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A Transverse Dynamic Deflection Model for Thin Plate Made of Saturated Porous Materials

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Abstract: In this article, a transverse dynamic deflection model is established for thin plate made of saturated porous materials. Based on the Biot's model for fluid-saturated porous media, using the Love–Kirchhoff hypothesis, the governing equations of transverse vibrations of fluid-saturated poroelastic plates are derived in detail, which take the inertial, fluid viscous, mechanical couplings, compressibility of solid, and fluid into account. The free vibration and forced vibration response of a simply supported poroelastic rectangular plate is obtained by Fourier series expansion method. Through numerical examples, the effect of porosity and permeability on the dynamic response, including the natural frequency, amplitude response, and the resonance areas is assessed.

Keywords: Biot's Theory; Classical Plate Theory; Forced Vibration; Free Vibration; Saturated Poroelastic Plate.

1 Introduction

Plates are key components in many structural and machinery applications. Considering transverse deflection of plates due to external loads is important in the design and analysis of engineering structures. Up to now, several thin and moderately thick plate theories have been proposed, such as the classical thin-plate theory, the first-order shear deformation theory [1, 2], and the higher-order shear deformation theory [3, 4]. Recently, some works on the mechanical behaviours of the nanoplate using nonlocal high shear deformation plate theory have been reported [5–8]. The saturated porous material is composed of two elements, one of which is solid and the other element is liquid. During the past several years, porous material structures, such as beams, plates, and shells, are widely used in geotechnical engineering, bioengineering, and material science and engineering [9–11]. Problem of deflection of the porous plates has been developed by many authors. Biot [12, 13] used Lagrange's equations to derive a set of coupled differential equations governing the motions of solid and fluid phases, extended the acoustic propagation theory in the wider context of the mechanics of porous media, in which the inertial, viscous, mechanical coupling, and compressibility of the solid particles and fluid were considered. Biot's theory has successfully predicted that there are generally two kinds of compression wave and a kind of shear wave in the macroscopic isotropic homogeneous porous medium, and the three kinds of wave are all with frequency dispersion characteristics. The parameters meaning in Biot's model has been explained and feasible experiment methods have been given by Biot [14, 15]. During the past several years, porous material structures, such as beams, plates, and shells, are used widely in structural design problems, including bone, heart, and urinary bladder. Compared with the research achievements for sing-phase elastic structures [3, 16–18], few initial-boundary value problems involving poroelastic structures have been reported in the literature. The buckling of a fluid-saturated poroelastic plate under axial compression has been investigated by Biot [19]. Wen [20] investigated on analytical solutions for deformation of a thick circular plate saturated by an incompressible fluid, in which the shear deformation theory is used.

Based on the Biot's poroelastic theory [14], Taber [21] presented a model for transverse deflection of fluid-saturated poroelastic plates. The governing equations are developed using the methods of the classical linear theory for bending of thin elastic plates. Another governing equations of flexural vibrations fo thin, fluid-saturated poroelastic plates were presented by Theodorakopoulos and Beskos [22], in which the inertia effects are included. Based on the classical theory of homogeneous plates and on Biot's stress–strain relations in an isotropic porous medium with a uniform porosity. A simple model of the transverse vibrations of a thin rectangular porous plate saturated by a fluid was proposed by Leclaire [23]. Two

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coupled dynamic equations of equilibrium relating the plate defection and the fluid/solid relative displacement are given, and the energy dissipation by viscous friction is included in the model.

Based on Biot’s three-dimensional system of equations in frequency domain, using the method of series expansions, a general framework for the development of assumption-free poroelastic plate theories is presented by Nagler and Schanz [24], which allows deriving poroelastic plate formulations of any desired level of approximation by power series expansions in thickness direction. However, these transverse vibration models of thin poroelastic plates are assumed that the normal stress component in plate thickness-direction vanishes. This is contradictory to hypothesis of classical plate theory that the displacement in plate thickness-direction is independent of thickness-direction coordination.

