Yan Wang, Yufeng Zhang* and Xiangzhi Zhang

A Few New 2+1-Dimensional Nonlinear Dynamics and the Representation of Riemann Curvature Tensors

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Abstract: We first introduced a linear stationary equation with a quadratic operator in ∂_x and ∂_y, then a linear evolution equation is given by N-order polynomials of eigenfunctions. As applications, by taking $N=2$, we derived a (2+1)-dimensional generalized linear heat equation with two constant parameters associative with a symmetric space. When taking $N=3$, a pair of generalized Kadomtsev-Petviashvili equations with the same eigenvalues with the case of $N=2$ are generated. Similarly, a second-order flow associative with a homogeneous space is derived from the integrability condition of the two linear equations, which is a (2+1)-dimensional hyperbolic equation. When $N=3$, the third second flow associative with the homogeneous space is generated, which is a pair of new generalized Kadomtsev-Petviashvili equations. Finally, as an application of a Hermitian symmetric space, we established a pair of spectral problems to obtain a new (2+1)-dimensional generalized Schrödinger equation, which is expressed by the Riemann curvature tensors.

Keywords: (2+1)-Dimensional Equation; Homogeneous Space; Symmetric Space.

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1 Introduction

It has been an important task to generate integrable systems in soliton theory. A great number of (1+1)-dimensional integrable systems had been found in the past decades by applying zero-curvature equations, Lax pairs, and other techniques (see, e.g. [1–12]). However, it is more difficult to search for (2+1)-dimensional integrable systems than the (1+1)-dimensional case [13–18]. Athorne and Dorfman [15] and Dorfman and Fokas [16] constructed Hamiltonian operators to generate (2+1)-dimensional integrable systems over noncommutation rings. Moreover, Fokas and Tu [17] and Tu et al. [18] introduced a residue operator over an associative algebra to generate the Kadomtsev-Petviashvili (KP) equation and the Davey-Stewartson (DS) equation. This method was proposed by Tu et al. [18], which was called the Tu-Andrushkiw-Huang scheme, briefly called the TAH scheme. By applying the TAH scheme, some (2+1)-dimensional hierarchies and their corresponding Hamiltonian structures were obtained by Zhang et al. [19–21]. However, there exists an open problem that the integrability of the (2+1)-dimensional hierarchies obtained by the TAH scheme cannot be determined. Another approach is that Ablowitz et al. [22] applied some reduced equations of the self-dual Yang-Mills equations to generate some (1+1)- and (2+1)-dimensional integrable equations, such as the Kortweg-de Vries equation and the KP equation. On the basis of this procedure, Zhang et al. [23, 24] generated some (2+1)-dimensional integrable systems, including a (2+1)-dimensional integrable coupling, which was the first result on (2+1)-dimensional integrable coupling, to our best knowledge. In addition, Athorne and Fordy [25] applied the symmetric and homogeneous spaces to generate the N-wave, the KP equation, and the DS equation. Actually, we once adopted such symmetric space to generate nonlinear integrable couplings and some (2+1)-dimensional integrable equations [26, 27]. In this paper, we first recalled some basic notions on the symmetric and homogeneous spaces, then we introduced a stationary linear equation with a quadratic operator in ∂_x and ∂_y. An evolution equation is also introduced whose compatibility with the stationary linear equation can generate higher dimensional integrable equations. In particular, a second-order flow, which is a (2+1)-dimensional matrix heat equation, is obtained, which is associative with the symmetric space. A third flow in (2+1) dimensions

