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Fractional Zero-Point Angular Momenta in Noncommutative Quantum Mechanics

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Abstract: The charged particle confined by a harmonic potential in a noncommutative planar phase space interacting with a homogeneous dynamical magnetic field and Aharonov-Bohm potentials is studied. We find that the canonical orbital angular momenta of the reduced models, which are obtained by setting the mass and a dimensionless parameter to zero, take fractional values. These fractional angular momenta are not only determined by the flux inside the thin long solenoid but also affected by the noncommutativities of phase space.

Keywords: Aharonov–Bohm Potential; Fractional Angular Momenta; Noncommutativities.

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The Aharonov–Bohm (AB) effect exhibits the shift of the interference pattern in the double-slit experiment with a thin long solenoid located between these two slits [1]. It indicates that quantum mechanically, charged particles are influenced by magnetic vectors in the region where the field strengths are zero. It is quite different from the classical mechanics since the classical motion of the charged particle is not affected by the AB potentials. Therefore, AB effect reveals the fundamental roles the magnetic potentials play in quantum theories. It is well known that contrary to three-dimensional space in which angular momenta can only take half-integer values (we set $\hbar=1$), angular momenta in a plane can take fractional values. It leads to an important concept in theoretical physics, namely, anyons [2–4].

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There are many methods to realise the fractional angular momentum. One of the most classical methods of realising fractional angular momentum is to couple a charged planar particle to the Abelian Chern-Simons gauge field [2–4].

Recently, the author of [5] proposed an interesting method to realise the fractional angular momentum. He considered a planar charged particle (take, for example, an ion) confined by a harmonic potential in the background of a dynamical magnetic field¹ and AB magnetic vector potentials. The author found that the angular momentum of the reduced model, which is obtained by setting the kinetic energy to one of its eigenvalues, i.e. the lowest energy level, takes the fractional values. This fractional angular momentum is induced by the AB vector potentials and is proportional to the magnetic flux inside the thin long solenoid. The dynamical magnetic field plays an interesting role: although it does not contribute to the fractional angular momentum, the fractional zero-point angular momentum will not appear in the absence of it.

On the other hand, spatial non-commutativity attracts much attention nowadays because of the string theory [6–8]. In fact, it had a long history in physics [9, 10]. It is known that the spatial noncommutativity arises naturally from string theory [11–14]. There are numerous papers about quantum field theories on the non-commutative space, including both perturbative and non-perturbative aspects [15–17].

Quantum mechanics on the noncommutative space has also been studied for both non-relativistic and relativistic models. The most popular method of studying quantum mechanics in noncommutative space is to map the noncommutative space to a commutative one by the Bopp shift (or the generalised Bopp shift) and then study them in the commutative space [18–23]. Most exactly solvable models were studied by using this method. Path integral formulation in noncommutative mechanics has also been investigated [24]. Based on the path integral formulation in noncommutative space, the authors of Refs. [25, 26] deduced an effective Lagrangian and gave the non-commutative corrections to the AB effect in noncommutative space. The application of path integral formulation to

¹ We name it dynamical magnetic field since it influences the classical motions of the charged particles.

solve the spectra of noncommutative quantum mechanical models has also been investigated [27–29].

In this paper, we shall generalise the work of Ref. [5] to the noncommutative planar phase space; i.e. both the coordinates and momenta are noncommutative simultaneously. We shall show that the fractional angular momenta can also be obtained in the noncommutative planar phase space if certain limits are taken.² The noncommutative plane is described by the commutators (the lowercase Latin indexes run from 1 to 2)

$$[x_i, x_j] = i\theta\epsilon_{ij}, [x_i, p_j] = i\delta_{ij}, [p_i, p_j] = 0, \quad (1)$$

where θ is the noncommutative parameter between coordinates which is taken as a constant throughout this paper, and ϵ_{ij} is the two-dimensional anti-symmetrical tensor.