This work deals with the problem of transverse vibrations of thin, fluid-saturated, Using Kirchhoffs theory of plates and Biot’s model, the governing equations of dynamic vibration are derived, which consider the inertial, fluid viscous, mechanical couplings, compressibility of solid skeleton, and fluid. The dynamic response of a rectangular, simply supported, poroelastic plate to harmonic lateral load is obtained and the effects of physical parameters, such as porosity and permeability, on the response are assessed. In this article, the plane stress hypothesis is not used and only considers the normal strain component in plate thickness-direction vanishes. These assuming can satisfy that the hypothesis of classical plate theory for the transverse displacement is constant over the thickness.

2 Basic Equations for Poroelastic Thin Plate

Let $\mathbf{u}^s(x, y, z, t) = u_x^s\mathbf{e}_x + u_y^s\mathbf{e}_y + u_z^s\mathbf{e}_z$ and $\mathbf{w}^f(x, y, z, t) = w_x^f\mathbf{e}_x + w_y^f\mathbf{e}_y + w_z^f\mathbf{e}_z$ be the solid and fluid displacements, respectively, at time t relative to the Cartesian coordinates (x, y, z). Consider a fluid-saturated poroelastic plate made of a material for which diffusion is possible in in-plane directions only, as shown in Figure 1. The plate with size of $a \times b \times h$ $(h << a,b)$ and subjected to distributed transverse load $p_1(x, y, t)$ and $p_2(x, y, t)$ on the upper surface $(z = h/2)$ and lower surface $(z = -h/2)$, respectively. According to the classical plate theory and Biot’s porosity model, the transverse displacement of the middle surface is $u_z^s(x, y, z, t) = w_s(x, y, 0, t)$. Then, the in-plane solid displacement components are given by

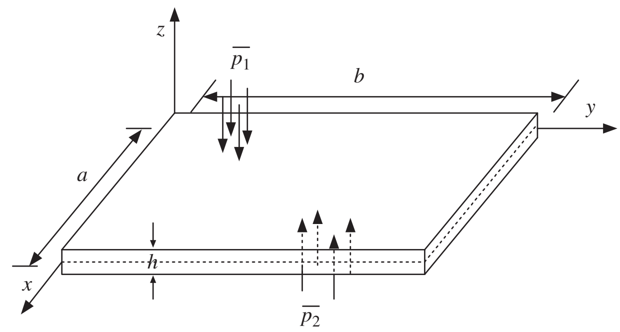

Figure 1: A thin rectangular fuild-saturated poroelastic plate model.

$$u_x^s = -z\frac{\partial w_s}{\partial x},\ u_y^s = -z\frac{\partial w_s}{\partial y} \tag{1}$$

Based on the small deformation assumption, the geometric relations are

$$\begin{aligned}
&\varepsilon_x^s = -z\frac{\partial^2 w_s}{\partial x^2}, \varepsilon_y^s = -z\frac{\partial^2 w_s}{\partial y^2}, \varepsilon_z^s = \frac{\partial w_s}{\partial z} = 0,\\
&\varepsilon_{xy}^s = -z\frac{\partial^2 w_s}{\partial x \partial y}, \varepsilon_{xz}^s = \varepsilon_{yz}^s = 0
\end{aligned} \tag{2}$$

For an isotropic medium with uniform porosity and in the absence of body force, the stress–strain relations can be written as [11]

$$\tau_{ij} = 2\mu\varepsilon_{ij}^s + (\lambda\varepsilon_{kk}^s - \alpha p)\delta_{ij} \tag{3}$$

$$p_f = M(\zeta - \alpha\varepsilon_{kk}^s) \tag{4}$$

According to Biot’s theory, the motion equations as

$$\tau_{ij \cdot j} = \rho\ddot{u}_i^s + \rho_f\ddot{w}_i^f \tag{5}$$

$$-p_{f,i} = \rho_f\ddot{u}_i^s + m_1\ddot{w}_i^f + \frac{1}{k_f}\dot{w}_i^f \tag{6}$$