***Corresponding author: Yufeng Zhang,** College of Sciences, China University of Mining and Technology, Xuzhou 221116, P.R. China, E-mail: zhangyfcumt@163.com
Yan Wang: College of Sciences, China University of Mining and Technology, Xuzhou 221116, P.R. China; and School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454000, China
Xiangzhi Zhang: College of Sciences, China University of Mining and Technology, Xuzhou 221116, P.R. China

is obtained, which is a generalized KP equation associated with the symmetric space. Under the framework of the homogeneous space, a second-order flow, which is a (2+1)-dimensional hyperbolic equation, is expressed by an element P of the Lie subalgebra m. For the third flow associated with the homogeneous space, we generated the $m^{\pm}$ components of the Lie subalgebra m, which is a generalized KP equation. It is an extended form of the result presented by Athorne and Fordy [25]. Finally, we extended a pair of spectral problems by Fordy and Kulish [28] with the quadratic operators with respect to the operators ∂_x and ∂_y to further derive a new (2+1)-dimensional nonlinear Schrödinger equation associative with the symmetric space, which generalizes a main result by Fordy and Kulish [28]. It is remarkable that the method for generating (2+1)-dimensional nonlinear equations presented here is different from the Adler-Gelfand-Dikii (AGD) scheme, the TAH scheme, and the binomial residue representation scheme by Zhang et al. [29, 30].

2 The Symmetric and Homogeneous Spaces

We first recall some basic notions on the Hermitian symmetric and reductive homogeneous spaces [25, 31]. A homogeneous space of a Lie group G is a differentiable manifold M on which G acts transitively. The subgroup of G that leaves a given point $p_0 \in M$ fixed is called the isotropy group at p_0 and is defined by

$$K \equiv K_{p_0} = \{g \in G : g \cdot p_0 = p_0\}.$$

Such manifold M can be identified with a coset space G/K. In this paper, we only consider the decompositions of the corresponding Lie algebras of the Lie group G and the isotropy group K.

Let g and k be the Lie algebras of G and K, respectively, and let m be the vector space complement of k in g. Then we have

$$g = k \oplus m,\ [k, k] \subset k,$$

where m is identified with the tangent space $T_{p_0}M$ of $M = G/K$ at point p_0. When g satisfies the following conditions:

$$g = k \oplus m,\ [k, k] \subset k,\ [k, m] \subset m,$$

then G/K is called a reductive homogeneous space [25]. Such space possesses the defined connections with curvature and torsion. At fixed point p_0, the curvature and torsion tensors are given by the Lie bracket operation on m:

$$(R(X, Y)Z)_{p_0} = -[[X, Y]_k, Z],\ T(X, Y)_{p_0} = -[X, Y]_m,\ \forall X, Y, Z \in m.$$

When g satisfies the following conditions:

$$g = k \oplus m,\ [k, k] \subset k,\ [k, m] \subset m,\ [m, m] \subset k,$$

then g is called a symmetric algebra and G/K is a symmetric space. At fixed point p_0, the curvature is given by

$$(R(X, Y)Z)_{p_0} = -[[X, Y], Z],\ \forall X, Y, Z \in m,$$

and the torsion is free. Assuming h is a Cartan subalgebra of g, there exists an element $A \in h$ such that

$$k = C_g(A) = \{B \in g : [B, A] = 0\}.$$

If A is regular, $C_g(A) = h$. Otherwise, $C_g(A) \supset h$. The operator representation $\alpha(A)$ has three distinct eigenvalues: $0, \pm a$. In particular, we have

$$m = m^+ + m^-,\ [A, k] = 0,\ [A, X^{\pm}] = \pm a X^{\pm}.$$

For any $X \in g$, $X = X^0 + X^+ + X^-$, and $X^+ = \sum_{\alpha} X_{\alpha} e_{\alpha}$, $X^- = \sum_{\alpha} X_{-\alpha} e_{-\alpha}$, where α is summed over a special subset θ^+ of the positive root system Φ^+. In Hermitian symmetric spaces, we have the convenient property that $[X^+, Y^+] = 0$, $\forall X^+, Y^+ \in m^+$, and similarly for m^-. Besides, $X^+Y^+ = 0$ for all pairs of elements of m^+, similarly for m^-. However, for the homogeneous space, we still have $m = m^+ \oplus m^-$, but each of $m^{\pm}$ is further split into blocks, each of which is an eigenspace of $\alpha(A)$. θ^+ splits into several θ_j^+ subsets, each of which $\alpha(A) = a_j$, $\alpha \in \theta_j^+$. Assuming $Q \in m$, we can write

