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The Successive Application of the Gauge Transformation for the Modified Semidiscrete KP Hierarchy

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Abstract: In this article, the successive application of three gauge transformation operators for the modified semidiscrete Kadomtsev–Petviashvili(mdKP) hierarchy has been provided. The commutativity of the Bianchi diagram of these gauge transformation operators is investigated.

Keywords: Gauge Transformation; Modified Semidiscrete KP Hierarchy; Successive Application.

Mathematics Subject Classifications(2000): 37K10; 37K40; 35Q51; 35Q55.

1 Introduction

There are many efforts focussed on understanding the implications of the discrete integrable systems in the last three decades. The area of application includes from arrays of nonlinear-optical waveguides [1] to Bose–Einstein condensates (BECs) in periodic potentials [2] and dynamical models of the DNA double strand [3]. The semidiscrete Kadomtsev–Petviashvili (dKP) hierarchy [4–9] is an attractive research object in the field of the discrete integrable systems. The dKP hierarchy is defined by the difference derivative Δ instead of the usual derivative ∂ with respect to x in the classical system [10, 11], and the continuous spatial variable x is replaced by the discrete variable n. By using a nonuniform shift of space variable, the τ-function of the KP hierarchy implies a special kind of τ-function for the dKP hierarchy [5]. The ghost symmetry of the dKP hierarchy is constructed in [7]. The dKP hierarchy possesses an infinite dimensional algebraic structure [8]. Very recently, the continuum limit of the dKP hierarchy is given in [9]. Another important discrete integrable hierarchy is the modified dKP (mdKP) hierarchy [4, 12, 13], which is gauge connected to the dKP hierarchy [12, 13] and related to the Kac–van Moerbeke systems [14, 15].

Gauge transformation is one kind of powerful method to construct the solutions of the integrable systems for both the continuous KP hierarchy [16–21] and the dKP hierarchy[13, 22, 23]. The multifold of gauge transformation is expressed directly by determinants [21, 22]. This transformation is also applicable to the cKP hierarchy [21, 24–28]. The gauge transformation of the mdKP hierarchy is discussed in [13]. However, the determinant representation of the gauge transformation of the mdKP hierarchy has not been considered in literature.

The purpose of this article is to construct the successive applications of the gauge transformation operators of the mdKP hierarchy. There are three types of the gauge transformation operators T_1, T_2, and T_3 for the mdKP hierarchy, where T_3 is the composition of T_1 and T_2. In this article, the successive applications of the three types of the gauge transformations have been discussed, and the communities of these gauge transformation operators are also investigated. The difficulty of the gauge transformation of the mdKP hierarchy is the discretisation of the operator of the mdKP hierarchy. The discretisation of the operator causes the different Leibniz rule from the classic case, which leads to the second high-order coefficient of the dKP hierarchy and the nontrivial highest-order coefficient of the mdKP hierarchy. So, this research is different from the one for the gauge transformtion of the continuous KP hierarchy in [21] and has much more interesting. As for the mdKP hierarchy, there are more types of the gauge transformation than the dKP hierarchy case. What's more, the successive application cannot be easily obtained through the Miura transformation from the dKP hierarchy case. Thus, the result of this article is different from the one in the dKP hierarchy case [22].

This article is organised as follows. Some basic results of the mdKP hierarchy are summarised in Section 2. Three types of the gauge transformations of the mdKP hierarchy have been reviewed in Section 3. The successive applications of three types of the gauge transformations T_1, T_2, and

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T_3 have been discussed in Section 4. Based on the Banichi diagram, the communities of these gauge transformation operators have also been investigated in Section 5. Section 6 is devoted to the conclusions and discussions.

2 The Modified Semidiscrete KP Hierarchy

Let us briefly recall some basic formulas about the mdKP hierarchy according to [4, 5]. Firstly, a space F, namely

$$F=\{f(n)=f(n, t_1, t_2, \cdots, t_j, \cdots); n\in\mathbb{Z}, t_i\in\mathbb{R}\} \tag{1}$$

is defined for the space of the dKP hierarchy. Λ and Δ are denoted for the shift operator and the difference operator, respectively. Their actions on the function $f(n)$ are defined as

$$\Lambda f(n)=f(n+1) \tag{2}$$

and

$$\Delta f(n)=f(n+1)-f(n)=(\Lambda-I)f(n) \tag{3}$$

respectively, where I is the identity operator.

