	dimensionless temperature	
x	distance measured from the fin tip	m
t	dimensionless coordinate	
T	temperature	K
k_a	thermal conductivity at the ambient fluid temperature	$Wm^{-1}K^{-1}$
P	fin perimeter	m
N	thermo-geometric fin parameter	
k_b	thermal conductivity at the base temperature	$Wm^{-1}K^{-1}$
Q	heat-transfer rate	W
k	thermal conductivity of the fin material	$Wm^{-1}K^{-1}$
n	Number of iterations used in the proposed method	
R_n	The residual error of the nth iteration	

Greek symbols

ϵ	variation of the thermal conductivity	
η	fin efficiency	
λ	slope of thermal conductivity–temperature curve	K^{-1}

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