$$Q^{\pm} = \sum_j Q_j^{\pm},\ [A, Q_j^{\pm}] = \pm a_j Q_j^{\pm}.$$

3 Applications of the Symmetric Space

Athorne and Fordy [25] once introduced the following operator:

$$L = D_x + A\partial_y + Q,$$

where Q is a matrix function, and D_x is a derivative with respect to x. Now we consider a stationary linear equation with a quadratic operator

$$L\psi = Q\psi,\ L = \partial_x^2 - A\partial_y^2 - B\partial_x\partial_y, \tag{1}$$

where A and B are diagonal matrices that have the same sizes with the matrix Q; here $Q \in m$.

A time evolution linear equation is given by

$$\psi_{T_N}=\sum_{i=0}^{N}S^{(N-i)}\partial_y^i\psi. \tag{2}$$

Equating coefficients of $\partial_{x_i}^i\partial_{x_j}^j\,(x_i,\,x_j=x,\,y;\,i+j=0,\,1,\,\ldots,\,N+1)$ in the commutator

$$[\partial_x^2-A\partial_y^2-B\partial_x\partial_y-Q,\,\partial_{T_N}-\sum_{i=0}^{N}S^{(N-i)}\partial_{x_i}^i\partial_{x_j}^j]=0 \tag{3}$$

can cause some evolution equations concerning $S^{(i)}(i=0,\,1,\,\ldots,\,N+1)$, A, B, Q, where A, B, and Q are the same-order matrices with the matrices $S^{(i)}$. In what follows, we consider the cases where $N=2$ and $N=3$ associated with the symmetric spaces.

By taking $N=2$ and by comparing the coefficients of $\partial_x^i\partial_y^j$, $i+j=0,\,1,\,2,\,3,\,4$, (3) leads to the following equations:

$$[A,\,S^{(0)}]=0, \tag{4}$$

$$[B,\,S^{(0)}]=0, \tag{5}$$

$$[A,\,S^{(1)}]+2AS_y^{(0)}+BS_x^{(0)}=0, \tag{6}$$

$$[B,\,S^{(1)}]=-Bs_y^{(0)}+2S_x^{(0)}+2S^{(0)}B_y, \tag{7}$$

$$[A,\,S^{(2)}]+2AS_y^{(1)}+AS_{yy}^{(0)}+BS_x^{(1)}+BS_{xy}^{(0)}+[Q,\,S^{(0)}]-S_{xx}^{(0)}=0, \tag{8}$$

$$[B,\,S^{(2)}]+BS_y^{(1)}-2S_x^{(1)}-S^{(1)}B_y-S^{(0)}B_{yy}=0, \tag{9}$$

$$[Q,\,S^{(1)}]+2AS_y^{(2)}+AS_{yy}^{(1)}+BS_x^{(2)}+BS_{yy}^{(1)}-S_{xx}^{(1)}-2S^{(0)}Q_y=0, \tag{10}$$

$$BS_y^{(2)}=2S_x^{(2)}, \tag{11}$$

$$Q_t=-AS_{yy}^{(2)}-BS_{xy}^{(2)}+[S^{(2)},\,Q]+S_{xx}^{(2)}+S^{(1)}Q_y+S^{(0)}Q_{yy}. \tag{12}$$

Assuming $A,\,B\in k$, $B=I$, we can take

$$S^{(0)}=\alpha A+\beta I, \tag{13}$$

where α and β are constants independent of x, y, and t. Equation (6) admits

$$S^{(1)}=\alpha A+\beta I, \tag{14}$$

and (8) gives

$$S^{(2)}=\alpha Q. \tag{15}$$

Equations (9–11) can be written as

$$\begin{cases}\alpha Q_x+\alpha[Q,\,A]=2\beta Q_y,\\ Q_y=2Q_x,\end{cases} \tag{16}$$

which indicates that there exist traveling-wave solutions for the function Q. Equation (12) can be written together with (16) as follows:

$$Q_t=\alpha Q_{xx}+(2\beta-\alpha)Q_{xy}+(\alpha A+\beta I)Q_y,$$

which is a (2+1)-dimensional generalized heat equation with parameters α and β. To our best knowledge, it is a new (2+1)-dimensional linear matrix equation.