For any $j\in\mathbb{Z}$, the Leibniz rule of Δ operation is,

$$\Delta^j\circ f=\sum_{i=0}^{\infty}\binom{j}{i}(\Delta^i f)(n+j-i)\Delta^{j-i},\quad \binom{j}{i}=\frac{j(j-1)\cdots(j-i+1)}{i!}. \tag{4}$$

Specially,

$$\Delta^{-1}\circ f=\Lambda^{-1}(f)\Delta^{-1}-\Delta(\Lambda^{-2}f)\Delta^{-2}+2!\Delta^2(\Lambda^{-3}f)\Delta^{-3} -3!\Delta^3(\Lambda^{-4}f)\Delta^{-4}+\cdots. \tag{5}$$

So an associative ring $F(\Delta)$ of formal pseudo difference operators is obtained, with the operation "+" and "$\circ$", namely $F(\Delta)=\left\{\mathcal{R}=\sum_{j=-\infty}^{d} f_j(n)\Delta^j, f_j(n)\in\mathbb{R}, n\in\mathbb{Z}\right\}$. The adjoint operator to the Δ operator is given by Δ^*,

$$\Delta^*\circ f(n)=(\Lambda^{-1}-I)f(n)=f(n-1)-f(n), \tag{6}$$

where $\Lambda^{-1}f(n)=f(n-1)$, and the corresponding "$\circ$" operation is

$$\Delta^{*j}\circ f=\sum_{i=0}^{\infty}\binom{j}{i}(\Delta^{*i}f)(n+i-j)\Delta^{*j-i}. \tag{7}$$

Then the adjoint ring $F(\Delta^*)$ of the $F(\Delta)$ is obtained, and the formal adjoint to $\mathcal{R}\in F(\Delta)$ is defined by $\mathcal{R}^*\in F(\Delta^*)$ as $\mathcal{R}^*=\sum_{j=-\infty}^{d}\Delta^{*j}\circ f_j(n)$. The "$*$" operation satisfies the rules as $(F\circ G)^*=G^*\circ F^*$ for two operators F and G and $f(n)^*=f(n)$ for a function $f(n)$.

There are some useful properties for the difference operators as follows:

Lemma 2.1. *For $f\in F$, Δ and Λ as above, the following identities hold* [23].

$$(1)\quad \Delta\circ\Lambda=\Lambda\circ\Delta, \tag{8}$$

$$(2)\quad \Delta^*=-\Delta\circ\Lambda^{-1}, \tag{9}$$

$$(3)\quad (\Delta^{-1})^*=(\Delta^*)^{-1}=-\Lambda\circ\Delta^{-1}, \tag{10}$$

$$(4)\quad \Delta^{-1}\circ f\circ\Delta^{-1}=(\Delta^{-1}f)\circ\Delta^{-1}-\Delta^{-1}\circ\Lambda(\Delta^{-1}f), \tag{11}$$

$$(5)\quad \Delta\circ f(n)=\Lambda(f(n))\circ\Delta+\Delta(f(n)). \tag{12}$$

Next, one can consider the algebra

$$g=\left\{\sum_{i\ll\infty}u_i\Delta^i\right\}=\left\{\sum_{i\ge k}u_i\Delta^i\right\}\oplus\left\{\sum_{i<k}u_i\Delta^i\right\}\triangleq g_{\ge k}\oplus g_{<k}. \tag{13}$$

of Δ-pseduo-difference operator (Δ-PDO). When $k=0, 1$, $g_{\ge k}$, and $g_{<k}$ are sub-Lie algebras of g: $[g_{\ge k}, g_{\ge k}]\subseteq g_{\ge k}$ and $[g_{<k}, g_{<k}]\subset g_{<k}$. The projections of $A=\sum_i u_i\Delta^i$ are

$$A_{\ge k}=\sum_{i\ge k}u_i\Delta^i,\ A_{<k}=\sum_{i<k}u_i\Delta^i. \tag{14}$$

According to the famous Adler–Kostant–Symes scheme [29], the commuting of the Lax equations on g can be constructed as

$$\frac{\partial L}{\partial t_l}=[(L^l)_{\ge k}, L],\ l=1, 2, \cdots, \tag{15}$$

where $k=0$, 1 are corresponding to the dKP hierarchies and the mdKP hierarchies, respectively [4, 5, 12, 13]. Here, the Lax operators L are given by

$$L=\begin{cases}\Delta+f_0(n)+\sum_{j=1}^{\infty}f_{-j}(n)\Delta^{-j},\ k=0,\\ v_1\Delta+v_0(n)+\sum_{j=1}^{\infty}v_{-j}(n)\Delta^{-j},\ k=1,\end{cases} \tag{16}$$

where $f_i=f_i(n, x, t)=f_i(n, x, t_1, t_2, \ldots)$ and $v_i=v_i(n, x, t)=v_i(n, x, t_1, t_2, \ldots)$.

For $k=0$, 1, if the functions $\Phi=\Phi(x, t)$ satisfy the linear equations

$$\Phi_{t_l}=(L^l)_{\ge k}(\Phi),\ l=1, 2, 3, \ldots, \tag{17}$$

then Φ are eigenfunctions of the hierarchy of the Lax equations (15).

There is a gauge transformation connecting the dKP hierarchy and mdKP hierarchy showed in the following Lemma.

