Jiang Guan and Yue Kai*

Asymptotic Analysis to Two Nonlinear Equations in Fluid Mechanics by Homotopy Renormalisation Method

DOI 10.1515/zna-2016-0210
Received May 26, 2016; accepted July 31, 2016; previously published online August 24, 2016

Abstract: By the homotopy renormalisation method, the global approximate solutions to Falkner-Skan equation and Von Kármá's problem of a rotating disk in an infinite viscous fluid are obtained. The homotopy renormalisation method is simple and powerful for finding global approximate solutions to nonlinear perturbed differential equations arising in mathematical physics.

Keywords: Asymptotic Solution; Falkner-Skan Equation; Homotopy Renormalisation Method; Renormalisation Method; Von Kármá's Problem.

1 Introduction

In early 1990s, Goldenfeld et al. initially applied the renormalisation group (RG) method instead of dimensional analysis [1], to nonlinear partial differential equations, such as Barenblatt's equation [2], modified porous medium equation [3], and turbulent-energy-balance equation [4]. The results showed that the method was more efficient and perhaps more accurate than the usual perturbation methods for asymptotic analysis because it avoided the necessity to perform asymptotic matching, they also anticipated that RG method would be useful in numerical calculations. After that, many scientists' work proved it is true. For example, recently, Tu and Cheng published series of papers about applying some improved RG method to perturbed partial differential equations [5–7] such as Barenblatt equation, etc., in a series of papers [8–11], Teiji Kunihiro gave a geometrical interpretation to RG method based on the classical theory of envelop in differential geometry. In [12], Kai applied the Kunihiro's geometrical formulation of RG method to deal with perturbed Burger's equation and perturbed Korteweg–de Vries equation, and obtained their global asymptotic solutions. However, the mathematical foundation of the RG method is still unclear. Recently, in [13], Liu formulated the mathematical foundation of the RG method by the traditional Taylor series, and he proved that the RG method was just based on the usual Taylor expansion, and all key assumptions in the standard RG method could be given as the natural facts in Taylor expansion. The mathematical foundation of Liu's proposed renormalisation method based on Taylor expansion (TR) was so simple and clear that we could easily understand it in theory and use it in practice. On the other hand, there also exist some weaknesses for the RG method and TR method, for example, the standard RG or TR method can not work or do not improve the global solutions to some equations [13]. To overcome these weaknesses, Liu [13] proposed a powerful method called homotopy renormalisation method (HTR) by combining the TR method and homotopy. By Liu's HTR method, we can actually solve some problems that traditional RG method shows little help. In the article [13], Liu dealt with some important and interesting problems including the forced duffing equation and Blasius equation and so on, the results showed that the HTR method is an important and powerful method in the asymptotic analysis.

In this article, we apply Liu's HTR method to get the global approximate solutions correct up to $O(\varepsilon^2)$ to the Falkner-Skan equation [14]

$$f'''(\eta)+f(\eta)f''(\eta)+\beta[1-f'^2(\eta)]=0, \tag{1}$$

and Von Kármá's problem of a rotating disk in an infinite viscous fluid [15]

$$\begin{cases} w'''-w''w+\dfrac{1}{2}w'^2-2g^2=0 \\ g''-wg'+w'g=0, \end{cases} \tag{2}$$

where β is a constant and f, w, g are functions to be determined. The Falkner-Skan equation was originally derived in 1931 by Falkner and Skan [14], is of central importance to the fluid mechanics of wall-bounded viscous flows. It is derived from the two-dimensional incompressible

***Corresponding author: Yue Kai,** Department of Mathematics, Northeast Petroleum University, Daqing 163318, China, Tel.: +18196587213, E-mail: kaiyue991103@126.com
Jiang Guan: Department of Mathematics, Northeast Petroleum University, Daqing 163318, China

Navier-Stokes equations for a one-sided bounded flow using a similarity analysis [16] and its solution describes the form of an external laminar boundary layer in the presence of an adverse or favourable stream wise pressure gradient. Ullah et al. [17] used the optimal homotopy asymptotic method [18] to Falkner-Skan equation and obtained some results. And Von Kármá's problem of a rotating disk in an infinite viscous fluid was originally considered by Von Kármá, which is also very important in the study of flows on rotating bodies, centrifugal pumps, viscometers etc [19–24].