The noncommutative planar phase space is defined by the commutation relations

$$[x_i, x_j] = i\theta\epsilon_{ij}, [x_i, p_j] = i\delta_{ij}, [p_i, p_j] = i\eta\epsilon_{ij}, \quad (2)$$

where η is the noncommutative parameter between momenta (it is also taken as a constant in this paper). Compared with the noncommutative plane (1), we can see that the momenta are also noncommutative in the noncommutative phase space. Quantum mechanical models on the noncommutative phase spaces which are characterised by the algebraic relations similar with (2) were widely studied in the past years [30–47].

We can construct the Lagrangian which gives the classical version (i.e. $\{ \ , \ } \rightarrow \frac{1}{i} [\ , \]$) of the noncommutative phase space (2). It is (the summation convention is used throughout this paper)

$$L = \frac{1}{\kappa} \left(p_i \dot{x}_i + \frac{1}{2} \eta \epsilon_{ij} x_i \dot{x}_j + \frac{1}{2} \theta \epsilon_{ij} p_i \dot{p}_j \right) - H \quad (3)$$

in which $\kappa = 1 - \theta\eta$, and H is a specific Hamiltonian. In this paper we shall focus on the harmonic potential. Thus, the Hamiltonian is

$$H = \frac{p_i^2}{2m} + \frac{K}{2} x_i^2 \quad (4)$$

with K being a constant. Note that θ and η have the dimensions of $\mathbf{L}^2$ and $\mathbf{L}^{-2}$, respectively, so κ is dimensionless. We should note that the noncommutativities, if they do exist,

will be extremely small [31, 32], i.e. $|\theta|, |\eta| \ll 1$. Therefore, there is no singularity in the Lagrangian (3).

There are two different kinds of magnetic vector potentials we shall introduce. One kind is the dynamical magnetic vector potentials. The other is the AB ones which are realised by putting a thin long solenoid vertical to the plane. We introduce these magnetic vector potentials into the model (3) by the minimal substitution [48–50]

$$p_i \rightarrow p_i + A_i + A_i^{AB} \quad (5)$$

where A_i and A_i^{AB} are magnetic vector potentials of the dynamical field and AB ones, respectively. Choosing the symmetrical gauge, we get

$$A_i = -\frac{B}{2} \epsilon_{ij} x_j \quad (6)$$

for the dynamical magnetic field with B being the field strength and

$$A_i^{AB} = -\frac{\Phi_0}{2\pi} \frac{\epsilon_{ij} x_j}{x_k x_k} \quad (7)$$

for the AB potentials outside the solenoid with Φ_0 being the flux inside the thin long solenoid.

Thus, the model we are interested in the present study is described by the Lagrangian (outside the solenoid)

$$L = \frac{1}{\kappa} \left[(p_i + A_i + A_i^{AB}) \dot{x}_i + \frac{1}{2} \eta \epsilon_{ij} x_i \dot{x}_j + \frac{1}{2} \theta \epsilon_{ij} p_i \dot{p}_j \right] - H. \quad (8)$$

Substituting the expressions of magnetic vector potentials (6, 7) into the above Lagrangian and writing the term $p_i \dot{x}_i$ in the symmetrical form $\frac{1}{2}(p_i \dot{x}_i - x_i \dot{p}_i)$, we get

$$L = \frac{1}{2\kappa} \left[p_i \dot{x}_i - x_i \dot{p}_i + B \epsilon_{ij} x_i \dot{x}_j + \frac{\Phi_0}{\pi} \frac{\epsilon_{ij} x_i \dot{x}_j}{x_k x_k} + \theta \epsilon_{ij} p_i \dot{p}_j \right] - H, \quad (9)$$

where H has been given in (4) and

$$B_T = B + \eta \quad (10)$$

is the combination of the dynamical magnetic field and the noncommutative parameter between momenta. Clearly, the noncommutative parameter between momenta contributes to the dynamical magnetic field.

We will show that the model (9) has two different reduced models which are obtained by setting the mass and a dimensionless parameter to zero. The angular momentum of each reduced model takes fractional values. Therefore, there are two different mechanisms to get the fractional angular momentum from the model (9).

² We noticed that the fractional angular momentum can also be produced by the spatial noncommutativity [30]. However, the mechanisms of getting angular momentum in Ref. [30] are different from those in the present paper.