Taking $N=3$, similar to the case of $N=2$, (3) admits some differential equations (see Appendix) and

$$Q_t=-AS_{yy}^{(3)}-BS_{xy}^{(3)}+[S^{(3)},\,Q]+S_{xx}^{(3)}+S^{(2)}Q_y+S^{(1)}Q_{yy}\\ +S^{(0)}Q_{yyy}. \tag{17}$$

Taking $B=I$, $S^{(0)}=A$, $A\in k$, the equations in the Appendix have the following special solutions:

$$\begin{cases}S^{(1)}=A,\;S^{(2)}=Q,\\ [A,S^{(3)}]=-2AQ_y-Q_y+[A,\,Q]+3AQ,\\ Q_y=2Q_x,\\ 2AS_y^{(3)}-2AQ_{yy}+S_x^{(3)}+Q_{xy}-Q_{xx}-2AQ_y=0,\\ S^{(3)}=2S_x^{(3)},\end{cases} \tag{18}$$

which implies that the solutions for Q, $S^{(3)}$ are traveling waves. Equation (17) becomes

$$Q_t+AS_{yy}^{(3)}+S_{xy}^{(3)}+[Q,\,S^{(3)}]-S_{xx}^{(3)}-AQ_{yy}-AQ_{yyy}-QQ_y=0. \tag{19}$$

Because $S^{(3)}\in g$, it can be decomposed into

$$S^{(3)}=S_0^{(3)}+S_+^{(3)}+S_-^{(3)},\;[A,\,S_0^{(3)}]=0.$$

Equation (18) can be presented as

$$[A,\,S_+^{(3)}]+[A,\,S_-^{(3)}]=-2AQ_y^+-2AQ_y^--Q_y^+-Q_y^-+[A,\,Q^+]\\ +[A,\,Q^-]+3AQ^++3AQ^-.$$

Therefore, we obtain the $m^{\pm}$ components of $S^{(3)}$ as follows:

$$S_+^{(3)}=\frac{1}{a}(-2AQ_y^+-Q_y^++[A,\,Q^+]+3AQ^+), \tag{20}$$

$$S_-^{(3)}=\frac{1}{a}(2AQ_y^-+Q_y^--[A,\,Q^-]-3AQ^-). \tag{21}$$

Hence, with the help of (20) and (21), $S^{(3)}$ can be written as

$$S^{(3)}=S_0^{(3)}+\frac{1}{a}[(-2A\partial_y-\partial_y+4A)(Q^+-Q^-)-(Q^+-Q^-)A]. \tag{22}$$

The k component of (19) reads

$$AS_{0,yy}^{(3)}+S_{0,xy}^{(3)}+[Q^+,\,S_-^{(3)}]+[Q^-,\,S_+^{(3)}]-S_{0,xx}^{(3)}=0. \tag{23}$$

Substituting (20) and (21) into (23) yields

$$(A\partial_y^2+\partial_x\partial_y-\partial_x^2)S_0^{(3)}=-\frac{1}{a}\{(2A\partial_y+\partial_y-4A)[Q^+,\,Q^-]$$
$$+[Q^+,\,Q^-]A+Q^+AQ^--Q^-AQ^+\}. \qquad (24)$$

