Adrian Sfarti*

Massive Particle Reflection from Moving Mirrors

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Abstract: We investigate the reflection of massive particles from moving mirrors. The adoption of the formalism based on the energy-momentum allowed us to derive the most general set of formulas, valid for massive and, in the limit, also for massless particles. We show that the momentum change of the reflecting particle always lies along the normal to the mirror, independent of the mirror speed. The subject is interesting not only to physicists designing concentrators for fascicles of massive particles and electron microscopes but also to computer scientists working in raytracing operating in the photon sector. The paper, far from being only theoretical, has profound and novel practical applications in both domains of engineering design and computer science.

Keywords: Energy-Momentum Four-Vector; Ideal Mirrors; Massive Particles; Photons; Relativistic Equations of Reflection; Special Relativity; Wave-Vector Four-Vector.

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1 Introduction

The reflection of light (massless particles) from moving mirrors has been investigated extensively in the specialty literature, starting very early on with Bateman and Varicak [1, 2], continuing with Pauli [3] and ending with more recent works [4–13]. Even Einstein was interested in the subject [14]. Our paper reprises the issue in order to introduce the energy/momentum four-vector formalism for the case of massive particles reflected off totally reflective (ideal) mirrors. The subject is important to physicists designing concentrators for fascicles of massive particles and electron microscopes [15, 16] because, as opposed to the vast majority of references, we are interested in the way massive particles, rather than light, reflect off moving mirrors.

***Corresponding author: Adrian Sfarti,** 387 Soda Hall, CS Department, University of California, Berkeley, CA 94720, USA, E-mail: egas@pacbell.net

2 Massive Particles Reflection Off Ideal Mirrors

In order to deal correctly with massive particles, we needed to transition from one type of four-vector, the wave-vector used in the treatment for photons, to another type, the energy-momentum four-vector. As we will realise later on, the formulas derived using this formalism reduce in the limit to the ones derived in prior literature for the photon sector. The energy-momentum four-vectors for the incident and the reflected particle are, respectively,

$$\mathbf{W}_i = (cp_i \cos\theta_i,\ cp_i \sin\theta_i,\ 0,\ E_i) \tag{1}$$

$$\mathbf{W}_r = (cp_r \cos\theta_r,\ cp_r \sin\theta_r,\ 0,\ E_r) \tag{2}$$

In the above, $\mathbf{p}_i$ is the momentum of the incident particle, $\mathbf{p}_r$ is its momentum after reflection, E_i, E_r are the total energy before and after reflection, θ_i, θ_r are the angles made by the incident, respectively reflected direction of the particle with the normal to the mirror and c is the speed of light (see Fig. 1), all measured in the frame S, co-moving with the mirror. In the following, the mirror is assumed to be rigid, as opposed to reference [17], which assumes photon reflection off elastic mirrors. While reference [17] deals with photon (laser) reflection, our paper deals with both massive and massless particles.

The ideal reflection equations of energy-momentum conservation show that in the frame S, co-moving with the mirror,

$$\begin{aligned} &p_r \cos\theta_r = -p_i \cos\theta_i \\ &p_r \sin\theta_r = p_i \sin\theta_i \\ &E_r = E_i \end{aligned} \tag{3}$$

So, in frame S, co-moving with the mirror, we can write

$$\begin{aligned} &p_r = -p_i = p \\ &\theta_r = \theta_i = \theta \\ &E_r = E_i = E \end{aligned} \tag{4}$$

