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Generalized Klein-Gordon and Dirac Equations from Nonlocal Kinetic Approach

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Abstract: In this note, I generalized the Klein-Gordon and the Dirac equations by using Suykens's nonlocal-in-time kinetic energy approach, which is motivated from Feynman's kinetic energy functional formalism where the position differences are shifted with respect to one another. I proved that these generalized equations are similar to those obtained in literature in the presence of minimal length based on the Quesne-Tkachuk algebra.

Keywords: Generalized Klein-Gordon and Dirac Equations; Higher-Order Euler-Lagrange Equations; Nonlocal-in-Time Kinetic Energy; Nonlocal in Space-Time; Spinor Field.

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1 Introduction

In his nonrelativistic approach to quantum mechanics, Feynman introduced the concept of nonlocal-in-time backward-forward coordinates where position differences of the particle are shifted with respect to one another [1]. This idea was used afterward by Nelson in his stochastic approach to quantum mechanics, which aims to recover quantum mechanics from a primary stochastic process by deriving the wave and the Schrödinger equations in forward and backward time directions from natural conditions on a diffusion Brownian process in configuration space [2, 3]. Actually, the reversible Schrodinger equation follows from the set of both the forward and the backward stochastic equations and the Newton-like equation of motion. Extensions of Nelson's theory were discussed in [4–6], and its generalizations to interacting particles were performed by Loffredo and Morato [7]. This approach was used also to derive the Klein-Gordon equation from classical relativistic action from stochastic arguments [8]. A stochastic formulation of the Dirac equation was also pointed out by Tiwari [9]. Other implications of the concept of nonlocal-in-time backward-forward coordinates are found in Nottale's scale relativity, which aims to combine fractal space-time with relativity principles [10]. Supplementary implications concern dissipative processes in quantum mechanics [11, 12], the extended Newtonian mechanics characterized by a nonlocal-in-time kinetic energy [13], nonconservative dynamical systems [14], nonstandard Lagrangian dynamics [15], complexified Lagrangian dynamics [16], local fractional derivative formulation of nondifferentiable functions [17], and fractional actionlike variational approach [18], among others. Moreover, a general setting for Nelson's stochastic processes associated with quantum fields was discussed by Albeverio et al. [19]. It should be stressed that the concept of nonlocal-in-time backward-forward coordinates leads to the materialization of higher-order derivatives due to the Taylor approximations of the coordinates system. These higher-order derivatives play a crucial role mainly in all branches of theoretical physics ranging from quantum field theory [20] to classical mechanics [21]. More particularly in quantum field theory, higher-order field equations such as the generalized Klein-Gordon equation were discussed in literature through different arguments, e.g. Lagrangian procedures [22, 23] and the Quesne-Tkachuk Lorentz-covariant deformed algebra [24]. In the present paper, we will use the concept of nonlocal-in-space-time backward-forward coordinates to generalize both Klein-Gordon and Dirac equations, which represent basic equations in quantum field theory. Our analysis is motivated from Suykens's nonlocal-in-time kinetic energy approach [13], which is motivated from Feynman argument [1]. Therefore, our approach will generalize one of the basic equations in quantum field theory by means of classical arguments and not from quantum and stochastic arguments. We will try to systematize in a straightforward way the mathematical procedures that can be followed for the development of the theory. It should be stressed that nonlocal field equations may be obtained by means of pseudo-differential operators acting on a space of analytic functions [25, 26]. These nonlocal equations play a crucial role in nonlocal quantum field theory and have many advantages in solving several problems arising in high

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energy physics and particle theory [27–30]. Besides, a field theory can also display nonlocal kinetic terms because of the presence of nonlocal interactions [31].

The paper is organized as follows: in Section 2, we introduce the basic setups of our model, mainly the nonlocal-in-space-time backward-forward Klein-Gordon field coordinates and the modified Klein-Gordon Lagrangian in the sense of Suykens's nonlocal kinetic term. In Section 3, a comparable nonlocal derivation of the generalized Dirac equation is derived from nonlocal spinor field. In the same section, we discuss the corresponding solutions and we compare our result with solutions obtained in literature through minimal length arguments. Finally, conclusions and perspectives are given in Section 4.