where $\tau_{ij}(i,\ j = x,\ y,\ z)$ are the total stress components of solid skeleton. λ and μ are Lame’s constants. $\lambda = \frac{\nu E}{1-\nu^2}$, $\mu = \frac{E}{2(1+\nu)}$, E and ν are the Young’s modulus and Poisson’s ratio, respectively. $\zeta = w_{i,i}^f$ is the coefficient of expansion. δ_{ij} is the Kronecker symbol. α and M are Biot’s parameters, $\alpha = 1 - K/K_s$, $1/M = (\alpha - \phi)K_s + \phi/K_f$, K, K_f, and K_s are the bulk modulus of solid skeleton, solid particles, and interstitial fluid, respectively. $\rho = (1-\phi)\rho_s + \phi\rho_f$ is the mass density of mixed media and ρ_s and ρ_f are the solid and liquid densities. ϕ is the porosity. p_f is the pore pressure. k_f is the effective permeability coefficient. $m_1 = \rho_f/\phi$ is the mass density and void geometry feature associated with void fluid.

3 Governing Equations of Transverse Vibration

Substituting (1) and (2) into the constitutive relation (3), the following expressions for stress components can be obtained

$$\tau_{xx} = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w_s}{\partial x^2} + \nu\frac{\partial^2 w_s}{\partial y^2}\right) - \alpha p_f \tag{7a}$$

$$\tau_{yy} = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w_s}{\partial y^2} + \nu\frac{\partial^2 w_s}{\partial v^2}\right) - \alpha p_f \tag{7b}$$

$$\tau_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w_s}{\partial x \partial y} \tag{7c}$$

Derivate (5) and (6) along the z direction

$$\tau_{ij,jk} = \rho \ddot{u}^s_{i,k} + \rho_f \ddot{w}^f_{i,k} \tag{8}$$

$$-p_{,ii} = \rho_f \ddot{u}^s_{i,i} + m_1 \ddot{w}^f_{i,i} + \frac{1}{k_f}\dot{w}^f_{i,i} \tag{9}$$

Sum of the (8) in the x and y direction

$$\frac{\partial^2 \tau_{xx}}{\partial x^2} + \frac{\partial^2 \tau_{yy}}{\partial y^2} + 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{yz}}{\partial y \partial z} + \frac{\partial^2 \tau_{xz}}{\partial x \partial z} = \rho\left(\frac{\partial \ddot{u}^s_x}{\partial x} + \frac{\partial \ddot{u}^s_y}{\partial y}\right) + \rho_f\left(\frac{\partial \ddot{w}^f_x}{\partial x} + \frac{\partial \ddot{w}^f_y}{\partial y}\right) \tag{10}$$

By making use of the expansion coefficient $\zeta = w^f_{i,i}$, (4) can be rewritten as

$$w^f_{i,i} = \alpha\varepsilon^s_{kk} + \frac{1}{M}p_f \tag{11}$$

Considering the fluid diffusion is in-plane direction only, substituting (11) into (9) and (10), yields

$$-p_{,ii} = \rho_f \ddot{u}^s_{i,i} + m_1\left(\alpha\ddot{\varepsilon}^s + \frac{1}{M}\ddot{p}_f\right) + \frac{1}{k_f}\left(\alpha\dot{\varepsilon}^s + \frac{1}{M}\dot{p}_f\right) \tag{12}$$

$$\frac{\partial^2 \tau_{xx}}{\partial x^2} + \frac{\partial^2 \tau_{yy}}{\partial y^2} + 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{yz}}{\partial y \partial z} + \frac{\partial^2 \tau_{xz}}{\partial x \partial z} = \rho\left(\frac{\partial \ddot{u}^s_x}{\partial x} + \frac{\partial \ddot{u}^s_y}{\partial y}\right) + \rho_f\left(\alpha\ddot{\varepsilon}^s + \frac{1}{M}\ddot{p}_f\right) \tag{13}$$

Substituting (2) into (13), and integrating the result across the thickness of plate