The m^+ component of (19) presents

$$Q_t^++AS_{+,yy}^{(3)}+S_{+,xy}^{(3)}+[Q^+,\,S_0^{(3)}]-S_{+,xx}^{(3)}-AQ_{yy}^+-AQ_{yyy}^+=0;$$

that is,

$$Q_t^+=-\frac{1}{a}[-2A^2Q_{yyy}^+-AQ_{yy}^++A[A,\,Q_{yy}^+]+3A^2Q_{yy}^+-2AQ_{xyy}^+$$
$$-Q_{xyy}^++[A,\,Q_{xy}^+]+3AQ_{xy}^++2AQ_{xxy}^++Q_{xxy}^+-[A,\,Q_{xx}^+]-3AQ_{xx}^+]$$
$$+AQ_{yy}^++AQ_{yyy}^+-[Q^+,\,S_0^{(3)}]. \qquad (25)$$

Similarly, the m^- component of (19) reads

$$Q_t^-=\frac{1}{a}(-2A^2Q_{yyy}^--AQ_{yyy}^-+A[A,\,Q_{yy}^-]+3A^2Q_{yy}^-)$$
$$+\frac{1}{a}(-2AQ_{xyy}^--Q_{xyy}^-+[A,\,Q_{xy}^-]+3AQ_{xy}^-)$$
$$+\frac{1}{a}(2AQ_{xxy}^-+Q_{xxy}^--[A,\,Q_{xx}^-]-3AQ_{xx}^-)$$
$$-[Q^-,\,S_0^{(3)}]+AQ_{yy}^-+AQ_{yyy}^-, \qquad (26)$$

where $S_0^{(3)}$ is determined by (24). Obviously, (25) and (26) are all nonlocal, which are generalized KP equations with the same eigenvalues.

4 Applications of the Homogeneous Space

As applications of (4–12), we first consider a second-order flow by taking $S^{(0)}=C\in k$, and introducing an element $P\in m$ so that $Q=[A,\,P]$. Set $B=I$, then (4–12) can be solvable. In terms of (6) and (8), we get the solutions

$$S^{(1)}=C,\; S^{(2)}=[P,\,C].$$

Equation (11) gives

$$P_y=2P_x.$$

Therefore, (12) can be written as

$$Q_t=-A[P_{yy},\,C]-[P_{xy},\,C]+[[P,\,C],\,[A,\,P]]_m+CQ_y+CQ_{yy}. \qquad (27)$$

Equation (27) is a (2+1)-dimensional hyperbolic equation, which is different from the N-wave equation presented by Athorne and Fordy [25].

In what follows, we shall discuss the third-order flow by taking $B=I$, $S^{(0)}=S^{(1)}=A$, $A\in k$, and $S^{(2)}=Q$ and introducing $P\in m$ such that $Q=[A,\,P]$. According to the equations in Appendix, we have

$$Q_t=-AS_{yy}^{(3)}-S_{xy}^{(3)}+[S^{(3)},\,Q]+S^{(2)}Q_y+S^{(1)}Q_{yy}+S^{(0)}Q_{yyy}. \qquad (28)$$

In the following, we want to discover the k component and the $m^\pm$ component of the matrix equation (28).

It is easy to see that

$$S^{(3)}=S_0^{(3)}+(-2A\partial_y-\partial_y+3A)P+Q, \qquad (29)$$

Because

$$Q^\pm=\sum_j Q_j^\pm,\;[A,\,Q_j^\pm]=\pm a_jQ_j^\pm.$$

Thus, the k component of (31) is that

$$-(A\partial_y^2+\partial_x\partial_y)S_0^{(3)}+\sum_j\{[(-2A\partial_y-\partial_y+3A)P_j^++Q_j^+,\,Q_j^-]$$
$$+[(-2A\partial_y-\partial_y+3A)P_j^-+Q_j^-,\,Q_j^+]+Q_j^+Q_{j,y}^-+Q_j^-Q_{j,y}^+\}=0. \qquad (30)$$