We shall quantise the model (9) canonically before studying its quantum properties. In doing so, we introduce the canonical momenta with respects to the variables (x_i, p_i) . They are

$$\begin{aligned}\pi_i^x &= \frac{\partial L}{\partial \dot{x}_i} = \frac{1}{2\kappa} \left(p_i - B_T \epsilon_{ij} x_j - \frac{\Phi_0}{\pi} \frac{\epsilon_{ij} x_j}{x_k x_k} \right), \\ \pi_i^p &= \frac{\partial L}{\partial \dot{p}_i} = \frac{1}{2\kappa} (-x_i - \theta \epsilon_{ij} p_j).\end{aligned}\quad (11)$$

The basic non-vanishing Poisson brackets among canonical variables $(x_i, p_i, \pi_i^x, \pi_i^p)$ are

$$\{x_i, \pi_j^x\} = \{p_i, \pi_j^p\} = \delta_{ij}. \quad (12)$$

The canonical orbital angular momentum, by definition, is

$$J = \epsilon_{ij} (x_i \pi_j^x + p_i \pi_j^p). \quad (13)$$

The introduction of canonical momenta (11) does not contain ‘velocity’ terms. It results in algebraic relations among canonical variables $(x_i, p_i, \pi_i^x, \pi_i^p)$. Therefore, they are primary constraints in the terminology of Dirac [51]. In fact, the Lagrangian (9) is in the first-order form; it is not surprising to get this result.

The primary constraints are

$$\begin{aligned}\phi_i^{(0)} &= \pi_i^x - \frac{1}{2\kappa} \left(p_i - B_T \epsilon_{ij} x_j - \frac{\Phi_0}{\pi} \frac{\epsilon_{ij} x_j}{x_k x_k} \right) \approx 0, \\ \psi_i^{(0)} &= \pi_i^p + \frac{1}{2\kappa} (x_i + \theta \epsilon_{ij} p_j) \approx 0,\end{aligned}\quad (14)$$

in which ‘ $\approx$ ’ means equivalent on the constraint hypersurface. For future convenience, we label the primary constraints $\phi_i^{(0)} \approx 0, \psi_i^{(0)} \approx 0$ in a unified way as

$$\Phi_I = (\phi_i, \psi_i) \approx 0, \quad I = 1, 2, 3, 4. \quad (15)$$

The Poisson brackets among the primary constraints Φ_I are

$$\{\Phi_I, \Phi_J\} = \begin{pmatrix} 0 & \frac{B_T}{\kappa} & -\frac{1}{\kappa} & 0 \\ -\frac{B_T}{\kappa} & 0 & 0 & -\frac{1}{\kappa} \\ \frac{1}{\kappa} & 0 & 0 & \frac{\theta}{\kappa} \\ 0 & \frac{1}{\kappa} & -\frac{\theta}{\kappa} & 0 \end{pmatrix}. \quad (16)$$

The determinant of this matrix is

$$\text{Det}\{\Phi_I, \Phi_J\} = \frac{\gamma^2}{\kappa^4}, \quad (17)$$

in which

$$\gamma = 1 - \theta B_T \quad (18)$$

is also a dimensionless parameter.

Clearly, the determinant of the matrix $\{\Phi_I, \Phi_J\}$ can be modulated by adjusting the magnitude of the dynamical magnetic field. It will vanish when the dimensionless parameter γ takes zero value (or when the magnitude of the dynamical magnetic field takes critical value $B = B_c = \frac{1}{\theta} - \eta$). One can anticipate intuitively that whether the determinant of this matrix takes zero value or not will lead to different results.

