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Exact Solutions for a Coupled Korteweg–de Vries System

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Abstract: Korteweg–de Vries (KdV)-type equation can be used to characterise the dynamic behaviours of the shallow water waves and interfacial waves in the two-layer fluid with gradually varying depth. In this article, by virtue of the bilinear forms, rational solutions and three kind shapes (soliton-like, kink and bell, anti-bell, and bell shapes) for the *N*th-order soliton-like solutions of a coupled KdV system are derived. Propagation and interaction of the solitons are analyzed: (1) Potential u shows three kind of shapes (soliton-like, kink, and anti-bell shapes); Potential v exhibits two type of shapes (soliton-like and bell shapes); (2) Interaction of the potentials u and v both display the fusion phenomena.

Keywords: Bilinear Forms; Multi-Soliton Solutions; Rational Solutions.

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1 Introduction

Nonlinear evolution equations (NLEEs) are used to describe the nonlinear physical phenomena in the nature. Among those models, Korteweg–de Vries (KdV)-type [1–3] equation can be used to characterise the dynamic behaviours of the shallow water waves, ion-acoustic waves in the plasmas, interfacial waves in the two-layer liquids with gradually varying depth, Alfv́en waves in the interaction plasmas, and acoustic waves in the anharmonic lattices [4–9].

Besides, several results of the KdV-type equation have been reported over the past few decades [10–15]. For example, the integrability aspects, perturbation theories, numerical solutions, conservation laws, analytical solutions, and various other aspects have all been addressed. Among those results, soltion solutions and rational solutions have been studied because these have some special properties. Rational solutions are used to describe the rogue waves in the fluid mechanics and nonlinear optics [16, 17]. A rogue wave is thought of as an isolated huge wave with the amplitude much larger than the average wave crests around it in the ocean [17] and also seen in the Bose–Einstein condensates, optics, and superfluids [18, 19]. A soliton is a solitary wave that preserves its velocity and shape after the interaction [20], i.e. the soliton can be considered as a quasi-particle [21, 22].

In this article, we will work on a coupled KdV system, which is impressive in [23–27] and interest in the fluid physics [25, 28]

$$u_t-2u_xv-(3\sigma-4)u_{xxx}-2uu_x-4uv_x+3(\sigma-2)^2v_{xxx}+2(\sigma-1)\sigma vv_x=0, \quad (1)$$

$$v_t-2u_xv-3u_{xxx}-2uv_x-(8-4\sigma)vv_x-(8-3\sigma)v_{xxx}=0, \quad (2)$$

where the potential functions u and v are both the real functions of the scaled temporal coordinate t and spatial coordinate x and σ is an arbitrary constant. Lax pair by virtue of the prolongation structure, Painlevé analysis and soliton solutions via the Bäcklund transformation of (1) have all been attained [28].

To our knowledge, soliton solutions and rational solutions of (1) via the bilinear forms have not been obtained. With the aid of symbolic computation [29–31], in Section 2, Bilinear forms, rational solutions and three kind shapes of the *N*th-order soliton-like solutions for (1) will be attained. In Section 3, wave propagation and interaction will be discussed. Section 4 will be the conclusions.

2 Bilinear Forms and Exact Solutions

Through the following dependent variable transformations,

$$u=\frac{3(3-\sigma)f_x^2}{f^2}+\frac{3(\sigma-2)f_{xx}}{f}, \quad (3)$$

$$v=3\log(f)_{xx}, \quad (4)$$

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where f is a function of the variables x and t, (1) becomes the trilinear forms as

$$(\sigma-2)(f_{xxt}-2f_{xxxxx})f^2-2f[(\sigma-3)f_x(f_{xt}-2f_{xxxx}) -(\sigma-2)f_{xx}f_{xxx}]+f_t[2(\sigma-3)f_x^2 -(\sigma-2)ff_{xx}]-4(\sigma-3)f_x^2f_{xxx}=0, \quad (5)$$

$$f_tf_x-2f_{xxx}f_x-ff_{xt}+2ff_{xxxx}=0. \quad (6)$$

We note that: When $\sigma=2$, Expression (5) can degenerate into Expression (6); When $\sigma\neq 2$, Expression (5) can be simplified as $\frac{(5)-2(\sigma-3)f_x(6)}{(\sigma-2)f}$, where (5) is Expression (5) and (6) is Expression (6). Thus, Expressions (5 and 6) can be simplified to the bilinear forms,