2 Basic Setups: Nonlocal-in-Space-Time Backward-Forward Field Coordinates and the Generalized Klein-Gordon Equation

In classical mechanics, one usually considers a particle of mass m at some position $x(t)$. The Lagrangian is defined by $L=T-U$, where $T=mx^2/2$ and U is the potential. The equation of motion is derived from the principle of least action, which states that the motion of the particle must be such that the action $S=\int_{t_0}^{t_f} L(x,\dot{x})\mathrm{d}t$ is minimum. For fixed boundary conditions, the requirement $\delta S=0$ leads to the Euler-Lagrange equation $\partial L/\partial x-(\mathrm{d}/\mathrm{d}t)(\partial L/\partial x)=0$ from which the equation of motion is derived directly. In Suykens's approach, the kinetic energy $T=mx^2/2$ is replaced by a nonlocal-in-time kinetic energy $T=(m\dot{x}/2)(\dot{x}(t+\tau)+\dot{x}(t-\tau))/2$, where τ is a tiny parameter [13]. It is easy to check that after expanding the second parenthesis factors in a Taylor series, the theory leads to a generalized Newton's second law of motion, which contains advanced and retarded terms in the limiting case. The Lagrangian in that way contains higher-order derivative terms, and a set of equations of motion is derived from a higher-order Euler-Lagrange equation: $\sum_{j=0}^{n+1}(-1)^j(\mathrm{d}^j/\mathrm{d}t^j)(\partial L_{\tau,n}/\partial x^{(j)})=0$ which is the stationary solution to the action functional $S=\int_{t_0}^{t_f} L_{\tau,n}\mathrm{d}t$ under the assumption that the action functional is subject to given boundary conditions $\delta x^{(j)}(t_0)=\delta x^{(j)}(t_f)=0$, $j=0, 1, 2, \ldots, N-1$ and by considering independent variables $x_t(t)$ such that $x_i=\dot{x}_{i-1}$, $i=1, 2, \ldots, N-1$, $x_0=x$ and $x=x_0$.

In field theory, the particle's trajectory $x(t)$ is replaced by the classical field $\phi(x,t)$, and the Lagrangian density $L(x,\dot{x})$ is replaced by $\mathcal{L}(\phi,\partial_\mu\phi)$, and usually t is replaced by x^μ, $\mu=0, 1, 2, 3$. The action is now defined by $S=\int\mathcal{L}(\phi,\partial_\mu\phi)\mathrm{d}^4x$ where the Lagrangian is given in terms of the Lagrangian density by $L(\phi,\partial_\mu\phi)=\int\mathcal{L}(\phi,\partial_\mu\phi)\mathrm{d}^3x$. The equation of motion is derived from the minimum principle $\delta S=0$, which gives $\partial\mathcal{L}/\partial\phi-\partial_\mu(\partial\mathcal{L}/\partial(\partial_\mu\phi))=0$. Here gradient operators are as follows: $\partial_\mu=\partial/\partial x^\mu=((1/c)\partial/\partial t,\nabla)$, $\partial^\mu=\partial/\partial x_\mu=((1/c)\partial/\partial t,-\nabla)$, where $x^\mu=(x^0,x^i)$; $x_\mu=(x^0,-x^i)$; $i=1$, 2, and 3; and $\partial_\eta=\eta_{\mu\nu}\partial^\eta$, with $\eta_{\mu\nu}=\eta^{\mu\nu}=(+1,-1,-1,-1)$ being the Minkowski metric tensor [32]. By considering the Lagrangian $\mathcal{L}(\phi,\partial_\mu\phi)=(1/2)\partial^\mu\phi\partial_\mu\phi-m^2c^2\phi^2/2\hbar^2$, it is easy to check that the Euler-Lagrange equation yields the Klein-Gordon equation $\Box\phi+m^2c^2\phi/\hbar^2=0$, where $\Box=\partial^\mu\partial_\mu$ is the d'Alembertien operator, $\hbar$ is the Planck constant, and c is the celerity of light.

