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Quasi-Exact Solutions for Generalised Interquark Interactions in a Two-Body Semi-Relativistic Framework

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Abstract: We consider the generalised Cornell, Song-Lin and Richardson interquark interactions in a semi-relativistic two-body basis which originates from the spinless Salpeter equation and is valid for heavy quark limit. In our calculations, due to the complicated nature of arising differential equations, we use the quasi-exact ansatz technique and thereby report the ground-state solution.

Keywords: Interquark Potential; Quasi-Exactly Solvable; Two-Body-System.

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1 Introduction

The few-body problems play a crucial role in many areas of physics including very tiny systems of particle physics such as mesons, which are quark-antiquark systems. To deal with such quark systems, the Bethe–Salpeter formalism is a quite reliable framework [1–4]. The formalism, however, possesses such a complicated mathematical structure that we often have to make some simplifications. A proposition to keep the essential physics of the formalism and simultaneously reduce the mathematical complexity of the equation appears in the form of the so-called spinless saltpeter equation (SSE) which neglects the spin degrees of freedom in the original formalism [5–8]. The SSE also bears some approximations which change the integral structure of the original equation into a simpler differential form [9–13]. This is not yet the end of the story as the resulting differential equation appears in nonlocal form. Therefore, to deal with the equation, we have to use numerical techniques or work on approximate analytical approaches, each of them having their own advantages and weaknesses. An interesting proposition to bring the equation into the form of an ordinary differential equation is the idea of expansion of the square roots. This solution is shown to work well for heavy-quark systems [10–13].

In very recent studies on this equation, the scattering states of the exponential Hulthén potential was studied in [14] in one spatial dimensions. A kink-like potential in an approximation of SSE was analysed in a conceptual approach by elementary considerations and inequalities of quantum mechanics [15]. The problem in a time-dependent basis was analysed via the Lewis-Riesenfeld dynamical invariant method in [16].

A definitely instructive survey on various analytical approximations and bounds on the nonlocal SSE can be found in [5–8]. On the other hand, the successful phenomenological interactions in particle physics ought to have confining term(s) besides the well-known non-confining terms such as coulomb interaction. In particular, the so-called Richardson [17–19], generalised Cornell [20, 21] and Song-Lin [22, 23] interactions have this characteristics and their predictions are consistent with experimental data. All these interactions, just like any successful interaction of bound quark systems, include both confining a non-confining terms and have already been used within the framework of potential model in connection with Charmonium, AdS/QCD and Quarkonium.

Here, we intend to study these interactions within a semi-relativistic two-body basis. In our calculations, as the forthcoming equations reveal, we have to use a quasi-exact approach as, to our best knowledge, the arising equations cannot be solved by other well-known exact techniques of mathematical physics such supersymmetry quantum mechanics, factorisation, Nikiforov-Uvarov (NU) technique, etc. In Section 2, we briefly review an approximation to the two-body SSE which is valid for the case of heavy quarks. We then introduce the simple but powerful quasi-exact ansatz technique [24–29] by which we solve all arising equations.

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2 An Approximation to the Spinless Salpeter Equation in Heavy-Quark Limit

The SSE in the two-body basis is written as [3–9]

$$\left(\sum_{i=1,2}\sqrt{-\Delta+m_i^2}+V(r)-M\right)\chi(\vec r)=0,\quad \Delta=\nabla^2, \tag{1}$$

where $\chi(\vec r)=Y_{lm}(\theta,\phi)R_{nl}(r)$. We can expand the square roots as [10–14]

$$\begin{aligned}\sum_{i=1,2}\sqrt{-\Delta+m_i^2}&=\sqrt{-\Delta+m_1^2}+\sqrt{-\Delta+m_2^2}\\&=m_1\left(1-\frac{1}{2}\frac{\Delta}{m_1^2}-\frac{1}{8}\frac{\Delta^2}{m_1^4}-\dots\right)+m_2\left(1-\frac{1}{2}\frac{\Delta}{m_2^2}-\frac{1}{8}\frac{\Delta^2}{m_2^4}-\dots\right)\\&=m_1+m_2-\frac{\Delta}{2}\left(\frac{m_1+m_2}{m_1m_2}\right)-\frac{\Delta^2}{8}\left(\frac{m_1^3+m_2^3}{m_1^3m_2^3}\right)-\dots,\end{aligned} \tag{2}$$