$$2f_{xxx}f_{xx}-f_tf_{xx}+ff_{xxt}-2ff_{xxxxx}=0, \quad (7)$$

$$f_tf_x-2f_{xxx}f_x-ff_{xt}+2ff_{xxxx}=0. \quad (8)$$

We expand f into the power serious of a small parameter ϵ as

$$f=1+\epsilon f_1+\epsilon^2 f_2+\cdots \quad (9)$$

where f_i's ($i=1, 2, \ldots$) are all the real functions of x and t.

2.1 Rational Solutions

In order to obtain the rational solutions for (1), we assume that

$$f_1=l_3x^3+l_2x^2+l_1x+l_0t+\xi,\ \ f_2=0, f_3=0, \cdots \quad (10)$$

where l_m's ($m=0, 1, 2, 3$) and ξ are all the real numbers. Substituting Expression (10) into Expressions (7 and 8), we derive out the rational solutions of (1),

$$u=\frac{6(\sigma-2)(l_2+3xl_3)f_1-3(\sigma-3)[l_1+x(2l_2+3xl_3)]^2}{(f_1)^2}, \quad (11)$$

$$v=\frac{3\{2(l_2+3xl_3)f_1-[l_1+x(2l_2+3xl_3)]^2\}}{(f_1)^2}. \quad (12)$$

2.2 Soliton Solutions

In order to attain the first-order soliton-like solutions of (1), we assume that

$$f_1=Q_1e^{\eta_1}+P_1e^{-\eta_1}+R\cos\gamma, \quad (13)$$

where $\eta_1=k_1x+w_1t+\delta_1$ and $\gamma=kx-wt$, while Q_1, P_1, R, k_1, w_1, k, w, and δ_1 are all the real constants. Substituting Expression (13) into Expressions (7 and 8), we derive out the relationship between k_1, w_1, k, and w as

$$w_1=2k_1^3,\ \ w=2k^3, \quad (14)$$

which is called the nonlinear dispersion relation [32].

Similarly, in order to derive the second-order soliton-like solutions, we set

$$f_1=Q_1e^{\eta_1}+Q_2e^{\eta_2}+P_1e^{-\eta_1}+P_2e^{-\eta_2}+R\cos\gamma, \quad (15)$$

where $\eta_j=k_jx+w_jt+\delta_j$ ($j=1, 2$), while Q_2, P_2, k_2, w_2, and δ_2 are all the real constants. Substituting Expression (15) into Expressions (7 and 8), we have

$$w_2=2k_2^3. \quad (16)$$

Thus, we can obtain the second-order soliton-like solutions for (1) as

$$u=\frac{u_{21}}{u_{22}^2}, \quad (17)$$

$$v=\frac{v_{21}}{u_{22}^2}, \quad (18)$$

where $u_{21}=3(3-\sigma)[-kR\sin[k(2kt+x)]+k_1(-e^{-xk_1-2tk_1^3-\delta_1}P_1 +e^{xk_1+2tk_1^3+\delta_1}Q_1)+k_2(-e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_2+2tk_2^3+\delta_2}Q_2)]^2+3(\sigma-2) [1+R\cos[k(2kt+x)]+e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_1+2tk_1^3+\delta_1} Q_1+e^{xk_2+2tk_2^3+\delta_2}Q_2][-k^2R\cos[k(2kt+x)]+k_1^2(e^{-xk_1-2tk_1^3-\delta_1}P_1 +e^{xk_1+2tk_1^3+\delta_1}Q_1)+k_2^2(e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_2+2tk_2^3+\delta_2}Q_2)]$,

$$u_{22}=\{1+R\cos[k(2kt+x)]+e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{-xk_2-2tk_2^3-\delta_2}P_2 +e^{xk_1+2tk_1^3+\delta_1}Q_1+e^{xk_2+2tk_2^3+\delta_2}Q_2\}^2,$$

$$v_{21}=3\{-[-kR\sin[k(2kt+x)]+k_1(-e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{xk_1+2tk_1^3+\delta_1}Q_1) +k_2(-e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_2+2tk_2^3+\delta_2}Q_2)]^2+[1+R\cos[k(2kt+x)] +e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_1+2tk_1^3+\delta_1}Q_1+e^{xk_2+2tk_2^3+\delta_2}Q_2] [-k^2R\cos[k(2kt+x)]+k_1^2(e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{xk_1+2tk_1^3+\delta_1}Q_1) +k_2^2(e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_2+2tk_2^3+\delta_2}Q_2)]\}.$$