It is easy to see that

$$[A,\,Q_j^+Q_{i,y}^-]=(a_j-a_i)Q_j^+Q_{i,y}^-.$$

Similar to the previously mentioned analysis, we can obtain the $m^\pm$ components of (28) as follows:

$$a_iQ_{i,t}^+=-A(-2A\partial_y-\partial_y+3A)Q_{i,yy}^++a_i[S_{0,i}^{(3)},\,Q_i^+]$$
$$-(-2A\partial_y-\partial_y+3A)Q_{i,xy}^+-a_iQ_{i,xy}^+$$
$$+[(-2A\partial_y-\partial_y+3A)Q_i^+,\,Q_i^+]_{m^+}+a_iQ_i^+Q_{i,y}^++a_iAQ_{i,yyy}^+, \qquad (31)$$

$$a_iQ_{i,t}^-=A(-2A\partial_y-\partial_y+3A)Q_{i,yy}^-+AQ_{i,yy}^-$$
$$+(-2A\partial_y-\partial_y+3A)Q_{i,xy}^-+P_{i,xy}^-+a_i[S_{0,i}^{(3)},\,Q_i^-]$$
$$-Q_i^-[A,\,Q_{i,y}^-]-A[A,\,P_{i,yy}^-]-A[A,\,P_{i,yyy}^-], \qquad (32)$$

where $S_0^{(3)}$ satisfies

$$(A\partial_y^2+\partial_x\partial_y)S_0^{(3)}=(I+3A)\sum_j[P_j^-,\,Q_j^+].$$

Equations (31) and (32) constitute a pair of new generalized KP equations, which they have various eigenvalues.

5 Applications of the Hermitian Symmetric Space

In this section, we shall introduce an isospectral Lax pair based on linear equations (1) and (2), whose integrability condition leads to a generalized nonlinear Schrödinger equation under the framework of the symmetric spaces, which can be expressed by the Reimann curvature tensors.

Consider the following isospectral problems:

$$\left(\partial_x^2+\frac{1}{4}B^2\partial_y^2-B\partial_x\partial_y\right)\psi=(\lambda D+Q)\psi, \tag{33}$$

$$\psi_t=P(x,y,t,\lambda)\psi\equiv P\psi, \tag{34}$$

where B, $D\in k$, and $P\in g$. The integrability condition of (33) and (34) reads

$$Q_t=\frac{1}{4}B^2P_{yy}-BP_{xy}+P_{xx}-[Q,P]-\lambda[D,P]\equiv TP-[Q,P]$$
$$-\lambda[D,P], \tag{35}$$

where $T=\frac{1}{4}B^2\partial_y^2-B\partial_x\partial_y+\partial_x^2$.

According to the definition of the symmetric space, (35) decouples to

$$\begin{cases} Q_t=TP_m-[Q,P_k]-\lambda[D,P_m],\\ TP_k=[Q,P_m],\end{cases} \tag{36}$$

where we have used $[D, P_k]=0$ because $k=C_g(D)$, $P=P_k+P_m$, $P_k\in k$, and $P_m\in m$. Because

$$P_k=T^{-1}[Q,P_m], \tag{37}$$

the first equation in (36) can be written as

$$Q_t=[T-\mathrm{ad}Q\bullet T^{-1}\mathrm{ad}Q-\lambda\mathrm{ad}D]P_m. \tag{38}$$

Assuming $P=\sum_{i=0}^{N}P^{(i)}\lambda^i$, the recursion operator appearing on the right of (38) is that

$$(T-\mathrm{ad}Q\bullet T^{-1}\mathrm{ad}Q)P_m^{(j)}=(\mathrm{ad}Q)P_m^{(j-1)}, \tag{39}$$

where $j=1, 2, \ldots, N-1$.

Substituting the P into the first equation in (36), we have

$$Q_t=TP_m^{(0)}-[Q,P_k^{(0)}]. \tag{40}$$

Because $Q\in m=\mathrm{span}\{e_{\pm a}\}$, we suppose

$$Q=\sum_{\alpha\in\theta^+}(q^\alpha e_\alpha+p^\alpha e_{-\alpha}). \tag{41}$$

Hence, we have derived from (41) and (42) that

$$\begin{cases} P_m^{(0)}=\frac{1}{a}\sum_{\alpha\in\theta^+}(Tq^\alpha e_\alpha+Tp^\alpha e_{-\alpha}),\\ P_k^{(0)}=\frac{1}{a}\sum_{\alpha,\beta\in\theta^+}T^{-1}(q^\alpha Tp^\beta-p^\beta Tq^\alpha)[e_\alpha,e_{-\beta}],\end{cases} \tag{42}$$

where we have used the property $[e_\alpha, e_\beta]=0$, $\forall\alpha,\beta\in\theta^+$.