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y} + \left(\frac{\rho h^3}{12} + \frac{\alpha\rho_f h^3}{12}\right)\left(\frac{\partial^2 \ddot{w}_s}{\partial x^2} + \frac{\partial^2 \ddot{w}_s}{\partial y^2}\right) - \frac{\rho_f}{M}M_p = 0 \tag{14}$$

where M_x and M_y are the bending moments around the x- and y-axes, respectively. M_{xy} is the twisting moment. M_p is the pore pressure p_f moment resultant over the cross section and Q_x and Q_y are the shear forces in z direction.

$$(M_x, M_y, M_{xy}, M_p) = \int_{-h/2}^{h/2}(\tau_{xx}, \tau_{yy}, \tau_{xy}, p_f)z\,dz \tag{15a}$$

$$(Q_x, Q_y) = \int_{-h/2}^{h/2}(\tau_{xy}, \tau_{xz})dz. \tag{15b}$$

Integrating (5) in the z direction over the thickness of plate gives

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_0 = \int_{-h/2}^{h/2}\rho\ddot{u}^s_z dz \tag{16}$$

where $p_0 = \bar{p}_2 - \bar{p}_1$

Substituting (7) into (15) yields

$$M_x = D(-w_{s,xx} - \nu w_{s,yy}) - \alpha M_p \tag{17a}$$

$$M_y = D(-w_{s,yy} - \nu w_{s,xx}) - \alpha M_p \tag{17b}$$

$$M_{xy} = D(\nu - 1)w_{s,xy} \tag{17c}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ denotes the plate flexural rigidity.

Substitution of (16) and (17) into (14) gives

$$D\nabla^4 w_s + \alpha\nabla^2 M_p - \left(\frac{\rho h^3}{12} + \frac{\alpha\rho_f h^3}{12}\right)\nabla^2\ddot{w}_s + \frac{\rho_f}{M}M_p + \rho h\ddot{w}_s = p_0 \tag{18}$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2d Laplace operator.

Substituting (2) into (12) and integrating the result over the thickness gives

$$\nabla^2 M_p = \left(\frac{\rho_f h^3}{12} + \frac{\alpha m_1 h^3}{12}\right)\nabla^2\ddot{w}_s + \frac{\alpha h^3}{12k_f}\nabla^2\dot{w}_s - \frac{m_1}{M}\ddot{M}_p - \frac{1}{Mk_f}\dot{M}_p \tag{19}$$

The partial differential equations system (18) and (19) is the dynamic governing equation for fluid-saturated poroelastic plate with in-plane diffusion, in which the transverse deflection of the mid-plane w_s and the equivalent moment of interstitial fluid M_p as unknown quantities.

For incompressible solid and fluid constituents, it leads to $\alpha \to 1$, $1/M \to 0$, (18) and (19) can be reduced as

$$D\nabla^4 w_s + \frac{h^3}{12}\left(\frac{\rho_f}{\phi} - \rho\right)\nabla^2 \ddot{w}_s + \frac{h^3}{12k_f}\nabla^2 \dot{w}_s + \rho h \ddot{w}_s = p_0 \quad (20)$$

For sing-phase elastic solid plate, there is no fluid-phase constituent, and the (20) can be reduced as follows:

$$D\nabla^4 w_s - \frac{\rho h^3}{12}\nabla^2 \ddot{w}_s + \rho h \ddot{w}_s = p_0 \quad (21)$$

4 Dynamic Analysis of Saturated Porous Plate with Four-Edge Simply Supported

Considering a rectangular fluid-saturated poroelastic plate subjected to uniformly distributed harmonic load p_0 with frequency ω, for four-edge simply supported and permeable boundary, the boundary conditions

$$\text{at } x = 0,\ a,\ w_s = 0,\ M_x = 0,\ M_p = 0 \quad (22a)$$

$$\text{at } y = 0,\ b,\ w_s = 0,\ M_y = 0,\ M_p = 0 \quad (22b)$$

Since the plate sides are simply supported, the solution is assumed to be expressible in double Fourier series, i.e.