This process can be continued for us to derive the third-order soliton-like solutions, which can be written as

$$u=\frac{u_{31}}{u_{32}^2},\tag{19}$$

$$v=\frac{v_{31}}{u_{32}^2},\tag{20}$$

where $u_{31}=3(3-\sigma)\{-kR\sin[k(2kt+x)]-e^{-xk_3-2tk_3^3-\delta_3}k_3P_3+k_1(-e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{xk_1+2tk_1^3+\delta_1}Q_1)+k_2(-e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_2+2tk_2^3+\delta_2}Q_2)+e^{xk_3+2tk_3^3+\delta_3}k_3Q_3\}^2+3(-2+\sigma)\{1+R\cos[k(2kt+x)]+e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{-xk_3-2tk_3^3-\delta_3}P_3+e^{xk_1+2tk_1^3+\delta_1}Q_1+e^{xk_2+2tk_2^3+\delta_2}Q_2+e^{xk_3+2tk_3^3+\delta_3}Q_3\}\{-k^2R\cos[k(2kt+x)]+e^{-xk_3-2tk_3^3-\delta_3}k_3^2P_3+k_1^2(e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{xk_1+2tk_1^3+\delta_1}Q_1)+k_2^2(e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_2+2tk_2^3+\delta_2}Q_2)+e^{xk_3+2tk_3^3+\delta_3}k_3^2Q_3\}$,

$u_{32}=1+R\cos[k(2kt+x)]+e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{-xk_3-2tk_3^3-\delta_3}P_3+e^{xk_1+2tk_1^3+\delta_1}Q_1+e^{xk_2+2tk_2^3+\delta_2}Q_2+e^{xk_3+2tk_3^3+\delta_3}Q_3$,

$v_{31}=3\{-(-kR\sin[k(2kt+x)]-e^{-xk_3-2tk_3^3-\delta_3}k_3P_3+k_1(-e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{xk_1+2tk_1^3+\delta_1}Q_1)+k_2(-e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_2+2tk_2^3+\delta_2}Q_2)+e^{xk_3+2tk_3^3+\delta_3}k_3Q_3)^2+(1+R\cos[k(2kt+x)]+e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{-xk_3-2tk_3^3-\delta_3}P_3+e^{xk_1+2tk_1^3+\delta_1}Q_1+e^{xk_2+2tk_2^3+\delta_2}Q_2+e^{xk_3+2tk_3^3+\delta_3}Q_3)(-k^2R\cos[k(2kt+x)]+e^{-xk_3-2tk_3^3-\delta_3}k_3^2P_3+k_1^2(e^{-xk_1-2tk_1^3-\delta_1}P_1+e^{xk_1+2tk_1^3+\delta_1}Q_1)+k_2^2(e^{-xk_2-2tk_2^3-\delta_2}P_2+e^{xk_2+2tk_2^3+\delta_2}Q_2)+e^{xk_3+2tk_3^3+\delta_3}k_3^2Q_3)\}$.

If we continue such process, the Nth-order soliton-like ($N\geq 1$) solutions of (1) are

$$u=\frac{3(3-\sigma)f_x^2}{f^2}+\frac{3(\sigma-2)f_{xx}}{f},\tag{21}$$

$$v=3\log(f)_{xx},\tag{22}$$

where $f=1+\sum_{j=1}^{N}(Q_je^{k_jx+2k_j^3t+\delta_j}+P_je^{-k_jx-2k_j^3t-\delta_j})+R\cos\gamma$,

while Q_j, P_j, k_j and δ_j ($j=1, 2, \dots, N$) are all the real constants.

We note that the values of Q_j, P_j, and R can affect the shape, propagation, and interaction of the solitons. Thus, three type shapes (soliton-like, kink and bell, anti-bell, and bell shapes) of the soliton solutions for (1) will be obtained in the following.