Lemma 2.2. *Let* $L\in g$ *be an Lax operator of the dKP hierarchy:* $\frac{\partial L}{\partial t_l}=[(L^l)_{\geq 0}, L]$. *If* $\Phi\neq 0$ *and* Ψ *are two eigenfunctions of the dKP hierarchy, then* $\tilde{L}=\Phi^{-1}L\Phi$ *satisfies the mdKP hierarchy:* $\frac{\partial \tilde{L}}{\partial t_l}=[(\tilde{L}^l)_{\geq 1}, \tilde{L}]$, *and* $\tilde{\Psi}=\Phi^{-1}\Psi$ *is an eigenfunction of the new mdKP hierarchy, i.e.* $\frac{\partial\tilde{\Psi}}{\partial t_l}=\tilde{L}^l_{\geq 1}(\tilde{\Psi})$

Proof. For any given Δ-PDO: $A\in g$,

$$\begin{aligned}(\Phi^{-1}A\Phi)_{\geq 1}&=(\Phi^{-1}A_{\geq 0}\Phi)_{\geq 1}\\&=(\Phi^{-1}A_{\geq 0}\Phi)_{\geq 0}-(\Phi^{-1}A_{\geq 0}\Phi)_{[0]}\\&=\Phi^{-1}A_{\geq 0}\Phi-\Phi^{-1}A_{\geq 0}(\Phi),\end{aligned}$$

where $A_{[0]}=u_0$ if $A=\sum_i u_i\Delta^{-i}$.

For $\tilde{L}=\Phi^{-1}L\Phi$ and $\tilde{\Psi}=\Phi^{-1}\Psi$, by means of the relationship of (15) and (17), then one can get

$$\begin{aligned}\tilde{L}_{t_l}-[(\tilde{L}^l)_{\geq 1}, \tilde{L}]&=\Phi^{-1}(L_{t_l}-[(L^l)_{\geq 0}, L])\Phi\\&\quad-[\Phi^{-1}(\Phi_{t_l}-(L^l)_{\geq 0}(\Phi)), L]=0,\end{aligned}\tag{18}$$

$$\begin{aligned}\tilde{\Psi}_{t_l}-(\tilde{L}^l)_{\geq 1}(\tilde{\Psi})&=-\Phi^{-2}\Psi(\Phi_{t_l}-(L^l)_{\geq 0}(\Phi))\\&\quad+\Phi^{-1}(\Psi_{t_l}-(L^l)_{\geq 0}(\Psi))=0.\end{aligned}\tag{19}$$

□

If the gauge transformation in Lemma 2.2 is applied to an Lax operator of the dKP hierarchy $L=\Delta+f_0(n)+\sum_{j=1}^{\infty}f_{-j}(n)\Delta^{-j}$, that is

$$\tilde{L}=\Phi^{-1}L\Phi=v_1\Delta+v_0(n)+\sum_{j=1}^{\infty}v_{-j}(n)\Delta^{-j},\tag{20}$$

where $\Phi\neq 0$ is an eigenfunction of the dKP hierarchy, then

$$v_1=\Phi^{-1}\Lambda(\Phi),\ v_0(n)=\Phi^{-1}\Delta(\Phi)+f_0(n),$$

$$\begin{aligned}&v_{-1}=f_{-1}(n)\Phi^{-1}\Lambda^{-1}(\Phi),\\&v_{-2}=f_{-2}(n)\Phi^{-1}\Lambda^{-2}(\Phi)-f_{-1}(n)\Phi^{-1}\Lambda^{-2}(\Delta\Phi).\end{aligned}\tag{21}$$

3 Gauge Transformations

We shall discuss the gauge transformations of the mdKP hierarchy in this section. For the mdKP hierarchy [see (15) and (16) with $k=1$], if there exists a Δ-PDO $T\in g$ and

$$L^{[1]}=T\circ L\circ T^{-1},\tag{22}$$

such that the transformed Lax operator $L^{[1]}$ satisfies

$$\frac{\partial L^{[1]}}{\partial t_l}=[(L^{[1]})^l_{\geq 1}, L^{[1]}],\tag{23}$$

then T is called a gauge transformation operator of the mdKP hierarchy. According to the definition of gauge transformation, we also have the following criterion.

Lemma 3.1. [23] *The operator T is a gauge transformation operator of the mdKP hierarchy, if*

$$(T\circ L^l\circ T^{-1})_{\geq 1}=T\circ(L^l)_{\geq 1}\circ T^{-1}+\frac{\partial T}{\partial t_l}\circ T^{-1}.\tag{24}$$

Similarly as [30], the following basic identities about the Δ-PDO are useful for the construction of the gauge transformation of mdKP hierarchy,.