$$p_0(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{0mn}\, e^{i\omega t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (23a)$$

$$w_s(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} w^s{}_{mn}\, e^{i\omega t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (23b)$$

$$M_p(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} M_{pmn}\, e^{i\omega t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (23c)$$

where p_{0mn}, $w^s{}_{mn}$, and M_{pmn} are the amplitudes of p_0, w_s, and M_p in (23), respectively. $p_{0mn} = \frac{4}{ab}\int_0^1\int_0^1 p_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{b} dxdy$

Substituting (23) into (18) and (19) yields

$$\left[D\xi^2 - \rho h\omega^2 - \xi\omega^2\left(\frac{\alpha\rho_f h^3}{12} + \frac{\rho h^3}{12}\right)\right] w^s_{mn} = \left(\alpha\xi - \frac{\rho_f}{M}\right) M_{pmn} + p_{0mn} \quad (24)$$

$$M_{pmn} = -\left(\frac{\rho_f h^3}{12} + \frac{\alpha m_1 h^3}{12}\right)\omega^2 w^s_{mn} + \frac{\alpha h^3 i\omega}{12k_f} w^s_{mn} + \left(\frac{i\omega}{Mk_f} - \frac{m_1\omega^2}{M\xi}\right) M_{pmn} \quad (25)$$

where $\xi = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

According to (24) and (25), we can obtain the displacement amplitudes $w^s{}_{mn}$ of fluid-saturated plate and interstitial fluid equivalent moment M_{pmn}. Then, combining the (23), the problem of transverse vibration will be solved.

5 The Numerical Results and Discussion

The dynamic response of plates made of porous material with in-plane diffusion under free and distributed transverse harmonic load is investigated. The effects of poroelastic parameters on free vibration frequency and force vibration amplitude are presented. The numerical parameters of the plate are chosen as shown in Table 1.

5.1 Free Vibration

For free vibration, the load amplitude $p_{0mn} = 0$ in (24) and (25), the natural frequency of fluid-saturated poroelastic

Table 1: Numerical computer parameters for saturated porous plate [24].

Lateral size	$a = 4$ m
Lateral size	$b = 4$ m
Thickness	$h = 0.2$ m
Poisson's ratio	$v = 0.25$
Solid density	$\rho_s = 2660$ kg/m^3
Fluid density	$\rho_f = 1000$ kg/m^3
Coefficient of bulk volume change	$\alpha = 0.8533$
The bulk modulus of solid particles	$K_s = 3.6 \times 10^8$ Pa
The bulk modulus of interstitial fluid	$K_f = 3.6 \times 10^7$ Pa
The Young's modulus	$E = 3.6 \times 10^8$ Pa

plate can be calculated. Under different values of the effective permeability coefficient k_f, the variation curve of first order frequency (m, $n=1$) with different porosity is plotted in Figure 2. It shows that the porosity and fluid permeability coefficient have influence significantly on free vibration frequency. Comparing of single elastic plate, due to the loss factor caused by the fluid–solid interactions in the fluid-saturated poroelastic plate, by increasing fluid permeability coefficient, the natural frequency of the plate decreases. In addition, increasing the porosity of the porous materials increases the natural frequency.

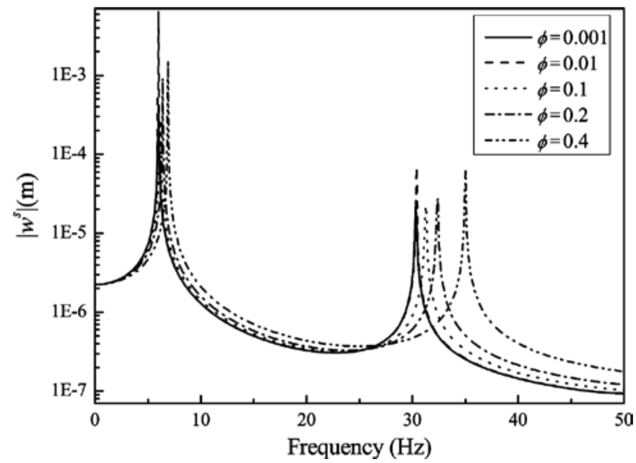

Figure 3: Resonance areas with respect to the frequency for different porosity.