Special case I

If $Q_jP_jR\neq 0$ in Expressions (21 and 22), the N-order soliton-like solutions of (1) will be obtained. For example, when $Q_1=P_1=R=1$, the first-order soliton-like solutions are

$$u=\frac{-3[(\sigma-3)(k\sin\gamma+2k_1\sinh\eta_1)^2+(\sigma-2)(\cos\gamma+2\cosh\eta_1)(k^2\cos\gamma-2k_1^2\cosh\eta_1)]}{(\cos\gamma+2\cosh\eta_1)^2},\tag{23}$$

$$v=\frac{3e^{\eta_1}\{-k^2[e^{\eta_1}+(1+e^{2\eta_1})\cos\gamma]-2kk_1(e^{2\eta_1}-1)\sin\gamma+[4e^{\eta_1}+(1+e^{2\eta_1})\cos\gamma]k_1^2\}}{(1+e^{2\eta_1}+e^{\eta_1}\cos\gamma)^2},\tag{24}$$

where $\gamma=kx+2k^3t$ and $\eta_1=k_1x+2k_1^3t+\delta_1$.

Special case II

If we take $R=P_j=f_j=0$ and $\epsilon=Q_j=1$ in Expressions (21 and 22), the Nth-order kink and bell-shape soliton-like solutions for (1) can be attained. Such as, the one-order solutions are

$$u=\frac{3e^{xk_1+2tk_1^3+\delta_1}(\sigma-2+e^{xk_1+2tk_1^3+\delta_1})k_1^2}{(1+e^{xk_1+2tk_1^3+\delta_1})^2},\tag{25}$$

$$v=\frac{3k_1^2}{2[1+\cosh(xk_1+2tk_1^3+\delta_1)]}.\tag{26}$$

Special case III

If we take $R=f_j=0$ and $\epsilon=Q_j=P_j=1$ in Expressions (21 and 22), the Nth-order anti-bell and bell-shape soliton solutions for (1) can be obtained. For example, the one solitons are

$$u=\frac{6k_1^2\{2\sigma-5+(\sigma-2)\mathrm{Cosh}(xk_1+2tk_1^3+\delta_1)+\mathrm{Cosh}[2(xk_1+2tk_1^3+\delta_1)]\}}{[1+2\mathrm{Cosh}(xk_1+2tk_1^3+\delta_1)]^2},\tag{27}$$

$$v=\frac{6k_1^2[2+\mathrm{Cosh}(xk_1+2tk_1^3+\delta_1)]}{[1+2\mathrm{Cosh}(xk_1+2tk_1^3+\delta_1)]^2}.\tag{28}$$

3 Propagation and Interaction of the Solitons

In this section, propagation and interaction of the solitons will be analysed. We note that Rational Solutions (11 and 12) have the singularities when $\xi+xl_1+x^2l_2+x^3l_3+12tl_3=0$. Thus, we will only analyse the dynamic behaviours of the others via the graphics in the following.

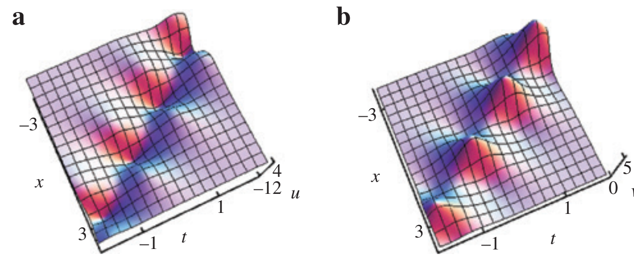

Figure 1: First-order soliton-like solutions for u and v via Expressions (23 and 24) with the parameters $k_1=0.9$, $k=1.2$, $\delta_1=\sigma=0$, $Q_1=P_1=1$, $R=0.6$, and (a) of u; (b) of v.

3.1 Propagation and Interaction of the Special Case I

Figure 1 displays that the propagation of the soliton-like solutions. Potential u and v are both have the periodic properties, which are caused by the term $R\cos\gamma$ in Expression (13).

Figure 2 shows the interaction of the second-order soliton-like solutions. Figures 1 and 2 display that the potential u is a dark soliton-like while the potential v is a bright soliton-like.