Lemma 3.2. *For an* Δ*-PDO operator* $A\in g$ *and an arbitrary difference function* Φ, *it has the following operator identities:*

$$\begin{aligned}&(1).\ (\Phi^{-1}A\Phi)_{\geq 1}=\Phi^{-1}A_{\geq 1}\Phi-\Phi^{-1}A_{\geq 1}(\Phi).\\&(2).\ ((\Delta\Phi)^{-1}\cdot\Delta A\Delta^{-1}\cdot(\Delta\Phi))_{\geq 1}=(\Delta\Phi)^{-1}(\Delta A_{\geq 1}\Delta^{-1})(\Delta\Phi)\\&\qquad-(\Delta\Phi)^{-1}(\Delta A)_{\geq 1}(\Phi).\end{aligned}$$

Proof. By a direction calculation, the first identity can be got by,

$$\begin{aligned}(\Phi^{-1}A\Phi)_{\geq 1}&=(\Phi^{-1}A_{\geq 1}\Phi)_{\geq 1}\\&=(\Phi^{-1}A_{\geq 1}\Phi)_{\geq 0}-(\Phi^{-1}A_{\geq 1}\Phi)_{[0]}\\&=\Phi^{-1}A_{\geq 1}\Phi-\Phi^{-1}A_{\geq 1}(\Phi).\end{aligned}$$

And the second identity can be derivation by the following:

$$\begin{aligned}&((\Delta\Phi)^{-1}\Delta A\Delta^{-1}(\Delta\Phi))_{\geq 1}\\&=((\Delta\Phi)^{-1}(\Delta A_{\geq 1}\Delta^{-1})_{\geq 1}(\Delta\Phi))_{\geq 1}\\&=((\Delta\Phi)^{-1}(\Delta A_{\geq}\Delta^{-1})_{\geq 1}(\Delta\Phi))_{\geq 0}-((\Delta\Phi)^{-1}(\Delta A_{\geq 1}\Delta^{-1})_{\geq 1}(\Delta\Phi))_{[0]}\\&=(\Delta\Phi)^{-1}\cdot\Delta A_{\geq 1}\Delta^{-1}\cdot(\Delta\Phi)-(\Delta\Phi)^{-1}(\Delta A_{\geq 1})(\Delta\Phi).\end{aligned}$$

Different from the two types of gauge transformation operators of the mdKP hierarchy [23], there are two others elementary types of gauge transformation operators of the mdKP hierarchy in the following lemma:

Lemma 3.3. *The mdKP hierarchy has two types of gauge transformation operators, namely,*

$$(1).\quad T_1(\Phi)=\Phi^{-1},\tag{25}$$

$$(2).\quad T_2(\Phi)=(\Delta(\Phi))^{-1}\Delta.\tag{26}$$

Here Φ is an eigenfunction of L in (16) with $k=1$, which is called a generating function of the gauge transformation.

Proof. From (17) for $k=1$, there are

$$\frac{\partial T_1(\Phi)}{\partial t_l}T_1(\Phi)^{-1}=-\Phi^{-1}\frac{\partial T_1(\Phi)^{-1}}{\partial t_l}=-(\Phi)^{-1}\frac{\partial \Phi}{\partial t_l}=-(\Phi)^{-1}(L^l)_{\geq 1}(\Phi) \quad (27)$$

$$\frac{\partial T_2(\Phi)}{\partial t_l}T_2(\Phi)^{-1}=-(\Delta(\Phi_{t_l}))^{-1}\Delta\cdot\Delta^{-1}(\Delta\Phi)=-(\Delta((L^l)_{\geq 1}(\Phi))^{-1}\Delta\Phi \quad (28)$$

One can find that $T_1(\Phi)$ and $T_2(\Phi)$ satisfied (24) by letting $A=L^n$ in Lemma 3.2. So it is proved for Lemma 3.3. □

For an eigenfunction of the mdKP hierarchy Ψ, the action of the three gauge transformation operators $T_1(\Phi)$, $T_2(\Phi)$, and $T_3(\Phi)$ can be derived by the following lemma.

Lemma 3.4. *Assume $\Phi\neq 0$ (Φ is nonconstant for T_2), Ψ be an eigenfunction of the mdKP hierarchy, then the action of the gauge transformation operator $T_1(\Phi)$ and $T_2(\Phi)$ generate new eigenfunctions of the transformed mdKP hierarchy, i.e.,*

$$\Psi\xrightarrow{T_1(\Phi)}\Psi^{[1]}=T_1(\Phi)\Psi=\Phi^{-1}\Psi, \quad (29)$$

$$\Psi\xrightarrow{T_2(\Phi)}\Psi^{[1]}=T_2(\Phi)\Psi=(\Delta(\Phi))^{-1}\Delta(\Psi). \quad (30)$$

Proof. Under the gauge transformation operator $T_1(\Phi)$, (29) can be got by using the first identity in Lemma 3.2,

$$\begin{aligned}\Psi^{[1]}_{t_l}-(L^{[1]})^l_{\geq 1}(\Psi^{[1]})&=-\Phi^{-2}\Phi_{t_l}\Psi+\Phi^{-1}\Psi_{t_l}-\Phi^{-1}L^l_{\geq 1}(\Psi)\\&\quad+\Phi^{-1}L^l_{\geq 1}(\Phi)(\Phi^{-1}\Psi)\\&=\Phi^{-1}(\Psi_{t_l}-L^l_{\geq 1}(\Psi))-\Phi^{-2}\Psi(\Phi_{t_l}-L^l_{\geq 1}(\Phi))=0.\end{aligned}$$