Thus, we shall investigate the cases of $\text{Det}\{\Phi_I, \Phi_J\} \neq 0$ and $\text{Det}\{\Phi_I, \Phi_J\} = 0$ (or, equivalently, cases of $\gamma \neq 0$ and $\gamma = 0$), respectively, in the following. We shall show that there are different mechanisms in both cases to get the fractional angular momenta. Let us study the case of $\gamma \neq 0$ firstly. In this case, the determinant of the matrix $\{\Phi_I, \Phi_J\}$ does not vanish. The inverse matrix of $\{\Phi_I, \Phi_J\}$ is

$$\{\Phi_I, \Phi_J\}^{-1} = \frac{\kappa}{\gamma} \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & B \\ 0 & -1 & -B & 0 \end{pmatrix}. \quad (19)$$

The consistency conditions of primary constraints are given by

$$\dot{\Phi}_I = \{\Phi_I, H + \lambda_J \Phi_J\} \approx 0, \quad (20)$$

in which λ_I are Lagrange multipliers. Substituting the Hamiltonian (4) and the explicit expressions of constraints Φ_I into the above equation, we find that the consistency conditions only determine the Lagrangian multipliers λ_I . Therefore, in the case of $\gamma \neq 0, \Phi_I \approx 0$ ($I = 1, 2, 3, 4$) exhausts all the constraints of the model (9), and they belong to the second class [51].

We must get the Dirac brackets among variables $(x_i, p_i, \pi_i^x, \pi_i^p)$ in order to quantise the model (9) canonically. Dirac bracket is defined by

$$\{F, G\}_D = \{F, G\} - \{F, \Psi_M\} \{\Psi_M, \Psi_N\}^{-1} \{\Psi_N, G\} \quad (21)$$

where F, G are two arbitrary functions of the canonical variables $(x_i, p_i, \pi_i^x, \pi_i^p)$ and Ψ_M, Ψ_N ($M, N = 1, 2, L, 2k$) stand for all the second-class constraints. After some direct calculations, we arrive at

$$\begin{aligned}\{x_i, x_j\}_D &= \frac{\kappa \theta \epsilon_{ij}}{\gamma}, \\ \{x_i, p_j\}_D &= \frac{\kappa \delta_{ij}}{\gamma}, \\ \{p_i, p_j\}_D &= \frac{\kappa B_T \epsilon_{ij}}{\gamma}.\end{aligned}\quad (22)$$

The quantum commutators are obtained by the replacement $[,] \rightarrow i\{, \}_D$ [51].

The constraints $\Phi_I \approx 0$ ($I=1, 2, 3, 4$) are second class; they are ‘strong’ equations and can be used to eliminate the redundant degrees of freedom in the angular momentum (13). Thus, we write the angular momentum (13) in terms of (x_i, p_i)

$$J = \frac{1}{2\kappa} \left(2\epsilon_{ij} x_i p_j + \theta p_i^2 + B_T x_i^2 + \frac{\Phi_0}{\pi} \right). \quad (23)$$

It can be checked directly by using the commutators [the quantum version of the Dirac brackets (22)] that

$$[J, H] = 0 \quad (24)$$

and

$$[J, x_i] = i\epsilon_{ij} x_j, \quad [J, p_i] = i\epsilon_{ij} p_j. \quad (25)$$

It shows that the angular momentum (23) is conserved and is the generator of rotation.

In Ref. [5], the author proposed an interesting method to obtain the fractional angular momentum from the commutative counterpart ($\theta = \eta = 0$) of (9). He found that the angular momentum of the reduced model, which was attained by setting the eigenvalues of the kinetic energy to the lowest energy level, took fractional values. This fractional value is proportional to the flux inside the thin long solenoid. The dynamical magnetic field plays an interesting role: although the dynamical magnetic field does not contribute to the fractional angular momentum, the fractional angular momentum does not appear in the absence of it.

We shall show that the fractional angular momenta can also be obtained from the noncommutative model (9) if certain limits are chosen. In fact, there are two different mechanisms to get the fractional angular momenta in the model (9).

Let us consider the first mechanism of getting the fractional angular momentum. It is to set the mass to zero, i.e. $m \rightarrow 0$ in the model (9). The zero-mass limit was first considered in Ref. [52] during the studies of Chern-Simons quantum mechanics. Then, it was generalised to noncommutative cases in Refs. [27, 29, 53].

In view of the Hamiltonian (4), we have to set

$$p_i = 0 \quad (26)$$

so as to avoid the divergency of the Hamiltonian. It means that the kinetic energy is neglected.