The outline of this article is as follows. In Section 2, we consider a simple multi-solutions problem as an illustration example to introduce the Liu's HTR method and show the advantages of this method. In Sections 3 and 4, we use HTR method to obtain the global approximate solutions to Falkner-Skan equation and Von Kármá's problem of a rotating disk in an infinite viscous fluid. In Section 5, we give a conclusion.

2 HTR Method

According to the article [13], we list basic ideas and steps of Liu's TR and HTR methods. Consider an equation

$$N(y)=\varepsilon M(y), \tag{3}$$

where N and M are in general linear or nonlinear operators. Assuming that the solution can be expanded as a power series of the small parameter ϵ

$$y=y_0+y_1\epsilon+\cdots+y_n\epsilon+\cdots, \tag{4}$$

and substituting it into the above equation yields the equations of y_n's such as

$$N(y_0)=0, \tag{5}$$

and

$$N_1(y_1)=M_1(y_0),\ \cdots,\ N_k(y_k)=M_k(y_{k-1}),\ \cdots \tag{6}$$

for some operators N_k and M_k. By the first equation, we give the general solution of y_0 including some integral constants A and B and so on. Then we find the particular solutions of y_k and expand them as the power series at a general point t_0,

$$y_k(t)=\sum_{m=0}^{+\infty} y_{km}(t_0)(t-t_0)^m,\ k=0,\ 1,\ \cdots. \tag{7}$$

Furthermore, by rearranging the summation of these series, we obtain the solution

$$y(t)=\sum_{n=0}^{+\infty} Y_n(t_0,\ \epsilon)(t-t_0)^n, \tag{8}$$

where

$$Y_n(t_0,\ \epsilon)=\sum_{k=0}^{+\infty} y_{kn}(t_0)\epsilon^k,\ n=0,\ 1,\ \cdots. \tag{9}$$

According to Liu's theory [13], we have

(i) Solution is just $y(t)=Y_0(t,\ \epsilon)$;

(ii) Renormalisation equations are
$Y_n(t,\ \epsilon)=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}Y_0(t,\ \epsilon),\ n=1,\ 2,\ \cdots;$

The advantages of Liu's method are obvious. Firstly, in Liu's theory, the secular terms need not be considered at all since the first terms Y_0 is just solution. Secondly, to determine the unknown integral constants, the renormalisation equation is just taken as the usual relations between the coefficients of the Taylor series. This is just Liu's TR method for asymptotic analysis. Then by combining the TR method and homotopy, Liu proposed a HTR method to overcome some weaknesses of RG method and TR method. Assume the aim equation is

$$N(t,\ y,\ y',\ \cdots)=0, \tag{10}$$

where N is a function. Take a simple linear equation

$$L(y)=0, \tag{11}$$

where L is in general a linear operator. Next we take the homotopy equation as follows

$$(1-\epsilon)L(y)+\epsilon N(t,\ y,\ y',\ \cdots)=0, \tag{12}$$

where the homotopy parameter ϵ satisfying $0\le\epsilon\le 1$. We can see that the homotopy equation changes from the simple equation $L(y)=0$ to the aim equation $N(t,\ y,\ y',\ \cdots)=0$ as ϵ changing from 0 to 1. Then we use TR method to deal with the homotopy equation by considering ϵ as a small parameter and taking $\epsilon=1$ in final. A large number of important applications of TR and HTR methods can be found in [13].

Remark 1. As emphasised in [13], we must also point out that if there are m unknown integral constants, in principle, we will need m renormalisation equations $Y'_{n-1}=nY_n$ to get a closed equations system to solve out m unknown functions for $n=1,\cdots,\ m$. But, in practice, we usually need only one renormalisation equation, that is, the first renormalisation equation $Y'_0=Y_1$ to get the closed equations system by some approximations or other balance relations.

As an illustration example, we consider the equation $y'(t)=\epsilon(1-y^2(t))$ which cannot be dealt with RG method and TR method. So, we apply HTR method to deal with it. According to [13], we can write the homotopy equation as

$$y'+y=1-\epsilon(y^2-y). \tag{13}$$

Expanding y as

$$y=\sum_{n=0}^{+\infty} y_n\epsilon^n, \tag{14}$$

and substituting it into the above homotopy equation yields

$$y_0'+y_0=1, \tag{15}$$

and

$$y_1'+y_1=y_0-y_0^2, \tag{16}$$

and so on. From these two equations, we have

$$y_0=A\exp(-t)+1, \tag{17}$$

$$y_1=-A(t-t_0)\exp(-t)+A^2\exp(-2t). \tag{18}$$

Therefore

$$y=A\exp(-t)+1+\epsilon(-A(t-t_0)\exp(-t)+A^2\exp(-2t))+O(\epsilon^2). \tag{19}$$