First, motivated from Suykens's nonlocal-in-time kinetic energy approach, we introduce the following nonlocal-in-space-time Lagrangian:

$$\mathcal{L}(\phi,\partial_\mu\phi)=\frac{1}{2}\partial^\mu\phi\frac{\partial^+_\mu\phi_++\partial^-_\mu\phi_-}{2}-\frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi^2, \tag{1}$$

where ϕ_+ and ϕ_- are the forward and backward classical fields, respectively, defined by their corresponding Taylor approximations:

$$\phi_+(t+\tau,x^i+\xi)\approx\phi(t,x^i)+\sum_{k=1}^{n}\frac{1}{k!}\left(\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x^i}\right)^k\phi(t,x^i), \tag{2}$$

$$\phi_-(t+\tau,x^i+\xi)\approx\phi(t,x^i)+\sum_{k=1}^{n}\frac{(-1)^k}{k!}\left(\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x^i}\right)^k\phi(t,x^i). \tag{3}$$

where t is the proper time, $x^i=(x,y,z)$ the proper distance, and (τ,ξ) are tiny constants where physical meanings will be addressed at the end of this section. In what follows, we assume for convenience that $\xi=c\tau$, and accordingly, we can rewrite (2) and (3) as follows:

$$\phi_+(t+\tau,x^i+\xi)\approx\phi(t,x^i)+\sum_{k=1}^{n}\frac{\tau^kc^k}{k!}(\partial_\mu)^k\phi(t,x^i), \tag{4}$$

$$\phi_-(t+\tau,x^i+\xi)\approx\phi(t,x^i)+\sum_{k=1}^{n}\frac{(-1)^kc^k\tau^k}{k!}(\partial_\mu)^k\phi(t,x^i). \tag{5}$$

where $\partial^+_\mu\phi_+$ and $\partial^-_\mu\phi_-$ are the corresponding forward and backward derivatives, respectively. The nonlocal-in-space-time kinetic energy based on the nth-order Taylor approximations (4) and (5) is as follows:

$$
\begin{aligned}
T_{\tau,n} &= \frac{1}{2}\partial^{\mu}\phi\frac{\partial_{\mu}^{+}\phi_{+}+\partial_{\mu}^{-}\phi_{-}}{2} \\
&= \frac{1}{2}\partial^{\mu}\phi\frac{1}{2}\left(\partial_{\mu}\phi+\sum_{k=1}^{n}\frac{\tau^{k}c^{k}}{k!}(\partial_{\mu})^{k+1}\phi+\partial_{\mu}\phi\right. \\
&\quad \left.+\sum_{k=1}^{n}\frac{(-1)^{k}\tau^{k}c^{k}}{k!}(\partial_{\mu})^{k+1}\phi\right), \\
&= \frac{1}{2}\partial^{\mu}\phi\frac{1}{2}\left(2\partial_{\mu}\phi+\sum_{k=1}^{n}\frac{\tau^{k}c^{k}}{k!}(\partial_{\mu})^{k+1}\phi+\sum_{k=1}^{n}\frac{(-1)^{k}\tau^{k}c^{k}}{k!}(\partial_{\mu})^{k+1}\phi\right), \\
&= \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi+\frac{1}{4}\partial^{\mu}\phi\sum_{k=1}^{n}\frac{(1+(-1)^{k})}{k!}\tau^{k}c^{k}(\partial_{\mu})^{(k+1)}\phi. \qquad (6)
\end{aligned}
$$

The Lagrangian of the theory is then written as follows:

$$
\begin{aligned}
\mathcal{L}_{\tau,n}(\phi,\partial_{\mu}\phi) &= T_{\tau,n}-V=\frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi \\
&\quad +\frac{1}{4}\partial^{\mu}\phi\sum_{k=1}^{n}\frac{(1+(-1)^{k})}{k!}\tau^{k}c^{k}(\partial_{\mu})^{(k+1)}\phi-\frac{1}{2}m^{2}\phi^{2}. \qquad (7)
\end{aligned}
$$

It is observable that the nonlocal Lagrangian contains higher-order derivative terms, and accordingly the equation of motion must be derived from the higher-order Euler-Lagrange equation:

$$
\begin{aligned}
&\frac{\partial\mathcal{L}_{\tau,n}}{\partial\phi}-\partial_{\mu}\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_{\mu}\phi)}\right)+\partial_{\mu}\partial_{\nu}\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_{\mu}\partial_{\nu}\phi)}\right) \\
&\quad -\partial_{\mu}\partial_{\nu}\partial_{\eta}\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_{\mu}\partial_{\nu}\partial_{\eta}\phi)}\right)+\ldots=0. \qquad (8)
\end{aligned}
$$