With the help of the second identity in Lemma 3.2, (30) can be got as follows:

$$\begin{aligned}\Psi^{[1]}_{t_l}-(L^{[1]})^l_{\geq 1}(\Psi^{[1]})&=((\Delta(\Phi))^{-1}\Delta(\Psi))_{t_l}-(L^{[1]})^l_{\geq 1}((\Delta(\Phi))^{-1}\Delta(\Psi))\\&=(\Delta(\Phi))^{-1}(\Psi_{t_l}-L^l_{\geq 1}(\Psi))-(\Delta(\Phi))^{-2}\Delta(\Psi)\Delta(\Phi_{t_l}-L^l_{\geq 1}(\Phi))=0.\end{aligned}$$

From the above-mentioned analysis, one can get a new gauge transformation operator [13]

$$T_3(\Phi)=T_2(1^{[1]})T_1(\Phi)=(\Delta(\Phi^{-1}))^{-1}\cdot\Delta\cdot\Phi^{-1}, \quad (31)$$

where Φ is nonconstant and $1^{[1]}=T_1(\Phi)(1)=\Phi^{-1}$. Under the $T_3(\Phi)$, an arbitrary eigenfunction Ψ of the mdKP hierarchy becomes $\Psi^{[1]}$ by the way

$$\Psi\xrightarrow{T_3(\Phi)}\Psi^{[1]}=T_3(\Phi)\Psi=(\Delta(\Phi^{-1}))^{-1}\cdot\Delta(\Phi^{-1}\Psi). \quad (32)$$

4 The Successive Applications of the Gauge Transformations

We will discuss the successive transformations by using the three types of gauge transformation operators of the mdKP hierarchy. For k nonzero independent eigenfunctions of the mdKP hierarchy, Φ_1, Φ_2, …, Φ_k, consider a chain of the following gauge transformation operators, starting from the initial Lax operator L,

$$\begin{aligned}&L\xrightarrow{T_i(\Phi_1)}L^{[1]}\xrightarrow{T_i(\Phi_2^{[1]})}L^{[2]}\xrightarrow{T_i(\Phi_3^{[2]})}L^{[3]}\rightarrow\cdots\\&\rightarrow L^{[n-1]}\xrightarrow{T_i(\Phi_n^{[n-1]})}L^{[n]},\ i=1,\ 2,\ 3.\end{aligned} \quad (33)$$

Here, the $\Phi^{[k]}_j$ is transformed by k-steps gauge transformations from Φ_j, that is,

$$\Phi^{[k]}_j=T_i(\Phi^{[k-1]}_k)\circ\cdots\circ T_i(\Phi^{[2]}_3)\circ T_i(\Phi^{[1]}_2)T_i(\Phi_1)(\Phi_j). \quad (34)$$

The Lax operator $L^{[k]}$ is transformed by the k-steps gauge transformations from the initial Lax operator L.

Now we firstly consider the successive application of the gauge transformations in (33). We define the operator as

$$T^{[k]}_i\triangleq T_i(\Phi^{[k-1]}_k)\circ\cdots\circ T_i(\Phi^{[2]}_3)\circ T_i(\Phi^{[1]}_2)T_i(\Phi_1),\ i=1,\ 2,\ 3. \quad (35)$$

Case $i=1$. From (25), it has

$$\Phi^{[k-2]}_{k-1}\Phi^{[k-1]}_j=\Phi^{[k-2]}_j \quad (36)$$

with $\Phi^{[k]}_j=T_1(\Phi^{[k-1]}_k)(\Phi^{[k-1]}_j)$. So

$$\begin{aligned}T^{[k]}_1&=T_1(\Phi^{[k-1]}_k)\circ\cdots\circ T_1(\Phi^{[2]}_3)\circ T_1(\Phi^{[1]}_2)\circ T_1(\Phi_1)\\&=(\Phi_1\Phi^{[1]}_2\Phi^{[2]}_3\ldots\Phi^{[k-3]}_{k-2}\Phi^{[k-2]}_{k-1}\Phi^{[k-1]}_k)^{-1}\\&=(\Phi_1\Phi^{[1]}_2\Phi^{[2]}_3\ldots\Phi^{[k-3]}_{k-2}\Phi^{[k-2]}_k)^{-1}\\&=(\Phi_1\Phi^{[1]}_2\Phi^{[2]}_3\ldots\Phi^{[k-3]}_k)^{-1}\\&\ldots\\&=(\Phi_k)^{-1}.\end{aligned} \quad (37)$$

Case $i=2$. Assume $T^{[n]}_2=a_1\Delta+a_2\Delta^2+a_3\Delta^3+\cdots+a_n\Delta^n$. Because $T^{[n]}_2(\Phi_i)=0$, $i=1,\ 2,\ \ldots,\ n-1$ and $T^{[n]}_2(\Phi_n)=1$, then

$$\begin{cases}a_1\Delta(\Phi_1)+a_2\Delta^2(\Phi_1)+a_3\Delta^3(\Phi_1)+\cdots+a_n\Delta^n(\Phi_1)=0,\\a_1\Delta(\Phi_2)+a_2\Delta^2(\Phi_2)+a_3\Delta^3(\Phi_2)+\cdots+a_n\Delta^n(\Phi_2)=0,\\\vdots\\a_1\Delta(\Phi_{n-1})+a_2\Delta^2(\Phi_{n-1})+a_3\Delta^3(\Phi_{n-1})+\cdots+a_n\Delta^n(\Phi_{n-1})=0,\\a_1\Delta(\Phi_n)+a_2\Delta^2(\Phi_n)+a_3\Delta^3(\Phi_n)+\cdots+a_n\Delta^n(\Phi_n)=1.\end{cases} \quad (38)$$