We assume that A is independent on t_0, that is

$$A=A(t_0). \tag{20}$$

The Liu's renormalisation equation is

$$A'(t_0)+\epsilon A=0, \tag{21}$$

which gives $A(t_0)=A_0\exp(-\epsilon t_0)$, where A_0 is a constant. Therefore, the global solution is

$$y=1+A_0\exp(-(1+\epsilon)t)+\epsilon A_0^2\exp(-2(1+\epsilon)t))+O(\epsilon^2). \tag{22}$$

By taking $\epsilon=1$, we have

$$y=1+A_0\exp(-2t)+A_0^2\exp(-4t))+O(1), \tag{23}$$

which is a non-trivial global solution.

For the terminal value condition, the exact solution is

$$y=\frac{1-A\exp(-2t)}{1+A\exp(-2t)}=1+2A\exp(-2t)+2A^2\exp(-4t)+\cdots. \tag{24}$$

Comparing the asymptotic solution with the exact solution, when t tends to infinity, these two solutions are the same one. This shows that the HTR method is efficient for this problem.

From the above, the procedure of HTR method can be summarised as follows:

Step 1. Write the homopoty equation.

Step 2. Using the TR method to deal with the homotopy equation by considering the homotopy parameter as a small parameter.

Step 3. Take the homotopy parameter equals to one and give the global asymptotic solution.

3 Application to Falkner-Skan Equation

In this Section, we use HTR method to solve the Falkner-Skan equation (1). First, we write the homotopy equation as

$$f'''+f''=\varepsilon(f''+\beta f'^2-\beta-ff''). \tag{25}$$

Expanding f as

$$f(\eta)=f_0(\eta)+\varepsilon f_1(\eta)+\varepsilon^2 f_2(\eta)+\ldots, \tag{26}$$

yields

$$f_0'''+f_0''=0, \tag{27}$$

and

$$f_1'''+f_1''=f_0''+\beta f_0'^2-\beta-f_0f_0'', \tag{28}$$

and so on, whose solutions are given by

$$f_0(\eta)=Ae^{-\eta}+B(\eta-\eta_0), \tag{29}$$

and

$$\begin{aligned} f_1(\eta)=&-\frac{AB}{2}(\eta-\eta_0)^2e^{-\eta}+(A-2AB\beta-2AB)(\eta-\eta_0)e^{-\eta}\\ &+(2A-4AB\beta-3AB)e^{-\eta}+\frac{B^2\beta-\beta}{2}(\eta-\eta_0)^2\\ &+\frac{A^2-A^2\beta}{4}e^{-2\eta}-B(\eta-\eta_0). \end{aligned} \tag{30}$$

Inserting formulas (29) and (30) into (26), we have

$$\begin{aligned} f(\eta)=&Ae^{-\eta}+B(\eta-\eta_0)\\ &+\varepsilon\Big[-\frac{AB}{2}(\eta-\eta_0)^2e^{-\eta}+(A-2AB\beta-2AB)(\eta-\eta_0)e^{-\eta}\\ &+(2A-4AB\beta-3AB)e^{-\eta}+\frac{B^2\beta-\beta}{2}(\eta-\eta_0)^2\\ &+\frac{A^2-A^2\beta}{4}e^{-2\eta}-B(\eta-\eta_0)\Big]+O(\varepsilon^2). \end{aligned} \tag{31}$$

According to the standard procedure of the TR method, we assume that A and B are both dependent on η_0, namely

$$A = A(\eta_0),\ B = B(\eta_0). \tag{32}$$

By taking $\varepsilon = 1$ and the Taylor expansions of $e^{-\eta}$ and $e^{-2\eta}$ at η_0, we have

$$Y_0(\eta_0,\varepsilon) \approx (3A - 4AB\beta - 3AB)e^{-\eta_0} + \frac{A^2 - A^2\beta}{4}e^{-2\eta_0}, \tag{33}$$

$$\begin{aligned} Y_1(\eta_0,\varepsilon) &\approx (A - 2AB\beta - 2AB)e^{-\eta_0} - (3A - 4AB\beta - 3AB)e^{-\eta_0} \\ &\quad - \frac{A^2 - A^2\beta}{2}e^{-2\eta_0} \end{aligned} \tag{34}$$

$$\begin{aligned} Y_2(\eta_0,\varepsilon) &\approx -\frac{AB}{2}e^{-\eta_0} + \frac{B^2-1}{2}\beta - (A - 2AB\beta - 2AB)e^{-\eta_0} \\ &\quad + \frac{1}{2}(3A - 4AB\beta - 3AB)e^{-\eta_0} + (A^2 - A^2\beta)e^{-2\eta_0}. \end{aligned} \tag{35}$$