5.2 Force Vibration

Considering the plate subjected a transverse unit uniform harmonic load, the response of the vertical deflection of the plate middle-point located at $x=y=2$ m as a function of frequency is shown in Figures 3 and 4 for various values porosity and permeability.

Under different values of permeability coefficient ϕ, the transverse deflection of the plate middle point with $k_f = 1.9\times10^{-6}$ is given in Figure 3. It is observed that the resonance areas are shifted to the right and these shifts are more distinguishable for higher frequency. This phenomenon is caused by the stiffness variation of plate because of porosity changed. Since according to the classical theory of plates, the resonance frequencies are inversely proportional to the density, a higher density of the porous plate will result in lower resonance frequencies. This shifting is due to the fact that the frequency is in general proportional to the square root of the stiffness to mass ratio and increasing values of porosity decrease the mass more than the stiffness of the plate.

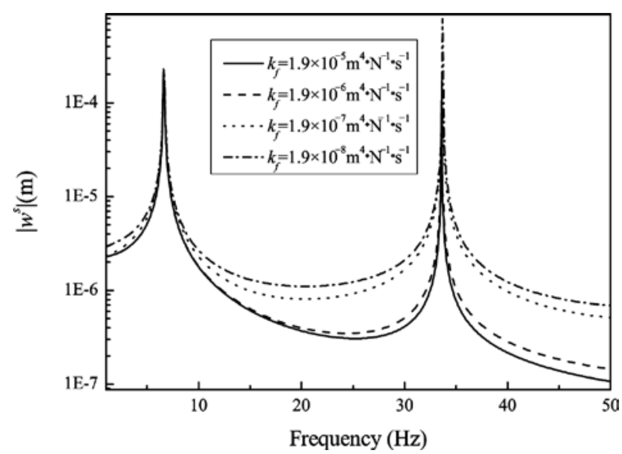

Figure 4: Resonance areas with respect to the frequency for different fluid permeability coefficient.

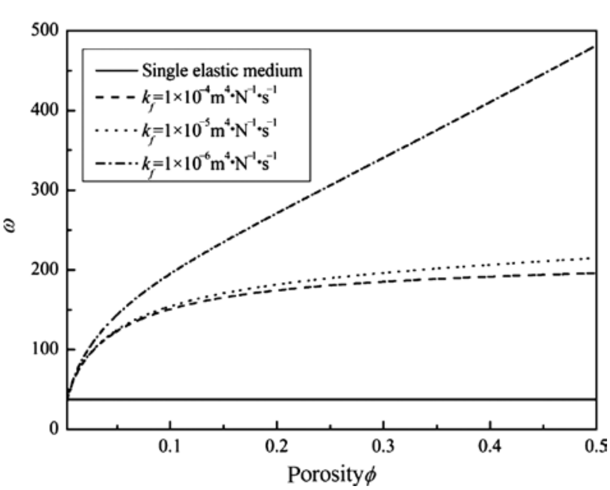

Figure 2: Variation of natural frequency with different values of porosity and permeability coefficient.

Under different values of permeability coefficient k_f, the transverse deflection of the plate middle point with $\phi=0.1$ is given in Figure 4. We can find that the scope of resonance frequency is almost equivalent in different permeability coefficient under the same porosity. A physical explanation of this phenomenon is that the permeability coefficient has a little effect on the free vibration frequency of the poroelestic plate with constant porosity.

6 Conclusions

The governing equations of transverse vibrations of thin fluid-saturated poroelastic plates were presented in detail based on the classical theory of homogenous plates and on Biot's model for porous media. These governing equations have wider scope of application compared with corresponding ones for poroelastic plate with incompressible solid and fluid constituents and sing-phase elastic solid plate. By using Fourier series expansion method, dynamic response of a poroelastic plate with the boundary condition of four edges simply supported and free permeable is analysed and the influences of porosity and permeability have been studied. The numerical example result shows that by increasing permeability or decreasing porosity, the natural frequency of the plate decreases. It was also found that the resonance frequency is an increasing function of porosity.

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