where

$$\begin{aligned}\left(\frac{m_1^3+m_2^3}{m_1^3m_2^3}\right)&=\left(\frac{m_1^3+m_2^3}{(m_1+m_2)^3}\right)\left(\frac{(m_1+m_2)^3}{m_1^3m_2^3}\right)\\&=\frac{1}{\mu^3}\frac{(m_1+m_2)^3-3m_1m_2(m_1+m_2)}{(m_1+m_2)^3},\\&=\frac{1}{\mu^3}\frac{\frac{(m_1m_2)^2}{\mu^2}-3m_1m_2}{\frac{(m_1m_2)^2}{\mu^2}}=\frac{1}{\mu^3}\frac{m_1m_2-3\mu^2}{m_1m_2}=\frac{1}{\eta^3},\end{aligned} \tag{3}$$

therefore,

$$\sum_{i=1,2}\sqrt{-\Delta+m_i^2}=m_1+m_2-\frac{\Delta}{2\mu}-\frac{\Delta^2}{8\eta^3}-\dots, \tag{4}$$

with

$$\begin{aligned}\mu&=\frac{m_1m_2}{m_1+m_2},\\ \eta&=\mu\left(\frac{m_1m_2}{m_1m_2-3\mu^2}\right)^{1/3}.\end{aligned} \tag{5}$$

From (2) to (5) in units where ($\hbar=c=1$), we have (in center of mass frame)

$$\left[-\frac{\Delta}{2\mu}-\frac{\Delta^2}{8\eta^3}+V(r)\right]R_{nl}(r)=E_{nl}R_{nl}(r). \tag{6}$$

From the well-known relations

$$\begin{aligned}\Delta^2&=p^4=4\mu^2(E_{n,l}-V(r))^2,\\ \Delta&=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{L^2}{r^2},\quad with\ L^2=l(l+1)\end{aligned} \tag{7}$$

Equation (6) appears as

$$\left[-\frac{1}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{l(l+1)}{r^2}\right)-\frac{\mu^2}{2\eta^3}(V(r)-E_{n,l})^2+(V(r)-E_{n,l})\right] R_{nl}(r)=0, \tag{8}$$

Applying the well-known transformation $R_{nl}(r)=\frac{\psi_{nl}(r)}{r}$, we obtain [10–14]

$$\left[\frac{-1}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)}{2\mu r^2}+W_{nl}(r)-\frac{W_{nl}^2(r)}{2\tilde m}\right]\psi_{nl}(r)=0. \tag{9}$$

where

$$\begin{aligned}W(r)&=V(r)-E_{nl},\\ \tilde m&=\eta^3/\mu^2=(m_1m_2\mu)/(m_1m_2-3\mu^2).\end{aligned} \tag{10}$$

We have thus succeeded in finding a Schrödinger-like equation.

3 A Class of Interquark Potentials

In this Section, we consider the generalised interquark terms including Richardson, generalised Cornell and Song-Lin terms in a quasi-exact approach and obtain the set of equations determining the ground-state wave function and associated energy.

3.1 The Richardson Potential

The Richardson potential includes cubic and inverse terms [17–19]:

$$V(r)=Ar^3-\frac{B}{r}, \tag{11}$$

Where A,B are constant parameters. Substituting the potential into (9) yields the equation

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{AB}{m}r^2+\left(\mathrm{A}+\frac{EA}{m}\right)r^3-\frac{A^2}{2m}r^6-\left(B+\frac{\mathrm{E}_{nl}B}{m}\right)\frac{1}{r}+\left(\frac{l(l+1)}{\hbar^2}-\frac{B^2}{2m}\right)\frac{1}{r^2}-\mathrm{E}_{nl}-\frac{\mathrm{E}_{nl}^2}{2m}\right]\psi_{nl}=0. \tag{12}$$

Equation (12), obviously, does not appear in the form of known equations of mathematical physics. Therefore, we introduce the ansatz solution [24–29]

$$\psi_{nl}(r)=h_n(r)\ \exp(s_l(r)), \tag{13}$$

Where

$$\mathrm{h}_n(r)=\begin{cases} 1, & if\ n=0 \\ \prod_{i=1}^{n}(r-\alpha_i^n), & if\ \ n\geq 1 \end{cases} \tag{14}$$

and the term in the exponent is determined by solving a corresponding Riccati equation. Considering the latter ansatz for (11) and solving the corresponding Riccati equation determines the term in the exponent as

$$s_l(r)=ar^4+br+c\ln(r), \tag{15}$$

Substituting the latter in (9) and equating the corresponding powers on both sides gives the set of equations

$$6a+4ac=\frac{\mu}{h^2}\frac{AB}{m} \tag{16a}$$

$$4ab=\frac{\mu}{h^2}\left(A+\frac{EA}{m}\right) \tag{16b}$$

$$16a^2=-\frac{\mu}{h^2}\frac{A^2}{m} \tag{16c}$$

$$bc=-\frac{\mu}{h^2}\left(B+\frac{E_{nl}B}{m}\right) \tag{16d}$$

$$c^2-c=l(l+1)-\frac{2\mu}{h^2}\frac{B^2}{2m} \tag{16e}$$

$$b^2=-\frac{2\mu}{h^2}\left(E_{nl}+\frac{E_{nl}{}^2}{2m}\right) \tag{16f}$$

which determine the spectrum for given values of potential parameters.