The Lagrangian (9) reduces to the form

$$L = \frac{1}{2\kappa} \left(B_T \epsilon_{ij} x_i \dot{x}_j + \frac{\Phi_0}{\pi} \epsilon_{ij} x_i \dot{x}_j \right) - \frac{K}{2} x_i^2 \quad (27)$$

when the limit $m \rightarrow 0$ is taken.

Therefore, besides the primary constraints $\Phi_I \approx 0$ ($I=1, 2, 3, 4$), there are two extra constraints, namely,

$$\chi_i = p_i \approx 0. \quad (28)$$

We label all the constraints as $\Psi_\alpha = (\Phi_I, \chi_i) \approx 0$ ($\alpha=1, 2, \dots, 6$). It can be verified that the constraints $\Psi_\alpha \approx 0$ are second class and there are no further constraints.

Since the constraints $\chi_i = p_i \approx 0$ are second class, they can be used to eliminate the redundant degrees of freedom. As a result, the angular momentum (23) is further simplified to the form

$$J = \frac{1}{2\kappa} \left(B_T x_i^2 + \frac{\Phi_0}{\pi} \right). \quad (29)$$

We must know the commutation relations between variables x_i before calculating the eigenvalues of the angular momentum (29). To this end, we shall calculate the Dirac brackets between x_i .

The explicit expression of the matrix $\{\Psi_\alpha, \Psi_\beta\}$ is

$$\{\Psi_\alpha, \Psi_\beta\} = \begin{pmatrix} 0 & \frac{B_T}{\kappa} & -\frac{1}{\kappa} & 0 & 0 & 0 \\ -\frac{B_T}{\kappa} & 0 & 0 & -\frac{1}{\kappa} & 0 & 0 \\ \frac{1}{\kappa} & 0 & 0 & \frac{\theta}{\kappa} & -1 & 0 \\ 0 & \frac{1}{\kappa} & -\frac{\theta}{\kappa} & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (30)$$

The inverse of this matrix is given by

$$\{\Psi_\alpha, \Psi_\beta\}^{-1} = \begin{pmatrix} 0 & -\frac{\kappa}{B_T} & 0 & 0 & 0 & -\frac{1}{B_T} \\ \frac{\kappa}{B_T} & 0 & 0 & 0 & \frac{1}{B_T} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{B_T} & -1 & 0 & 0 & -\frac{\gamma}{\kappa B_T} \\ \frac{1}{B_T} & 0 & 0 & -1 & \frac{\gamma}{\kappa B_T} & 0 \end{pmatrix}. \quad (31)$$

After some direct calculations, we get

$$\{x_i, x_j\}_D = -\frac{\kappa}{B_T} \epsilon_{ij}. \quad (32)$$

Therefore, the commutation relations among the variables x_i are

$$[x_i, x_j] = -i \frac{\kappa}{B_T} \epsilon_{ij}. \quad (33)$$

In order to express the angular momentum (29) lucidly, we introduce a new pair of variables (X, P) which satisfy $[X, P] = i$. They are

$$X = \sqrt{\frac{B_T}{\kappa}} x_2, \quad P = \sqrt{\frac{B_T}{\kappa}} x_1. \quad (34)$$

In terms of variables (X, P) , we rewrite the angular momentum (29) in the form

$$J = \frac{1}{2}(X^2 + P^2) + \frac{\Phi_0}{2\kappa\pi}. \quad (35)$$

By considering the commutation relation between X and P , we find that the first part in angular (35) is nothing but the Hamiltonian of a one-dimensional harmonic oscillator with unit mass and frequency. Thus, its eigenvalues can be written down directly. They are

$$J_n = n + \frac{1}{2} + \frac{\Phi_0}{2\kappa\pi}. \quad (36)$$

Obviously, there is an extra term in the eigenvalues of the angular momentum which is proportional to the flux inside the solenoid. It can take fractional values.

We have to emphasise that there is an exception in the case of $\gamma \neq 0$. To illustrate it clearly, let us consider the situation $B_T = 0$. It implies that the dynamical magnetic field cancels the effect of noncommutativity between the momenta.