In frame S', moving with speed V with respect to frame S,

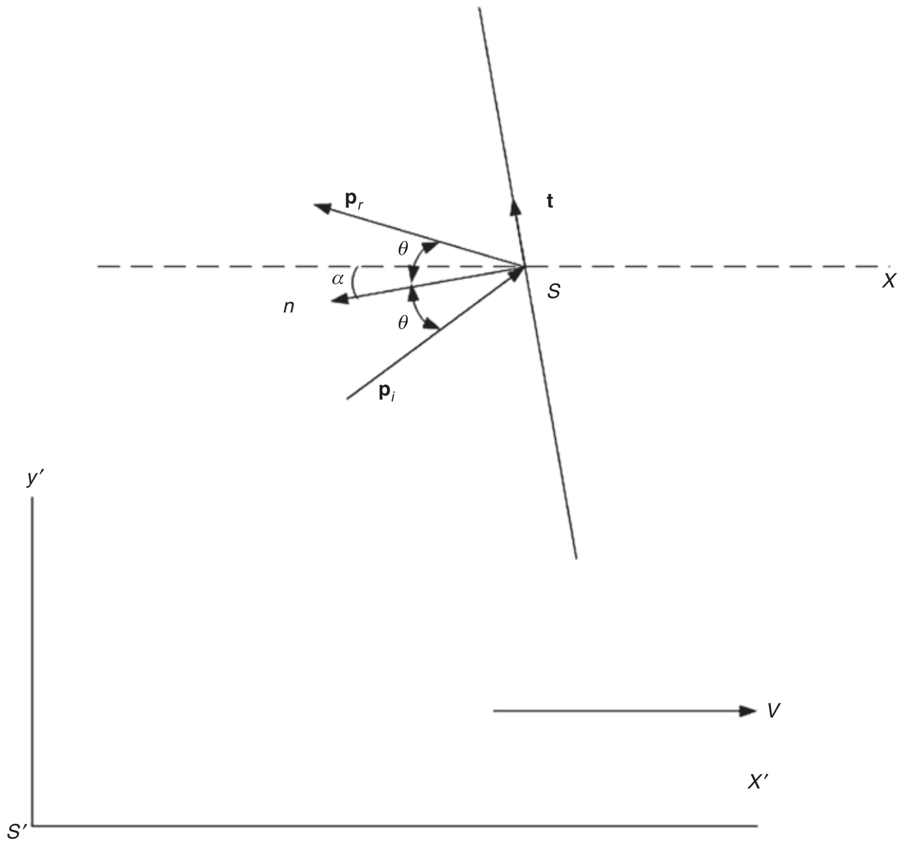

Figure 1: Reflection of a massive particle off a moving mirror.

$$
\begin{aligned}
&cp'_{ix}=\gamma(V)(cp_{ix}+\beta E_i)=\gamma(V)(cp\cos\theta+\beta E)\\
&cp'_{iy}=cp_{iy}=cp\sin\theta\\
&E'_i=\gamma(V)(E_i+\beta cp_{ix})=\gamma(V)(E+\beta cp\cos\theta)\\
&cp'_{rx}=\gamma(V)(cp_{rx}+\beta E_r)=\gamma(V)(-cp\cos\theta+\beta E)\\
&cp'_{ry}=cp_{ry}=-cp\sin\theta\\
&E'_r=\gamma(V)(E_r+\beta cp_{rx})=\gamma(V)(E-\beta cp\cos\theta)\\
&\beta=\frac{V}{c}\\
&\gamma(V)=\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}
\end{aligned}
\tag{5}
$$

We now have all the information necessary to derive the angles of incidence and reflection in frame S':

$$
\cos\theta'_i=\frac{p'_{ix}}{\sqrt{p'^2_{ix}+p'^2_{iy}}}=\frac{\cos\theta+\beta\frac{E}{pc}}{\sqrt{1+\beta^2\frac{m_0^2c^2}{p^2}+2\beta\frac{E}{pc}\cos\theta+\beta^2\cos^2\theta}}
\tag{6}
$$

$$
\cos\theta'_r=\frac{p'_{rx}}{\sqrt{p'^2_{rx}+p'^2_{ry}}}=\frac{-\cos\theta+\beta\frac{E}{pc}}{\sqrt{1+\beta^2\frac{m_0^2c^2}{p^2}-2\beta\frac{E}{pc}\cos\theta+\beta^2\cos^2\theta}}
\tag{7}
$$

Formulas (6) and (7) give the predicted angles of incidence and reflection in frame S' as a function of quantities $\mathbf{p}_i$, $\mathbf{p}_r$, E_i, E_r, θ_i and θ_r measured in the frame co-moving with the mirror. As the mirror moves with respect to the observer (with speed V), the formulas provide all the information necessary in finding out the incidence and reflection angle θ'_i, θ'_r in the frame of the lab.

From (6) and (7) we can generalise immediately for the more general case whereby the mirror makes an arbitrary angle α measured in the mirror frame with its direction of motion, as shown in Figure 1.

$$
\cos\theta'_i=\frac{\cos(\theta+\alpha)+\beta(V)\frac{E}{pc}}{\sqrt{1+\beta^2(V)\frac{m_0^2c^2}{p^2}+2\beta(V)\frac{E}{pc}\cos(\theta+\alpha)+\beta^2(V)\cos^2(\theta+\alpha)}}
\tag{8}
$$

$$\cos\theta_r' = \frac{-\cos(\theta-\alpha)+\beta(V)\dfrac{E}{pc}}{\sqrt{1+\beta^2(V)\dfrac{m_0^2c^2}{p^2}-2\beta(V)\dfrac{E}{pc}\cos(\theta-\alpha)+\beta^2(V)\cos^2(\theta-\alpha)}} \quad (9)$$

Formulas (8) and (9) give the predicted angles of incidence and reflection in frame S' as a function of quantities $\mathbf{p}_i$, $\mathbf{p}_r$, E_i, E_r, α, θ_i and θ_r measured in the frame co-moving with the mirror.

A quick sanity test proves that for $m_0=0$ and $E=pc$ we should recover the formulas derived for massless particles. For example, it is interesting to explore if massive particles behave the same way as photons when the mirror moves within its own plane, the simpler case studied in prior literature [1–3], i.e. for $\alpha=\frac{\pi}{2}$:

$$\cos\theta_i' = \frac{\beta-\sin\theta}{1-\beta\sin\theta} = \cos\theta_r' \quad (10)$$

The above result is used extensively in raytracing applications, where rays of light bounce off moving objects [9, 10].