3.2 The Generalised Cornell Potential

In this Section, we consider the generalization of the Cornell potential [20, 21]

$$V(r)=v_0+kr-\frac{e}{r}+\frac{f}{r^2}, \tag{17}$$

in which v_0, k, e and f are constant parameters. The corresponding equation then reads

$$\left[\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2}+\left(k-\frac{kv_0}{m}+\frac{E_{nl}k}{m}\right)r-\frac{k^2r^2}{2m}+\left(-e+\frac{ev_0}{m}-\frac{kf}{m}-\frac{E_{nl}e}{m}\right)\frac{1}{r}\right.$$
$$+\left(f-\frac{fv_0}{m}-\frac{e^2}{2m}+\frac{E_{nl}f}{m}+\frac{l(l+1)\hbar^2}{2\mu}\right)\frac{1}{r^2}+\frac{ef}{mr^3}$$
$$\left.-\frac{f^2}{2mr^4}+v_0-E_{nl}-\frac{v_0^2}{2m}+\frac{ke}{m}+\frac{E_{nl}v_0}{m}-\frac{E^2}{2m}\right]\psi_{nl}(r)=0 \tag{18}$$

and the solution of the associated Riccati equation gives

$$s_l(r)=ar^2+br+\frac{c}{r}+d\ln(r), \tag{19}$$

By the same token of the previous subsection, we obtain the set

$$4ab=\frac{2\mu}{\hbar^2}\left(k-\frac{kv_0}{m}+\frac{E_{nl}k}{m}\right) \tag{20a}$$

$$4a^2=-\frac{\mu k^2}{\hbar^2 m} \tag{20b}$$

$$-4ac+2bd=\frac{2\mu}{\hbar^2}\left(-e+\frac{ev_0}{m}-\frac{kf}{m}-\frac{E_{nl}e}{m}\right) \tag{20c}$$

$$-d-2cb+d^2=l(l+1)+\frac{2\mu}{\hbar^2}\left(f-\frac{fv_0}{m}-\frac{e^2}{2m}+\frac{E_{nl}f}{m}\right) \tag{20d}$$

$$c-cd=\frac{\mu}{\hbar^2}\frac{ef}{m} \tag{20e}$$

$$c^2=-\frac{\mu f^2}{\hbar^2 m} \tag{20f}$$

$$2a+4ad+b^2=\frac{2\mu}{\hbar^2}\left(v_0-E_{nl}-\frac{v_0^2}{2m}+\frac{ke}{m}+\frac{E_{nl}v_0}{m}-\frac{E^2}{2m}\right) \tag{20g}$$

3.3 The Song-Lin Potential

In this section, we consider the term proposed by Song and Lin to analyse the quark systems [22, 23]. The potential includes the square root and its inverse

$$V(r)=Ar^{\frac{1}{2}}+Br^{-\frac{1}{2}}. \tag{21}$$

with A and B being constant parameters. In this case, we have to deal with the differential equation

$$\left[\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2}-\frac{A^2}{2m}r+\left(A+\frac{EA}{m}\right)r^{\frac{1}{2}}+\left(B+\frac{EB}{m}\right)r^{-\frac{1}{2}}\right.$$
$$\left.-\frac{B^2}{2mr}+\frac{l(l+1)\hbar^2}{2\mu r^2}-E_{nl}-\frac{AB}{m}-\frac{E^2}{2m}\right]\psi_{nl}(r)=0 \tag{22}$$

and the term in the exponent is calculated to be

$$s_l(r) = ar^{\frac{3}{2}} + br + c\ln(r). \tag{23}$$

The latter gives the required set of equations as

$$\frac{9}{4}\mathrm{a}^2 = \frac{-\mu}{\hbar^2}\frac{A^2}{m} \tag{24a}$$

$$3\mathrm{ab} = \frac{2\mu}{\hbar^2}\left(A + \frac{EA}{m}\right) \tag{24b}$$

$$\frac{3}{4}\mathrm{a} + 3\mathrm{ac} = \frac{2\mu}{\hbar^2}\left(B + \frac{EB}{m}\right) \tag{24c}$$

$$2bc = \frac{-\mu}{\hbar^2}\frac{B^2}{m} \tag{24d}$$

$$\mathrm{c}^2 - \mathrm{c} = l(l+1) \tag{24e}$$

$$\mathrm{b}^2 = -\frac{2\mu}{\hbar^2}\left(E_{nl} + \frac{AB}{m} + \frac{E^2}{2m}\right) \tag{24f}$$

which can be solved numerically to report the energy for given potential parameters.