By limiting our analysis to $n=2$, we find

$$
\begin{aligned}
\mathcal{L}_{\tau,2}(\phi,\partial_{\mu}\phi) &= T_{\tau,2}-V=\frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi \\
&\quad +\frac{1}{4}c^{2}\tau^{2}\partial^{\mu}\phi\partial_{\mu}\partial_{\nu}\partial_{\eta}\phi-\frac{1}{2}\frac{m^{2}c^{2}}{\hbar^{2}}\phi^{2}. \qquad (9)
\end{aligned}
$$

This gives after some algebra:

$$
c^{2}\tau^{2}\Box\Box\phi+\Box\phi+\frac{m^{2}c^{2}}{\hbar^{2}}\phi=0. \qquad (10)
$$

Equation (10) is the generalized Klein-Gordon equation obtained from nonlocal arguments. The first term can be considered as nonlocal corrections. Assuming a plane wave solution $\phi\propto e^{-ikx}$, it is easy to check that a possible solution exists if and only if $c^{2}\tau^{2}k^{4}-k^{2}+\lambda\!\!\!^{-}{}^{-2}=0$, $\lambda\!\!\!^{-}=\hbar/mc$ being the reduced Compton wavelength and $k^{2}c^{2}\equiv c^{2}k_{\mu}k^{\mu}=\omega^{2}-c^{2}\vec{k}\cdot\vec{k}$ [24]. It is easy to check that the previous quartic equation admits two different solutions: $k_{+}^{2}=\lambda\!\!\!^{-}{}_{+}^{-2}$ and $k_{-}^{2}=\lambda\!\!\!^{-}{}_{-}^{-2}$ where $\lambda\!\!\!^{-}{}_{+}=\hbar/m_{+}c$ and $\lambda\!\!\!^{-}{}_{-}=\hbar/m_{-}c$, with

$$
m_{\pm}=\frac{\sqrt{1+2\frac{mc^{2}\tau}{\hbar}}\pm\sqrt{1-2\frac{mc^{2}\tau}{\hbar}}}{2\frac{c^{2}\tau}{\hbar}}. \qquad (11)
$$

It is easy to check that the parameter $\tau<\hbar/2mc^{2}$ to obtain real masses. Expanding (11) in a series expansion up to order τ^{2}, we find the following:

$$
m_{+}=\frac{\hbar}{c^{2}\tau}-\frac{m^{2}c^{2}\tau}{2\hbar}, \qquad (12)
$$

$$
m_{-}=m+\frac{m^{3}c^{4}\tau^{2}}{2\hbar^{2}}, \qquad (13)
$$

where $\tau=0$, $m_{-}=m$, and m_{+} diverges. The generalized nonlocal energy-momentum relations are given by $E_{p}^{\pm 2}=m_{\pm}^{2}c^{4}+|\vec{p}|^{2}c^{2}$ and therefore $E_{p}^{-2}=m_{-}^{2}c^{4}+|\vec{p}|^{2}c^{2}=m^{2}c^{4}+|\vec{p}|^{2}c^{2}+m^{4}c^{8}\tau^{2}/\hbar^{2}$. This is the only physical state, whereas the positive case corresponds to a ghost. The solution of (10) is then the superposition of plane waves, and accordingly our results are closely similar to the findings obtained by Moayedi et al. [24], which are based on the concept of minimal length derived from quantum fluctuations of the gravitational field. This proves the importance of nonlocal-in-space-time kinetic energy in quantum field theory.