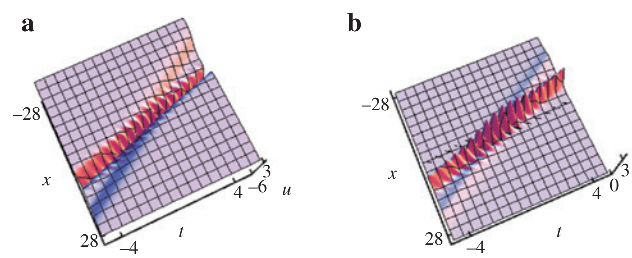

Figure 2: Second-order soliton-like solutions for u and v via Expressions (21 and 22) with the parameters $N=2$, $k_1=0.5$, $k_2=1.1$, $k=1.6$, $\sigma=\delta_1=\delta_2=0$, $Q_1=Q_2=P_1=P_2=1$, $R=0.6$, and (a) of u; (b) of v.

3.2 Propagation and Interaction of the Special Case II

Figure 3 shows that the shape of u is affected by the parameter σ. Figure 3(b) exhibits that the potential u is a shock wave when $\sigma=2$, while the potential u is composed of two shock waves when $\sigma=0$ as displayed in Figure 3(a) and (c). Besides, the direction of the one soliton-like propagation is decided by the sign of k_1, i.e. Figure 3(a) and (c) have the counter directions of the soliton-like propagation, when k_1 has the different sign.

Figure 4 shows that the parameters k_1 and σ have no effect on the shape and direction of the soliton-like propagation for the potential v. Besides, potential v is a bell shape soliton-like.

Figures 5 and 6 display the interaction of the two soliton-like solutions. Figures 5 and 6 exhibit that the potential u and v both have the fusion phenomena, when the parameters are taken as above.

3.3 Propagation and Interaction of the Special Case III

Figure 7 shows that the potential u is the anti-bell shape while potential v is a bell-shape soliton. Figures 1(a), 3, and 7(a) exhibit that the shape of the potential u changes with the values of Q_1, P_1, and R, while the potential v is a bright soliton in Figures 1(b), 4, and 7(b).

Figure 8 exhibits that the interaction of the two solitons are elastic, i.e. the velocity and direction are invariable before and after interaction.

4 Conclusions

Among the NLEEs, KdV-type equation can be used to characterise the dynamic behaviours of the shallow

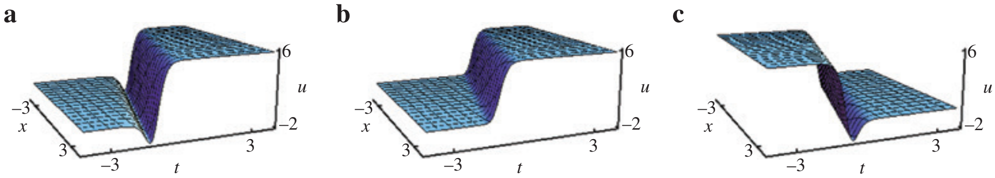

Figure 3: One soliton-like solutions for u via Expression (25) with the parameters $\delta_1 = 1.3$ and (a) of $k_1 = 1.4$, $\sigma = 0$; (b) of $k_1 = 1.4$, $\sigma = 2$; and (c) of $k_1 = -1.4$, $\sigma = 0$.

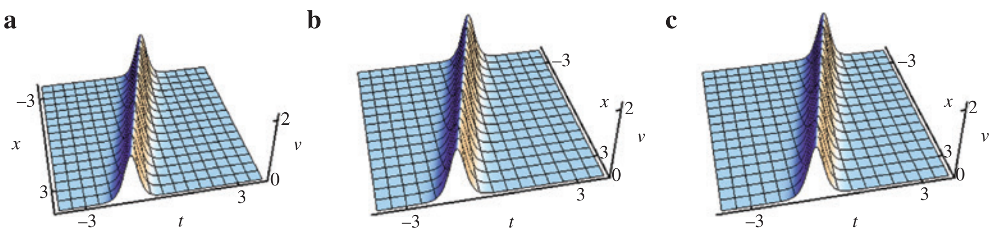

Figure 4: One soliton-like solutions for v via Expression (26) with the the same parameters as Figure 3, accordingly.