We decompose (40) into the following form:

$$\sum_{\alpha\in\theta^+}q_t^\alpha e_\alpha=\frac{1}{a}\Big(\sum_{\alpha\in\theta^+}T(Tq^\alpha)e_\alpha-\sum_{\beta,\gamma,\delta\in\theta^+}q^\beta T^{-1}(q^\gamma Tp^\delta-p^\gamma Tq^\delta)[e_\beta,$$
$$[e_\gamma,e_{-\delta}]]\Big), \tag{43}$$

$$\sum_{\alpha\in\theta^+}p_t^\alpha e_{-\alpha}=\frac{1}{a}\Big(\sum_{\alpha\in\theta^+}T(Tp^\alpha)e_{-\alpha}-\sum_{\beta,\gamma,\delta\in\theta^+}p^\beta T^{-1}(q^\gamma Tp^\delta-p^\gamma Tq^\delta)$$
$$[e_{-\beta},[e_\gamma,e_{-\delta}]]\Big). \tag{44}$$

According to the definition of the Riemann curvature tensor, (43) and (44) can be expressed by the Riemann curvature tensors as follows:

$$\begin{cases} aq_t^\alpha=T(Tq^\alpha)-\sum_{\beta,\gamma,\delta\in\theta^+}R^\alpha_{\beta\gamma-\delta}q^\beta T^{-1}(q^\gamma Tp^\delta-p^\gamma Tq^\delta),\\ ap_t^\alpha=T(Tp^\alpha)-\sum_{\beta,\gamma,\delta\in\theta^+}R^{-\alpha}_{-\gamma\delta-\beta}p^\beta T^{-1}(q^\gamma Tp^\delta-p^\gamma Tq^\delta),\end{cases} \tag{45}$$

which is the generalized nonlinear Schrödinger equation. If we take $T=\partial_x$, (45) reduces to the result of Fordy and Kulish [28].

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Appendix

In the case of $N=3$, the admitting equations of (3) are as follows:

$$[A,S^{(0)}]=0,\ [B,S^{(0)}]=0,$$
$$[A,S^{(1)}]+2AS_y^{(0)}+BS_x^{(0)}=0,$$
$$[B,S^{(1)}]+BS_y^{(0)}-2S_x^{(0)}-3S^{(0)}B_y=0,$$
$$[A,S^{(2)}]+[Q,S^{(0)}]+2AS_y^{(1)}+AS_{yy}^{(0)}+BS_x^{(1)}+BS_{xy}^{(0)}-S_{xx}^{(0)}=0,$$
$$[B,S^{(2)}]+BS_y^{(1)}-2S_x^{(1)}-2S^{(1)}B_y-3S^{(0)}B_{yy}=0,$$
$$[A,S^{(3)}]+[Q,S^{(1)}]+2AS_y^{(2)}+AS_{yy}^{(1)}+BS_y^{(2)}+BS_{xy}^{(1)}-S_{xx}^{(1)}-3S^{(0)}Q=0,$$
$$B_t+[B,S^{(3)}]+BS_y^{(2)}-2S_x^{(2)}-S^{(1)}B_{yy}-S^{(0)}B_{yyy}=0,$$
$$2AS_y^{(3)}+AS_{yy}^{(2)}+BS_x^{(3)}+BS_{xy}^{(2)}+[Q,S^{(2)}]-S_{xx}^{(2)}-2S^{(1)}Q_y-3S^{(0)}Q_{yy}=0,$$
$$BS_y^{(3)}=2S_x^{(3)},$$

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