The Lagrangian (27) is further simplified to the form

$$L = \frac{\Phi_0}{2\kappa\pi} \frac{\epsilon_{ij} x_i \dot{x}_j}{x_k x_k} - \frac{K}{2} x_i^2 \quad (37)$$

when $B_T = 0$.

We introduce the canonical momenta with respect to x_i

$$\pi_i^x = \frac{\partial L}{\partial \dot{x}_i} = -\frac{\Phi_0}{2\kappa\pi} \frac{\epsilon_{ij} x_j}{x_k x_k}. \quad (38)$$

Again, the introduction of the canonical momenta leads to primary constraints. We label them as

$$\bar{\phi}_i^{(0)} = \pi_i^x + \frac{\Phi_0}{2\kappa\pi} \frac{\epsilon_{ij} x_j}{x_k x_k} \approx 0. \quad (39)$$

Different from the situation $B_T \neq 0$ in which there are no secondary constraints, the consistency conditions of

the primary constraints $\bar{\phi}_i$ will result in the secondary constraints. They are

$$\bar{\phi}_i^{(1)} = \dot{\bar{\phi}}_i^{(0)} = \dot{x}_i \approx 0. \quad (40)$$

The constraints $p_i \approx 0$ and $x_i \approx 0$ means that there are no dynamics. Therefore, we cannot get the fractional angular momentum.

In the following, we shall analyse the other mechanism to get the fractional angular momentum from the model (9). We shall show that the fractional angular momenta will arise naturally when the dimensionless parameter $\gamma = 0$ (or, equivalently, when the magnetic field takes the value $B = B_c = \frac{1}{\theta} - \eta$).

In the case of $\gamma = 0$, the matrix (16) is singular. We should verify whether there are secondary constraints. The consistency conditions of the primary constraints are

$$\dot{\phi}_i^{(0)} = \{\phi_i, H + \mu_i \phi_i^{(0)} + \nu_i \psi_i^{(0)}\}, \quad (41)$$

$$\dot{\psi}_i^{(0)} = \{\psi_i, H + \mu_i \phi_i^{(0)} + \nu_i \psi_i^{(0)}\}, \quad (42)$$

where μ_i, ν_i are Lagrange multipliers. These consistency conditions lead to

$$\gamma \mu_i = \kappa \left(\frac{p_i}{m} + \theta K \epsilon_{ij} x_j \right), \quad (43)$$

$$\gamma \nu_i = \kappa \left(-K x_i + \frac{B_T}{m} \epsilon_{ij} p_j \right). \quad (44)$$

The above equations determine the Lagrange multipliers μ_i and ν_i uniquely if $\gamma \neq 0$. Therefore, there are no secondary constraints in the case of $\gamma \neq 0$. However, when the parameter γ takes zero value, the above equations will lead to secondary constraints. It is straightforward to check that (43) and (44) are not independent when $\gamma = 0$. They are equivalent up to an overall constant. Thus, we choose

$$x_i = -K x_i + \frac{B_T}{m} \epsilon_{ij} p_j \approx 0 \quad (45)$$

as the secondary constraints.

The consistency conditions of the secondary constraints (45) do not lead to further constraints. And the constraints (45) are also second class. They are the 'strong' equation and can be used to eliminate the redundant variables in the expression of angular momentum. Thus, the angular momentum (23) becomes

$$J = \frac{1}{2\theta\kappa} (1 + m\theta^2 K)^2 x_i^2 + \frac{\Phi_0}{2\kappa\pi}. \quad (46)$$

We must determine the commutation relation among the variables x_i in order to get the eigenvalues of the angular momentum (46). The calculation of the Dirac brackets among x_i is unavoidable.