Since Y_0, Y_1, and Y_2 are all finite terms and hence are approximate, so the renormalisation equations $Y_0' = Y_1$ and $Y_1' = 2Y_2$ are in general incompatible each other. And hence, further, in order to give a closed renormalisation equations system as simple and nontrivial as possible, we often take only one renormalisation equation and ignore some terms such as these terms including $e^{-2\eta_0}$. For this example, we take $\frac{\partial Y_0}{\partial \eta_0} = Y_1$ as the renormalisation equation to have

$$\begin{aligned} &(3A' - (4\beta+3)(AB)')e^{-\eta_0} + \frac{1-\beta}{2}AA'e^{-2\eta_0} \\ &= (A - 2AB\beta - 2AB)e^{-\eta_0}. \end{aligned} \tag{36}$$

Since $e^{-2\eta_0}$ is very small than $e^{-\eta_0}$ as η_0 tends to positive infinity, so we remove the last term in the left side of the above renormalisation equation, and get

$$3A' - (4\beta+3)(AB)' = A - 2AB\beta - 2AB. \tag{37}$$

For the purpose of further simplicity, we assume that B is a constant and hence get a closed renormalisation equations system

$$\begin{cases} (3 - 4B\beta - 3B)A' = A(1 - 2B\beta - 2B) \\ B = B_0. \end{cases} \tag{38}$$

Solving the above equations give

$$A(\eta) = A_0 e^{\frac{1 - 2B_0\beta - 2B_0}{3 - 4B_0\beta - 3B_0}\eta}, \tag{39}$$

where A_0 is a constant. Therefore the global asymptotic solution is represented by

$$\begin{aligned} f(\eta) &= (3 - 4B_0\beta - 3B_0)e^{-\frac{(1+2\beta)B_0 - 2}{(3+4\beta)B_0 - 3}\eta} \\ &\quad + \frac{1-\beta}{4}A_0^2 e^{-2\left(\frac{(1+2\beta)B_0 - 2}{(3+4\beta)B_0 - 3}\right)\eta} + O(1). \end{aligned} \tag{40}$$

When the constant B_0 satisfies $B_0 > \frac{2}{1+2\beta}$ or $B_0 < \frac{3}{3+4\beta}$, we have $\frac{(1+2\beta)B_0 - 2}{(3+4\beta)B_0 - 3} > 0$ since $\frac{2}{1+2\beta} > \frac{3}{3+4\beta}$ if we assume that $\beta > 0$ without loss of generality, and hence the solutions is fast decreasing at positive infinity. It is clear to see that we need not consider the secular terms of formula (31) at all, and a global approximate asymptotic solution is obtained by the HTR method.

Remark 2. In [17], Ullah et al. used the OHAM to solve the Falkner-Skan equation and give the corresponding power series approximate solution under the boundary conditions at two points 0 and 5. Our solution has an exponential function form, thus it is suitable for the condition at infinity.

Remark 3. Form the above example, we can see that an important problem is how to choose a closed renormalisation equations system. We have no a unique way to get it and have no a strict theory to tell us how to know which terms can be ignored. This problem needs further deep study in future.

4 Application to Von Kármá's Problem of a Rotating Disk in an Infinite Viscous Fluid

In this Section, we use HTR method to solve the Von Kármá's problem (2) of a rotating disk in an infinite viscous fluid. First, we write the homotopy equation as

$$\begin{cases} w''' + w' = \varepsilon\left(w''w - \frac{1}{2}w'^2 + 2g^2 + w'\right), \\ g'' + g = \varepsilon(wg' - w'g + g). \end{cases} \tag{41}$$