It can be got

$$a_i=(-1)^{i+n}\begin{vmatrix}\Delta(\Phi_1) & \Delta(\Phi_2) & \cdots & \Delta(\Phi_{n-1})\\ \Delta^2(\Phi_1) & \Delta^2(\Phi_2) & \cdots & \Delta^2(\Phi_{n-1})\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{i-1}(\Phi_1) & \Delta^{i-1}(\Phi_2) & \cdots & \Delta^{i-1}(\Phi_{n-1})\\ \Delta^{i+1}(\Phi_1) & \Delta^{i+1}(\Phi_2) & \cdots & \Delta^{i+1}(\Phi_{n-1})\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{n}(\Phi_1) & \Delta^{n}(\Phi_2) & \cdots & \Delta^{n}(\Phi_{n-1})\end{vmatrix} \Bigg/ \begin{vmatrix}\Delta(\Phi_1) & \Delta(\Phi_2) & \cdots & \Delta(\Phi_{n})\\ \Delta^2(\Phi_1) & \Delta^2(\Phi_2) & \cdots & \Delta^2(\Phi_{n})\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{n}(\Phi_1) & \Delta^{n}(\Phi_2) & \cdots & \Delta^{n}(\Phi_{n})\end{vmatrix}. \tag{39}$$

Then

$$T_2^{[n]}=\frac{\begin{vmatrix}\Delta(\Phi_1) & \cdots & \Delta(\Phi_{n-1}) & \Delta\\ \Delta^2(\Phi_1) & \cdots & \Delta^2(\Phi_{n-1}) & \Delta^2\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{i}(\Phi_1) & \cdots & \Delta^{i}(\Phi_{n-1}) & \Delta^{i}\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{n}(\Phi_1) & \cdots & \Delta^{n}(\Phi_{n-1}) & \Delta^{n}\end{vmatrix}}{\begin{vmatrix}\Delta(\Phi_1) & \Delta(\Phi_2) & \cdots & \Delta(\Phi_{n})\\ \Delta^2(\Phi_1) & \Delta^2(\Phi_2) & \cdots & \Delta^2(\Phi_{n})\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{n}(\Phi_1) & \Delta^{n}(\Phi_2) & \cdots & \Delta^{n}(\Phi_{n})\end{vmatrix}}. \tag{40}$$

Case $i=3$. Assume $T_3^{[n]}=\Lambda(\Phi^{-1})(\Delta(\Phi^{-1}))^{-1}\Delta+1=\Phi(\Delta(\Phi))^{-1}\Delta+1$, it can be let $T_3^{[n]}=1+b_1\Delta+b_2\Delta^2+b_3\Delta^3+\cdots+b_n\Delta^n$. For considering $T_3^{[n]}(\Phi_i)=0$, $i=1, 2, \ldots, n$, then

$$\begin{cases} b_1\Delta(\Phi_1)+b_2\Delta^2(\Phi_1)+b_3\Delta^3(\Phi_1)+\cdots+b_n\Delta^n(\Phi_1)=-\Phi_1\\ b_1\Delta(\Phi_2)+b_2\Delta^2(\Phi_2)+b_3\Delta^3(\Phi_2)+\cdots+b_n\Delta^n(\Phi_2)=-\Phi_2\\ \cdots\\ b_1\Delta(\Phi_{n-1})+b_2\Delta^2(\Phi_{n-1})+b_3\Delta^3(\Phi_{n-1})+\cdots+b_n\Delta^n(\Phi_{n-1})=-\Phi_{n-1}\\ b_1\Delta(\Phi_n)+b_2\Delta^2(\Phi_n)+b_3\Delta^3(\Phi_n)+\cdots+b_n\Delta^n(\Phi_n)=-\Phi_n. \end{cases} \tag{41}$$

Solving the linear equations group,

$$b_i=(-1)^{i}\frac{\begin{vmatrix}\Phi_1 & \Phi_2 & \cdots & \Phi_n\\ \Delta(\Phi_1) & \Delta(\Phi_2) & \cdots & \Delta(\Phi_{n})\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{i-1}(\Phi_1) & \Delta^{i-1}(\Phi_2) & \cdots & \Delta^{i-1}(\Phi_{n})\\ \Delta^{i+1}(\Phi_1) & \Delta^{i+1}(\Phi_2) & \cdots & \Delta^{i+1}(\Phi_{n})\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{n}(\Phi_1) & \Delta^{n}(\Phi_2) & \cdots & \Delta^{n}(\Phi_{n})\end{vmatrix}}{\begin{vmatrix}\Delta(\Phi_1) & \Delta(\Phi_2) & \cdots & \Delta(\Phi_{n})\\ \Delta^2(\Phi_1) & \Delta^2(\Phi_2) & \cdots & \Delta^2(\Phi_{n})\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{n}(\Phi_1) & \Delta^{n}(\Phi_2) & \cdots & \Delta^{n}(\Phi_{n})\end{vmatrix}}. \tag{42}$$