There are six constraints in total. They are $\phi_i^{(0)} \approx 0$, $\psi_i^{(0)} \approx 0$ and $\chi_i \approx 0$ which we label as $\bar{\Phi}_\alpha \approx 0$ ($\alpha=1, 2, \dots, 6$). The matrix of the Poisson brackets among the constraints $\bar{\Phi}_\alpha \approx 0$ is

$$\{\bar{\Phi}_\alpha, \bar{\Phi}_\beta\} = \begin{pmatrix} 0 & \frac{B_T}{\kappa} & -\frac{1}{\kappa} & 0 & K & 0 \\ -\frac{B_T}{\kappa} & 0 & 0 & -\frac{1}{\kappa} & 0 & K \\ \frac{1}{\kappa} & 0 & 0 & \frac{\theta}{\kappa} & 0 & \frac{B_T}{m} \\ 0 & \frac{1}{\kappa} & -\frac{\theta}{\kappa} & 0 & -\frac{B_T}{m} & 0 \\ -K & 0 & 0 & \frac{B_T}{m} & 0 & 0 \\ 0 & -K & -\frac{B_T}{m} & 0 & 0 & 0 \end{pmatrix}. \quad (47)$$

The inverse matrix is given by

$$\{\bar{\Phi}_\alpha, \bar{\Phi}_\beta\}^{-1} = \frac{1}{(B_T^2 + Km)^2} \times \begin{pmatrix} 0 & -B_T^3 \kappa & B_T^2 Km \kappa & 0 & -m(B^2 + Km) & 0 \\ -B_T^3 & 0 & 0 & B_T^2 Km \kappa & 0 & -m(B_T^2 + Km) \\ -B_T^2 Km \kappa & 0 & 0 & B_T K^2 m^2 \kappa & 0 & -m B_T (B^2 + Km) \\ 0 & -B_T^2 Km \kappa & B_T^2 Km \kappa & 0 & B_T m (B_T^2 + Km) & 0 \\ m(B_T^2 + Km) & 0 & 0 & -B_T m (B_T^2 + Km) & 0 & 0 \\ 0 & m(B_T^2 + Km) & -\frac{B_T}{m} & 0 & 0 & 0 \end{pmatrix}. \quad (48)$$

The Dirac brackets among x_i can be derived by using the matrix (48) directly. They are

$$\{x_i, x_j\}_D = -\frac{\theta \kappa \epsilon_{ij}}{(1 + m\theta^2 K)^2}. \quad (49)$$

So the commutation relations among the variables x_i become

$$[x_i, x_j] = -i \frac{\theta \kappa \epsilon_{ij}}{(1 + m\theta^2 K)^2}. \quad (50)$$

We introduce a pair of variables $(\tilde{X}, \tilde{P})$ which satisfy $[\tilde{X}, \tilde{P}] = i$ to substitute x_i . They are

$$\tilde{X} = \frac{1 + m\theta^2 K}{\sqrt{\theta K}} x_2, \quad \tilde{P} = \frac{1 + m\theta^2 K}{\sqrt{\theta K}} x_1. \quad (51)$$

The angular momentum (46) becomes

$$J = \frac{1}{2}(\tilde{X}^2 + \tilde{P}^2) + \frac{\Phi_0}{2\kappa\pi} \quad (52)$$

in terms of variables $(\tilde{X}, \tilde{P})$. Thus, the eigenvalues of the angular momentum (52) are identical to the ones (36).

It shows that when the dimensionless parameter γ takes zero value (or when the magnetic field takes the value $B_c = \frac{1}{\theta} - \eta$), the angular momentum of the model (9) will take fractional values.

Compared with the result of the commutative counterpart ($\theta, \eta \rightarrow 0$) in Ref. [5], we find that the fractional angular momenta in the noncommutative model (9) are modified by the noncommutative parameter η . As a consistency check, it is easy to see that our result (36) is identical to the one in Ref. [5] when the noncommutative parameters θ, η take zero values.

To summarise, we generalise the studies in Ref. [5] to the planar noncommutative phase space. we find

that there are two different reduced models of (9). They are obtained by setting the mass and the dimensionless parameter $\gamma = 1 - \theta B_T$ to zero from (9), respectively. We pay our attention to the angular momenta of the reduced models. We find that the angular momentum of each reduced model takes fractional value. Compared with its commutative counterpart [5], we find that the fractional angular momentum is not only proportional to the flux inside the thin long solenoid but also affected by the noncommutativities of coordinates and momenta.

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