Similar to Section 3, expanding w and g as

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots,\ g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots, \tag{42}$$

then substituting them into (41) yields

$$w_0''' + w_0' = 0, \tag{43}$$

$$w_1''' + w_1' = w_0'' w_0 - \frac{1}{2} w_0'^2 + 2g_0^2 + w_0', \tag{44}$$

and

$$g_0'' + g_0 = 0, \tag{45}$$

$$g_1'' + g_1 = w_0 g_0' - w_0' g_0 + g_0. \tag{46}$$

The solutions of (43–46) are given by

$$w_0 = a_1 \sin(t+\varphi_1) + b_1, \tag{47}$$

$$g_0 = a_2 \sin(t+\varphi_2), \tag{48}$$

$$\begin{aligned} w_1 =& -\frac{a_1^2}{24}\sin(2t+2\varphi_1) + \frac{a_2^2}{6}\sin(2t+2\varphi_2) \\ &+ (t-t_0)\left(-\frac{a_1}{2}\cos(t+\varphi_1) + \frac{a_1 b_1}{2}\sin(t+\varphi_1) + a_2^2 - \frac{3}{4}a_1^2\right) \\ &+ \frac{a_1 b_1}{2}\cos(t+\varphi_1) + \frac{a_1}{2}\sin(t+\varphi_1), \end{aligned} \tag{49}$$

and

$$g_1 = a_1 a_2 \sin(\varphi_1 - \varphi_2) + (t-t_0)\left(\frac{a_2 b_1}{2}\sin(t+\varphi_2) - \frac{a_2}{2}\cos(t+\varphi_2)\right). \tag{50}$$

Inserting them into (42), we have

$$\begin{aligned} w =& a_1 \sin(t+\varphi_1) + b_1 + \varepsilon\left\{-\frac{a_1^2}{24}\sin(2t+2\varphi_1) + \frac{a_2^2}{6}\sin(2t+2\varphi_2)\right. \\ &+ (t-t_0)\left(-\frac{a_1}{2}\cos(t+\varphi_1) + \frac{a_1 b_1}{2}\sin(t+\varphi_1) + a_2^2 - \frac{3}{4}a_1^2\right) \\ &\left.+ \frac{a_1 b_1}{2}\cos(t+\varphi_1) + \frac{a_1}{2}\sin(t+\varphi_1)\right\} + O(\varepsilon^2), \end{aligned} \tag{51}$$

and

$$\begin{aligned} g =& a_2 \sin(t+\varphi_2) + \varepsilon\{a_1 a_2 \sin(\varphi_1 - \varphi_2) \\ &+ (t-t_0)(\frac{a_2 b_1}{2}\sin(t+\varphi_2) - \frac{a_2}{2}\cos(t+\varphi_2))\} + O(\varepsilon^2). \end{aligned} \tag{52}$$

For simplicity, we first assume that $\varphi_1 = \varphi_2$. Then according to the procedure of the TR method, let a_1, a_2, b_1, φ_1 and φ_2 be both dependent on t_0, namely $a_1 = a_1(t_0)$, $a_2 = a_2(t_0)$, $b_1 = b_1(t_0)$ and $\varphi_1(t_0) = \varphi_2(t_0) = \varphi(t_0)$. By taking $\varepsilon = 1$, and ignoring the terms $\frac{a_1^2}{24}\sin(2t+2\varphi_1)$ and $\frac{a_2^2}{6}\sin(2t+2\varphi_2)$, the renormalisation equation gives

$$\begin{aligned} &\frac{3}{2}a_1' - \frac{1}{2}a_1 b_1 \varphi' = \frac{a_1 b_1}{2},\ \frac{1}{2}(a_1 b_1)' + \frac{3}{2}a_1 \varphi_1' = -\frac{a_1}{2}, \\ &b_1' = a_2^2 - \frac{3}{4}a_1^2,\ a_2' = \frac{a_2 b_1}{2},\ a_2 \varphi_2' = -\frac{a_2}{2},\ \varphi_1 = \varphi_2. \end{aligned} \tag{53}$$

But this equations system is incompatible, so we must remove some equation(s). A reasonable choice is to remove the equation $b_1' = a_2^2 - \frac{3}{4}a_1^2$ since Y_0 and Y_1 are approximate and it is impossible in general that these two terms are exactly equal. The remaining equations are compatible and form a closed equations system which is further reduced to the following form

$$a_1' = \frac{a_1 b_1}{6},\ (a_1 b_1)' = \frac{1}{2}a_1,\ a_2' = \frac{a_2 b_1}{2},\ \varphi_1' = \varphi_2' = -\frac{1}{2}. \tag{54}$$