The general equations for mirrors moving at arbitrary angles, as derived in [9, 10], are recovered as well:

$$\cos\theta_i' = \frac{\cos(\theta+\alpha)+\beta}{\sqrt{1+2\beta\cos(\theta+\alpha)+\beta^2\cos^2(\theta+\alpha)}} \quad (11)$$

$$\cos\theta_r' = \frac{-\cos(\theta-\alpha)+\beta}{\sqrt{1-2\beta\cos(\theta-\alpha)+\beta^2\cos^2(\theta-\alpha)}} \quad (12)$$

And yes, the massive particles behave exactly in the same manner as in the case of photons; the reflection angle is equal to the incident angle for the case of the mirror moving within its plane in the case of a massive particle, just like in the photon case.

Finally, we ask ourselves the question "Is the change in momentum $\Delta\mathbf{p}'$ perpendicular to the plane of the mirror in the moving frame S'?" This would mean that the dot product $\Delta\mathbf{p}'\cdot\mathbf{t}'$ is null, where $\mathbf{t}'=\left(\frac{\sin\alpha}{\gamma}, -\cos\alpha\right)$ is the tangent vector to the mirror in frame S' [because the vector tangent to the mirror in the frame S is obviously $\mathbf{t}=(\sin\alpha, -\cos\alpha)$].

$$\begin{aligned}\Delta\mathbf{p}'\cdot\mathbf{t}' &= \Delta p_x'\frac{\sin\alpha}{\gamma}-\Delta p_y'\cos\alpha\\ &= p\sin\alpha(\cos(\theta+\alpha)+\cos(\theta-\alpha))-p\cos\alpha(\sin(\theta+\alpha)\\ &\quad -\sin(\theta-\alpha))\\ &= 2p\sin\alpha\cos\alpha\cos\theta-2p\cos\alpha\cos\theta\sin\alpha=0\end{aligned} \quad (13)$$

Indeed, the change of momentum is perpendicular to the plane of the mirror.

The above shows that the change in momentum is perpendicular to the mirror independent of the angle of incidence, independent of the orientation of the mirror and independent of the inertial frame S', a generalisation of the result derived in [8, 17]. We can conclude that the momentum change of the reflecting particle always lies along the normal to the mirror, independent of the mirror speed. The present work concludes that this principle is also valid for the massive particles.

3 Physical, Practical Applications

The reflection of massive particles from the relativistically moving mirror was experimentally realised [15] for the case of electrons reflecting off mirrors made from nanoscale foils for the case of coherent frequency upshift to the extreme ultraviolet regime. The formulas obtained in the current paper are useful in the analysis of the reflection of an electron beam by a photon mirror [16]. In this case, $\beta(V)=1$, $\beta(v)=\frac{v}{c}$ and (8) and (9) yield the following interesting result:

$$\cos\theta_i' = \frac{\cos(\theta+\alpha)+\dfrac{1}{\beta(v)}}{\sqrt{1-\beta^2(v)+\left(\cos(\theta+\alpha)+\dfrac{1}{\beta(v)}\right)^2}} \quad (14)$$

$$\cos\theta_r' = \frac{-\cos(\theta-\alpha)+\dfrac{1}{\beta(v)}}{\sqrt{1-\beta^2(v)+\left(-\cos(\theta-\alpha)+\dfrac{1}{\beta(v)}\right)^2}} \quad (15)$$

Only for $\alpha=\frac{\pi}{2}$ the angle of reflection is equal to the angle of incidence:

$$\cos\theta_i' = \frac{\frac{1}{\beta(v)} - \sin\alpha}{\sqrt{1-\beta^2(v) + \left(\frac{1}{\beta(v)} - \sin\alpha\right)^2}} = \cos\theta_r' \tag{16}$$

In general, the two angles are not equal; the angle difference is largest for $\alpha = 0$:

$$\cos\theta_i' = \frac{\cos\theta + \frac{1}{\beta(v)}}{\sqrt{1-\beta^2(v) + \left(\cos\theta + \frac{1}{\beta(v)}\right)^2}} \tag{17}$$

$$\cos\theta_r' = \frac{-\cos\theta + \frac{1}{\beta(v)}}{\sqrt{1-\beta^2(v) + \left(-\cos\theta + \frac{1}{\beta(v)}\right)^2}} \tag{18}$$

The case of photon reflection comes up in the case of applications of raytracing [9, 10] since the procedures deal with light being reflected by (moving) objects.

4 Conclusions

In the present paper we presented a general method for solving the relativistic equations of reflection from a moving mirror for both the cases of massive and massless particles. The formalism based on the energy-momentum allowed us to derive the most general set of formulas, valid for massive and, in the limit, also for massless particles (photons). We have found that the change in momentum is perpendicular to the mirror independent of the angle of incidence, independent of the orientation of the mirror and independent of the inertial frame S', a generalisation of the result derived in [8]. We can conclude that the momentum change of the reflecting particle always lies along the normal to the mirror, independent of the mirror speed. The present work concludes that this principle is also valid for the massive particles. The subject is interesting not only to physicists designing concentrators for fascicles of massive particles but also to computer scientists working in raytracing operating in the photon sector [9, 10].

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