3 Generalized Dirac Equation

The derivation of the generalized Dirac equation for the case of a spinor field is obtained based on parallel arguments. Normally, for particles of spin, one half of the Dirac Lagrangian density for a spinor field ψ is given as follows: $\mathcal{L}(\psi,\bar{\psi},\partial_{\mu}\psi,\partial_{\mu}\bar{\psi})=(i\hbar c/2)(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi)-mc^{2}\psi\bar{\psi}$. Here γ^{μ} is the Dirac matrix, and $\bar{\psi}=\psi^{+}\gamma^{0}$ is the adjoint spinor [32]. The equations of motion are derived from the minimum principle, which gives $\partial\mathcal{L}/\partial\psi-\partial_{\mu}(\partial\mathcal{L}/\partial(\partial_{\mu}\psi))=0$, and $\partial\mathcal{L}/\partial\bar{\psi}-\partial_{\mu}(\partial\mathcal{L}/\partial(\partial_{\mu}\bar{\psi}))=0$ which gives $i\hbar\gamma^{\mu}\partial_{\mu}\psi-mc\psi=0$ and $i\hbar(\partial_{\mu}\bar{\psi})\gamma^{\mu}+mc\bar{\psi}=0$ respectively. Motivated from Suykens's approach, we generalize the Dirac Lagrangian density as follows:

$$
\begin{aligned}
&\mathcal{L}(\psi,\bar{\psi},\partial_{\mu}\psi,\partial_{\mu}\bar{\psi}) \\
&= \frac{i\hbar c}{2}\left(\bar{\psi}\gamma^{\mu}\frac{\partial_{\mu}^{+}\psi_{+}+\partial_{\mu}^{-}\psi_{-}}{2}-\frac{\partial_{\mu}^{+}\bar{\psi}_{+}+\partial_{\mu}^{-}\bar{\psi}_{-}}{2}\gamma^{\mu}\psi\right)-mc^{2}\psi\bar{\psi}, \qquad (14)
\end{aligned}
$$

where ψ_{+} and ψ_{-} are the forward and the backward spinor fields, respectively, defined by their corresponding Taylor approximations (2) and (3) with $\phi_{\pm}(t\pm\tau, x^i+\xi)\rightarrow\psi_{\pm}(t+\tau, x^i\pm\xi)$, where yet again we assume $\xi=c\tau$. Accordingly, (14) takes the general form as follows:

$$\begin{aligned}&\mathcal{L}_{\tau,n}(\psi,\bar{\psi},\partial_\mu\psi,\partial_\mu\bar{\psi})\\&=\frac{i\hbar c}{2}\left(\frac{1}{2}\bar{\psi}\gamma^\mu\left\{2\partial_\mu\psi+\sum_{k=1}^{n}\frac{(1+(-1)^k)}{k!}\tau^k c^k(\partial_\mu)^{(k+1)}\psi\right\}\right.\\&\quad\left.-\left\{2\partial_\mu\bar{\psi}+\sum_{k=1}^{n}\frac{(1+(-1)^k)}{k!}\tau^k c^k(\partial_\mu)^{(k+1)}\bar{\psi}\right\}\frac{\gamma^\mu\psi}{2}\right)-mc^2\psi\bar{\psi},\\&=\frac{i\hbar c}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi-(\partial_\mu\bar{\psi})\gamma^\mu\psi)-mc^2\psi\bar{\psi}\\&\quad+\frac{i\hbar c}{4}\left(\bar{\psi}\gamma^\mu\left\{\sum_{k=1}^{n}\frac{(1+(-1)^k)}{k!}\tau^k c^k(\partial_\mu)^{(k+1)}\psi\right\}\right.\\&\quad\left.-\left\{\sum_{k=1}^{n}\frac{(1+(-1)^k)}{k!}\tau^k c^k(\partial_\mu)^{(k+1)}\bar{\psi}\right\}\gamma^\mu\psi\right).\end{aligned}\tag{15}$$

It is evident the occurrence of higher-order derivative terms and, consequently, the generalized equations of motion are derived from the following higher-order Euler-Lagrange equations:

$$\begin{aligned}&\frac{\partial\mathcal{L}_{\tau,n}}{\partial\psi}-\partial_\mu\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_\mu\psi)}\right)+\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_\mu\partial_\nu\psi)}\right)\\&\quad-\partial_\mu\partial_\nu\partial_\eta\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_\mu\partial_\nu\partial_\eta\psi)}\right)+\ldots=0,\end{aligned}\tag{16}$$

$$\begin{aligned}&\frac{\partial\mathcal{L}_{\tau,n}}{\partial\bar{\psi}}-\partial_\mu\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_\mu\bar{\psi})}\right)+\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_\mu\partial_\nu\bar{\psi})}\right)\\&\quad-\partial_\mu\partial_\nu\partial_\eta\left(\frac{\partial\mathcal{L}_{\tau,n}}{\partial(\partial_\mu\partial_\nu\partial_\eta\bar{\psi})}\right)+\ldots=0.\end{aligned}\tag{17}$$