4 Conclusions

The Richardson, Song-Lin and especially the generalised Cornell interaction are among interactions of particle physics to model meson systems. In our work, we considered these interactions within the framework of an approximation to spinless Salpeter equation, which is acceptable in heavy quark limit. To solve the arising differential equation, we used the quasi-exact ansatz technique which is based on solving a Riccati equation and thereby reported the ground-state solutions. To our best knowledge, this is first time these interactions have been considered in an analytical manner in two-body basis. In addition, none of our common tools of mathematical physics such as supersymmetry quantum mechanics, factorisation and NU techniques can solve these equations. It is also worth-noting that although we only derived the ground state, the higher states can be similarly obtained by choosing $h_1(r) = (r - \alpha_1^1)$ for the first state, $h_2(r) = (r - \alpha_1^2)(r - \alpha_2^2)$ for the second state, etc. Nevertheless, it should be mentioned that the approach, just like any quasi-exact technique, does have its disadvantages: the first is that it puts a restriction on the choice of potential parameters and becomes more difficult for higher states. However, we when recall that, the problem cannot be solved by other common techniques and even in one its special cases, namely the special case of Cornell interaction, appears in the form of Heun differential equation, the limitation becomes worthy. Having calculated the wave function and the spectrum, the results can be used to study heavy mesons systems properties including the mass spectrum, binding energy, scattering states, decay rates, slope parameter, convexity parameter, charge radius, form factors, etc.

References

[1] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[2] L. Chang and C. D. Roberts, Phys. Rev. Lett. **103**, 081601 (2009).
[3] W. Lucha and F. F. Schoberl, Int. J. Mod. Phys. A. **14**, 2309 (1999).
[4] P. Maris and C. D. Roberts, Phys. Rev. C. **56**, 3369 (1997).
[5] W. Lucha and F. F. Schöberl, Phys. Rev. D. **50**, 5443 (1994).
[6] R. Hall, W. Lucha, and F. F. Schöberl, J. Phys. A. **34**, 5059 (2001).
[7] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A. **29**, 1450057 (2014).
[8] W. Lucha and F. F. Schöberl, J. Phys. G. **31**, 1133 (2005).
[9] S. Zarrinkamar, A. A. Rajabi, B. H. Yazarloo, and H. Hassanabadi, Few-Body Syst. **54**, 2001 (2013).
[10] S. M. Ikhdair and R. Sever, Z. Phys. C: Particles and Fields. **56**, 155 (1992).
[11] S. M. Ikhdair and R. Sever, Int. J. Mod. Phys. A. **20**, 6509 (2005).
[12] G. Jaczko and L. Durand, Phys. Rev. D. **58**, 114017 (1998).
[13] J. Nickisch, L. Durand, and B. Durand, Phys. Rev. D. **30**, 660 (1984).
[14] S. Hassanabadi, M. Ghominejad, and K-E. Thylwe, Comm. Theor. Phys. **63**, 423 (2015).
[15] W. Lucha and F. F. Schoberl, Int. J. Mod. Phys. A. **30**, 1550062 (2015).
[16] H. Sobhani and H. Hassanabadi, Adv. High Energy Phys. **2016**, 3647392 (2016), 5 pages.
[17] G. S. Bali, Phys. Rep. **343**, 1 (2001).
[18] J. L. Richardson, Phys. Lett. B. **82**, 272 (1976).
[19] S. Banerjee, Phys. Lett. B. **611**, 27 (2005).
[20] O. Andreev and V. I. Zakharov, Phys. Rev. D. **74**, 025023 (2006).
[21] C. D. White, Phys. Lett. B. **652**, 79 (2007).
[22] X. Song and H. Lin, Zeit. Phys. C. **34**, 223 (1987).
[23] L. P. Fulcher, Phys. Rev. D. **60**, 074006 (1999).
[24] H. Hassanabadi, S. F. Forouhandeh, H. Rahimov, S. Zarrinkamar, and B. H. Yazarloo, Can. J. Phys. **90**, 299 (2012).
[25] S. H. Dong, Int. J. Theor. Phys. **40**, 569 (2001).
[26] S. H. Dong and Z. Q. Ma, J. Phys. A: Mathematical and General **31**, 9855 (1998).
[27] M. S. Child, S. H. Dong, and X. Wang, J. Phys. A: Mathematical and General. **33**, 5653 (2000).
[28] S. H. Dong, Physica Scripta **65**, 289 (2002).
[29] S. H. Dong, Wave equations in higher dimensions, Springer, Netherland 2011.