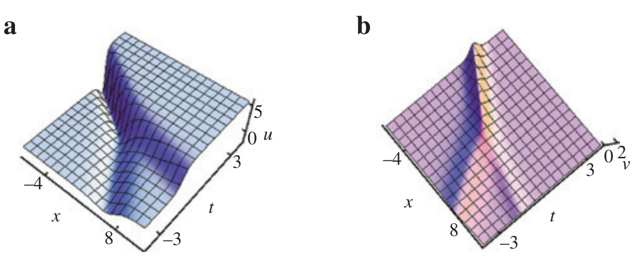

Figure 5: Two soliton-like solutions for u and v via Expressions (17 and 18) with the parameters $k_1 = 1.4$, $k_2 = 0.9$, $Q_1 = Q_2 = 1$, $P_1 = P_2 = R = \sigma = \delta_1 = \delta_2 = 0$, and (a) of u; (b) of v.

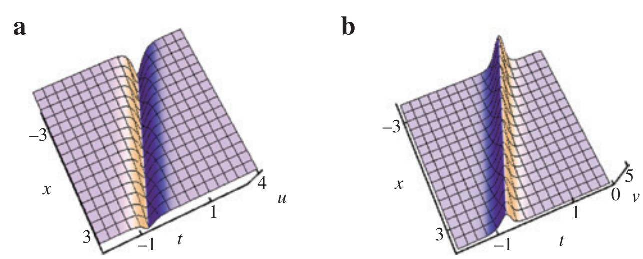

Figure 7: One-soliton solutions for u and v via Expressions (27 and 28) with the parameters $k_1 = 1.6$, $\sigma = 0$, and (a) of u; (b) of v.

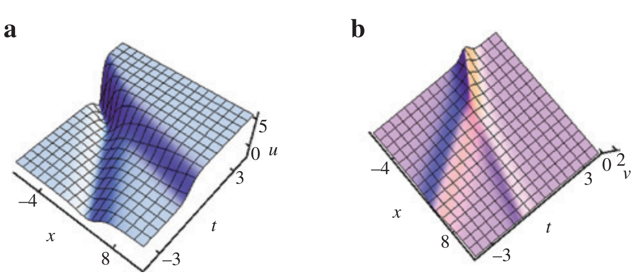

Figure 6: Two soliton-like solutions for u and v via Expressions (17 and 18) with the parameters $k_1 = 1.4$, $k_2 = 0.9$, $Q_1 = Q_2 = 1$, $\sigma = 2$, $P_1 = P_2 = R = \delta_1 = \delta_2 = 0$, and (a) of u; (b) of v.

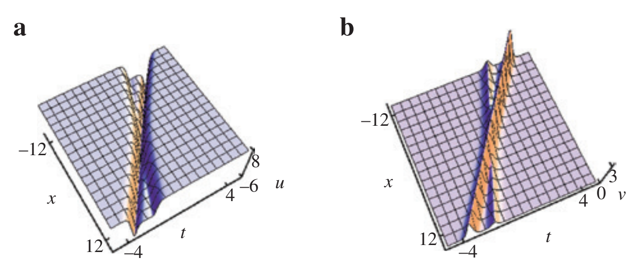

Figure 8: Two-soliton solutions for u and v via Expressions (21 and 22) with the parameters $n = 2$, $k_1 = 1.6$, $k_2 = 0.5$, $Q_1 = Q_2 = P_1 = P_2 = 1$, $\sigma = R = \delta_1 = \delta_2 = 0$, and (a) of u; (b) of v.

water waves and Alfv́en waves. In this article, a coupled KdV system (1) has been studied. Our main results as follows:

1. By virtue of the Bilinear Forms (7 and 8), Rational Solutions (11 and 12), *N*th-order Soliton-Like Solutions (13), Kink-Bell-Shape Solutions (25 and 26) and Bell-Bell-Shape Solutions (27 and 28) have all been attained.
2. Propagation and interaction of the solitons have been analysed:

As the one-soliton pictures, potential u has exhibited three kind shapes, i.e. Figure 1(a) has shown the soliton-like shape, Figure 3 has displayed the kink shape, and Figure 7(a) has shown the anti-bell shape; Potential v has exhibited two type shapes, i.e. Figure 1(b) has exhibited the soliton-like shape, while Figures 4 and 7(b) have both shown the bell shape.

As the two-soliton pictures, potentials u and v have displayed the fusion phenomena in Figures 5 and 6, while potentials u and v have exhibited the elastic collision in Figures 2 and 8.

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