Then

$$T_3^{[n]}=(-1)^{n}\frac{\begin{vmatrix}\Phi_1 & \cdots & \Phi_n & 1\\ \Delta(\Phi_1) & \cdots & \Delta(\Phi_{n}) & \Delta\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{i}(\Phi_1) & \cdots & \Delta^{i}(\Phi_{n}) & \Delta^{i}\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{n}(\Phi_1) & \cdots & \Delta^{n}(\Phi_{n} & \Delta^{n})\end{vmatrix}}{\begin{vmatrix}\Delta(\Phi_1) & \Delta(\Phi_2) & \cdots & \Delta(\Phi_{n})\\ \Delta^2(\Phi_1) & \Delta^2(\Phi_2) & \cdots & \Delta^2(\Phi_{n})\\ \vdots & \vdots & \ddots & \vdots\\ \Delta^{n}(\Phi_1) & \Delta^{n}(\Phi_2) & \cdots & \Delta^{n}(\Phi_{n})\end{vmatrix}}. \tag{43}$$

5 The Commutativity of the Bianchi Diagram

For the Lax operator L of the mdKP hierarchy and two nonzero independent eigenfunctions Φ_1 and Φ_1, we can consider the following diagram

$$\begin{array}{ccc} L,\ \Phi_1,\ \Phi_2 & \xrightarrow{T_i(\Phi_2)} & \bar{L},\ \bar{\Phi}_1\\ \Big\downarrow {\scriptstyle T_i(\Phi_1)} & & \Big\downarrow {\scriptstyle T_i(\bar{\Phi}_1)}\\ \tilde{L},\tilde{\Phi}_2 & \xrightarrow{T_i(\tilde{\Phi}_2)} & \tilde{\tilde{L}}\overset{?}{=}\bar{\bar{L}} \end{array}$$

where

$$\tilde{L}=T_i(\Phi_1)\circ L\circ T_i(\Phi_1)^{-1},\ \tilde{\Phi}_2=T_i(\Phi_1)(\Phi_2),$$
$$\bar{L}=T_i(\Phi_2)\circ L\circ T_i(\Phi_2)^{-1},\ \bar{\Phi}_1=T_i(\Phi_2)(\Phi_1),$$
$$\tilde{\tilde{L}}=T_i(\tilde{\Phi}_2)\circ \tilde{L}\circ T_i(\tilde{\Phi}_2)^{-1},\ \bar{\bar{L}}=T_i(\bar{\Phi}_1)\circ \bar{L}\circ T_i(\bar{\Phi}_1)^{-1}.$$

Whether this diagram will commute or not? That is $T_i^{[2]}(\Phi_1, \Phi_2)=T_i^{[2]}(\Phi_2, \Phi_1)$, $i=1, 2, 3$. It is an interesting question to discuss.

By means of (37), (40), and (43), we have

$$T_1^{[2]}(\Phi_1, \Phi_2)-T_1^{[2]}(\Phi_2, \Phi_1)=\Phi_2^{-1}-\Phi_1^{-1}\neq 0, \tag{44}$$

$$T_2^{[2]}(\Phi_1, \Phi_2) - T_2^{[2]}(\Phi_2, \Phi_1) = \frac{\begin{vmatrix} \Delta(\Phi_1+\Phi_2) & \Delta \\ \Delta^2(\Phi_1+\Phi_2) & \Delta^2 \end{vmatrix}}{\begin{vmatrix} \Delta(\Phi_1) & \Delta(\Phi_2) \\ \Delta^2(\Phi_1) & \Delta^2(\Phi_2) \end{vmatrix}} \neq 0, \tag{45}$$

$$T_3^{[2]}(\Phi_1, \Phi_2) - T_3^{[2]}(\Phi_2, \Phi_1) = \frac{\begin{vmatrix} \Phi_1 & \Phi_2 & 1 \\ \Delta(\Phi_1) & \Delta(\Phi_2) & \Delta \\ \Delta^2(\Phi_1) & \Delta^2(\Phi_2) & \Delta^2 \end{vmatrix}}{\begin{vmatrix} \Delta(\Phi_1) & \Delta(\Phi_2) \\ \Delta^2(\Phi_1) & \Delta^2(\Phi_2) \end{vmatrix}} - \frac{\begin{vmatrix} \Phi_2 & \Phi_1 & 1 \\ \Delta(\Phi_2) & \Delta(\Phi_1) & \Delta \\ \Delta^2(\Phi_2) & \Delta^2(\Phi_1) & \Delta^2 \end{vmatrix}}{\begin{vmatrix} \Delta(\Phi_2) & \Delta(\Phi_1) \\ \Delta^2(\Phi_2) & \Delta^2(\Phi_1) \end{vmatrix}} = 0. \tag{46}$$

So it is clearly that the diagram commutes only in the third gauge transformation.