Solving the closed system gives a set of special solutions

$$a_1(t) = \bar{A}e^{-\frac{\sqrt{3}t}{6}},\ a_2 = \bar{B}e^{-\frac{\sqrt{3}t}{2}},\ b_1 = -\sqrt{3},\ \varphi_1 = \varphi_2 = -\frac{t}{2} + \bar{\varphi}, \tag{55}$$

where $\bar{A}$, $\bar{B}$ and $\bar{\varphi}$ are constants. Under this condition, the global asymptotic approximate solutions to (41) are given by

$$\begin{aligned} w =& \frac{3\bar{A}}{2}e^{-\frac{\sqrt{3}}{6}t}\sin\left(\frac{t}{2}+\bar{\varphi}\right) - \sqrt{3} - \frac{1}{24}\bar{A}^2 e^{-\frac{\sqrt{3}}{3}t}\sin(t+2\bar{\varphi}) \\ &+ \frac{1}{6}\bar{B}^2 e^{-\sqrt{3}t}\sin(t+2\bar{\varphi}) - \frac{\sqrt{3}}{2}\bar{A}e^{-\frac{\sqrt{3}}{6}t}\cos\left(\frac{t}{2}+\bar{\varphi}\right) + O(1), \end{aligned} \tag{56}$$

and

$$g = \bar{B}e^{-\frac{\sqrt{3}}{2}t}\sin\left(\frac{t}{2}+\bar{\varphi}\right) + O(1). \tag{57}$$

It is easy to see that we needn't consider the secular terms of formulas (51) and (52) at all, and global approximate asymptotic solutions are obtained by HTR method.

5 Conclusion

In this paper, we apply the HTR method to Falkner-Skan equation and Von Kármá's problem of a rotating disk in an infinite viscous fluid, and obtain their global asymptotic solutions. The results show that the HTR method is simple and powerful to give the global approximate solutions since the secular terms needn't be considered at all and the renormalisation equation is very simple and natural.

Acknowledgements: The authors would like to thank the referees for their helpful comments and suggestions. This project is supported by Scientific Research Fund of Education Department of Heilongjiang Province of China under Grant No:12541083.

References

[1] E. Buckingham, Phys. Rev. **14**, 617 (1914).
[2] N. Goldenfeld, O. Martin, Y. Oono, and F. Liu, Phys. Rev. Lett. **64**, 1361 (1990).
[3] L. Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. **A44**, 6455 (1991).
[4] L. Y. Chen and N. Goldenfeld, Phys. Rev. **A45**, 5572 (1992).
[5] T. Tao, G. Cheng, and J. W. Liu, Commun. Theor. Phys. **42**, 290 (2004).
[6] T. Tao, G. Cheng, and J. W. Liu, Commun. Theor. Phys. **37**, 741 (2002).
[7] T. Tao, G. Cheng, and J. W. Liu, Commun. Theor. Phys. **36**, 617 (2001).
[8] T. Kunihiro, Prog. Theor. Phys. **94**, 835 (1995).
[9] T. Kunihiro, Jpn. J. Ind. Appl. Math. **14**, 51 (1995).
[10] T. Kunihiro, Prog. Theor. Phys. **97**, 179 (1996).
[11] T. Kunihiro, Prog. Theor. Phys. **97**, 459 (1998).
[12] K. Yue, Phy. Scr. **91**, 025202 (2016).
[13] C. Liu. arXiv: 1605.02888v1 [math. ph].
[14] V. M. Falkner, S. W. Skan, Phil. Mag. **12**, 865 (1931).
[15] T. Von Kármá, ZAMM. **1**, 233 (1921).
[16] T. Cebeci and P. Bradshaw, Momentum Transfer in Boundary Layers, Series in Thermal and Fluids Engineering, Hemisphere Publishing Corporation, Washington 1977.
[17] H. Ullah, S. Islam, M. Idrees, and M. Arif, Abstract and Applied Analysis, Hindawi Publishing Corporation, Cairo, New York 2013.
[18] Ullah, H, Islam, S, Idrees, M, and Nawaz, R, Math. Prob. Engineer. **2013**, Article ID 362816.
[19] C. Duque-Daza, D. Lockerby, C. Galeano, J. Pro. Den. **28**, 379 (1972).
[20] W. G. Cochran, Pro. Camb. Phil Soc. **30**, 365 (1934).
[21] H. E. Fettis, Proc. 4th Midwestern Conf. on Fluid Mech., Perdue 1955.
[22] M. H. Roger and G. N. Lance, J. Fluid Mech. **7**, 617 (1960).
[23] E. R. Benton, J. Fluid Mech. **24**, 781 (1966).
[24] J. B. Mcleod, Arch. Ration. Mech. Anal. **33**, 91 (1969).