By limiting our analysis once more to $n=2$, we obtain from (15):

$$\begin{aligned}\mathcal{L}_{\tau,2}(\psi,\bar{\psi},\partial_\mu\psi,\partial_\mu\bar{\psi})&=\frac{i\hbar c}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi-(\partial_\mu\bar{\psi})\gamma^\mu\psi)-mc^2\psi\bar{\psi}\\&\quad+\frac{i\hbar}{4}\tau^2c^3(\bar{\psi}\gamma^\mu(\partial_\mu\partial_\nu\partial_\eta\psi)-(\partial_\mu\partial_\nu\partial_\eta\bar{\psi})\gamma^\mu\psi).\end{aligned}\tag{18}$$

From (16) and (17), we find, respectively, after some algebra:

$$i\hbar\gamma^\mu(1+\tau^2c^2\Box)\partial_\mu\bar{\psi}+mc\bar{\psi}=0,\tag{19}$$

$$i\hbar\gamma^\mu(1+\tau^2c^2\Box)\partial_\mu\psi-mc\psi=0.\tag{20}$$

Equations (19) and (20) and their corresponding solutions are similar to those obtained by Moayedi et al. [33] based on the Quesne-Tkachuk algebra and minimal length phenomenology. When $\tau=0$, these equations are reduced to the standard Dirac equations. Assuming a plane wave solution of (19) and following the arguments of Moayedi et al. [33], it is easy to check that (11) still holds for nonlocal spinor fields augmented by the constraint $\tau<\hbar/2mc^2$. Up to order τ^2, (12) and (13) still hold together with the corresponding Einstein relations $E_p^{\pm 2}=m_\pm^2c^4+|\bar{p}|^2c^2$ and therefore $E_p^{-2}=m_-^2c^4+|\bar{p}|^2c^2=m^2c^4+|\bar{p}|^2c^2+m^4c^8\tau^2/\hbar^2$. The theory admits two particles of masses $m\pm$ given by (12) and (13). When $\tau=0$, $m_-=m$ while m_+ diverges, yet as long as $0<\tau<\hbar/2mc^2$, the theory is divergence-free and obeys all physical constraints.

4 Conclusions

In this work, we have generalized Klein-Gordon and Dirac equations by using a nonlocal approach motivated from Suykens's nonlocal-in-time kinetic energy framework. We have limited our Taylor expansion to order τ^2, which results to third-order covariant derivatives. Surprisingly, the generalized field equations are closely similar to those equations obtained within the framework of minimal length based on the Quesne-Tkachuk algebra. These field equations contain higher-order derivatives and describe two massive real particles unless $\tau<\hbar/2mc^2$. Let us mention at the end that by comparing our results with findings obtained by Moayedi et al. [24, 33] based on minimal length arguments and Quesne-Tkachuk algebra, mainly (10), (19), and (20), we may introduce the following representation:

$$\{x^\mu,\partial^\mu\}\rightarrow\{x^\mu,(1+\tau^2c^2\Box)\partial^\mu\},$$

up to $n=2$. Therefore, the quantum momentum operator $p^\mu=i\hbar\partial^\mu$ is generalized to $p^\mu=i\hbar(1+\tau^2c^2\Box)\partial^\mu$. Consequently, the nonlocal generalized momentum operator may be defined by $P^\mu=i\hbar(1-\tau^2c^2p^2/\hbar^2)p^\mu$. In that way, we conjecture that there exists a nonlocal modified Heisenberg algebra with the extra quadratic p^2 term, which may lead to a nonlocal minimal length. This modification was found in string theory, black hole physics, and doubly special relativity, among others [34–36]. It is therefore interesting to obtain modified quantum algebra from simple nonlocal arguments. Works in this direction is under progress.

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