6 Conclusions and Discussions

In this article, we have reviewed three types of the gauge transformation operators of the mdKP hierarchy as T_1 [see (25)], T_2 [see (26)], and T_3 [see (31)]. T_3 is the composition of T_1 and T_2. The successive application of these three types gauge transformations has been discussed, and the successive applications of the gauge transformations operators for the mdKP hierarchy are also given. In (37), (40), and (43), the result is different from the successive applications of the gauge transformtion of the continuous KP hierarchy [21, 22]. Then the communities of these gauge transformation operators are also investigated in (44–46). Only the operator T_3 commutes in the Bianchi diagram.

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References

[1] F. Lederer, G. I. Stegeman, D. N. Christodoulides, G. Assanto, M. Segev, et al., Phys. Rep. **463**, 1 (2008).
[2] O. Morsch and M. Oberthaler, Rev. Mod. Phys. **78**, 179 (2006).
[3] M. Peyrard, Nonlinearity **17**, R1 (2004).
[4] B. A. Kupershimidt, Astérisque **123**, 1 (1985).
[5] L. Haine and P. Iliev, Int. Math. Res. Not. **6**, 281 (2000).
[6] S. W. Liu and Y. Cheng, J. Phys. A: Math. Theor. **43**, 135202 (2010).
[7] C. Z. Li, J. P. Cheng, K. L. Tian, M. H. Li, and J. S. He, Monatshefte Für Mathematik, **11**, 1 (2015).
[8] X. L. Sun, D. J. Zhang, X. Y. Zhu, and D. Y. Chen, Mod. Phys. Lett. B **24**, 1033 (2010).
[9] W. Fu, L. Huang, K. M. Tamizhmani, and D. J. Zhang, Nonlinearity **26**, 3197 (2013).
[10] E. Date, M. Kashiwara, M. Jimbo, and T. Miwa, Nonlinear Integrable Systems – Classical and Quantum Theory, World Scientific, Singapore 1983, Vol. 39.
[11] L. A. Dickey, Soliton Equations and Hamiltonian Systems, 2nd ed., World Scintific, Singapore 2003.
[12] K. M. Tamizhmani and S. Kanaga Vel, Chaos Soliton Fract. **11**, 137 (2000).
[13] W. Oevel, in: Nonlinear Physics: Theory and Experiment, (Eds. E. Alfinito, L. Martina, and F. Pempinelli), World Scientific, Singapore 1996, p. 233.
[14] M. J. Ablowitz and P. A. Clarkson. Solitons, Nonlinear Equations and Inverse Scattering, Cambridge University Press, Cambridge 1991.
[15] J. Weiss, in: NATO ASI Series C: Mathematical and Physical Sciences, Vol. 310 (Eds. R. Conte and N. Boccara), Kluwer, Dordrecht 1990, p. 375.
[16] L. L. Chau, J. C. Shaw, and H. C. Yen, Commun. Math. Phys. **149**, 263 (1992).
[17] W. Oevel and W. Schief, in: Application of Nonlinear Differential Equations (Ed. P. A. Clarkson), Dordrecht, Kluwer Academic Publisher 1993, p. 193.
[18] J. J. Nimmo, in: Nonlinear Evolution Equation and Dynamical Systems (Eds. V. G. Makhankov and O. K. Pashaev), World Scientific, Singapore 1995, p. 168.
[19] W. Oevel and C. Rogers, Rev. Math. Phys. **5**, 299 (1993).
[20] W. X. Ma, Lett. Math. Phys. **39**, 33 (1997).
[21] J. S. He, Y. S. Li, and Y. Cheng, Chin. Ann. Math. B **23**, 475 (2002).
[22] S. W. Liu, Y. Cheng, and J. S. He, Sci. China Math. **53**, 1195 (2010).
[23] M. H. Li, J. P. Cheng, and J. S. He, Mod. Phys. Lett. B **27**, 1350043 (2013).
[24] W. Oevel, Phys. A **195**, 533 (1993).
[25] H. Aratyn, E. Nissimov, and S. Pacheva, Int. J. Mod. Phys. A **12**, 1265 (1997).
[26] L. L. Chau, J. C. Shaw, and M. H. Tu, J. Math. Phys. **38**, 4128 (1997).
[27] R. Willox, I. Loris, and C. R. Gilson, Inver. Prob. **13**, 849 (1997).
[28] J. S. He, Z. W. Wu, and Y. Cheng, J. Math. Phys. **48**, 113519(16pp) (2007).
[29] M. Adler, Invent. Math. **50**, 219 (1979).
[30] J. P. Cheng, J. Z. Wang, and X.Y. Zhang, J. Non. Math. Phys. **21.4**, 533 (2014).