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Nonlocal Symmetries, Explicit Solutions, and Wave Structures for the Korteweg–de Vries Equation

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Abstract: From the known Lax pair of the Korteweg–de Vries (KdV) equation, the Lie symmetry group method is successfully applied to find exact invariant solutions for the KdV equation with nonlocal symmetries by introducing two suitable auxiliary variables. Meanwhile, based on the prolonged system, the explicit analytic interaction solutions related to the hyperbolic and Jacobi elliptic functions are derived. Figures show the physical interaction between the cnoidal waves and a solitary wave.

Keywords: Explicit Solution; KdV Equation; Lax Pair; Nonlocal Symmetry; Wave Structure.

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1 Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in various fields of nature science, especially nonlinear physics [1–3]. Since the source of a nonlinear problem is closely linked with reality, it contains rich in content and broad, and the corresponding nonlinear model is also complicated. Because of the complexity of the nonlinear system, there are still a lot of nonlinear equations for which the exact solutions are not easy to be obtained. Therefore, how to get exact solutions or more new accurate analytical solutions for a given nonlinear equation has become an important topic to scientists. In recent decades, many effective methods of solving differential equations have been constructed [2–4]. Among these methods, in 1989, Clarkson and Kruskal [5] proposed a simple direct method (CK direct method) which is different from the Lie group method, since it can find all the possible similarity reductions without using any group theory. Soon after, Lou optimised the above-mentioned approach and presented the improved CK method [6, 7]. Most recently, Lou and Ma [8] applied the direct method of symmetry transformation group for Lax integrable system in place of the traditional method of solving symmetry transformation group.

As we all know, the nonlocal symmetry has closed relation with the integrable model and is beneficial to the enlarge the class of the symmetry that provides the chance of obtaining the exact solution. However, the nonlocal symmetry canot be used to construct solution directly. In other words, only nonlocal symmetry is not enough unless we localised nonlocal symmetry into the local ones with the closed prolonged system. Very recently, from the nonlocal symmetry related to Darboux transformation, Lou et al. [9] obtained the explicit analytic interaction solutions between cnoidal waves and solitary wave for the well-known KdV equation. Furthermore, by studying the new exact solutions of the equations such as mKdV equation, ANKS system and Boussinesq equation, Xin [10–12] proved the effectiveness of this proposed method.

In this article, the structure of the layout is as follows: In Section 2, under the infinitesimal transformation of variable u, a vector symmetry which contains the classical Lie point symmetry and the nonlocal symmetry is derived directly. In Section 3, with the aid of the Lax pair of KdV equation, the nonlocal symmetry is localised in the properly prolonged system by introducing two suitable auxiliary variables. As a result, the finite transformation and the general Lie point symmetry of the prolonged system can be obtained. In Section 4, on the basis of the prolonged system, some new explicit solutions are constructed through similarity reductions. The different wave structure pictures give a visual representation of these solutions. Section 5 is a simple conclusion of this article.

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2 Nonlocal Symmetry of the KdV Equation

The KdV equation in canonical form can be expressed as (1, 2)

$$u_t+6uu_x+u_{xxx}=0, \tag{1}$$

which is widely regarded as a good model for the description of weakly nonlinear long waves in many branches of physics and engineering field. In (1), $u\equiv u(x, t)$ is an appropriate field variable, t is time, and x is a space coordinate in the direction of propagation. This well-known model describes how waves evolve under the competing but comparable effects of weak nonlinearity and weak dispersion [3]. In fact, if it is supposed that x-derivatives scale as ϵ where ϵ is the small parameter characterising long waves (i.e. typically the ratio of a relevant background length scale to a wavelength scale), then the amplitude scales as ϵ^2 and the time evolution takes place on a scale of ϵ^{-3}. Based on the Bell polynomials scheme [4], the Lax pair of KdV equation (1) has

$$\psi_{xx}+(u-\lambda)\psi=0, \tag{2}$$

$$\psi_t+\psi_{xxx}+3(u+\lambda)\psi_x-\mu\psi=0, \tag{3}$$

under the compatibility condition $\psi_{xx,t}=\psi_{t,xx}$.

Under the transformation (with the infinitesimal parameter ε)

$$u\to u+\varepsilon\sigma_1, \tag{4}$$

the symmetry σ_1 of the KdV equation (1) is a solution of the following linearised equation

$$\sigma_{1,t}+6(u\sigma_1)_x+\sigma_{1,xxx}=0. \tag{5}$$

Supposing the symmetry σ_1 with the auxiliary variables ψ and its one-order partial derivative ψ_x

$$\sigma_1=\xi(x,t,u,\psi,\psi_x)u_x+\tau(x,t,u,\psi,\psi_x)u_t-U(x,t,u,\psi,\psi_x), \tag{6}$$

then substituting (6) into (5) and solving the determining equations, the vector symmetry can be derived

$$\sigma_1=\left(\frac{c_1x}{2}-6c_3t-c_5\right)u_x+\left(\frac{3c_1t}{2}-c_4\right)u_t+c_1u+c_2\psi\psi_xe^{-2\mu t}+c_3, \tag{7}$$

where $c_i(i=1, \dots, 5)$ are five arbitrary constants. Equation (7) contains the classical Lie point symmetry $\sigma_{11}=\left(\frac{c_1x}{2}-6c_3t-c_5\right)u_x+\left(\frac{3c_1t}{2}-c_4\right)u_t+c_1u+c_3$ and the nonlocal symmetry $\sigma_{12}=c_2\psi\psi_x\mathrm{e}^{-2\mu t}$.

3 Localisation of the Nonlocal Symmetry

For the vector symmetry (7), letting $c_1=c_3=c_4=c_5=0$ and $c_2=1$, we have the nonlocal symmetry

$$\sigma_1=\psi\psi_xe^{-2\mu t}. \tag{8}$$

At the same time, the linearised equations

$$\sigma_{2,xx}+\sigma_2u+\sigma_1\psi-\lambda\sigma_2=0, \tag{9}$$

$$\sigma_{2,t}+\sigma_{2,xxx}+3\sigma_1\psi_x+3\sigma_{2,x}u+3\lambda\sigma_{2,x}-\mu\sigma_2=0, \tag{10}$$

are the direct results of (2) and (3), respectively, under the symmetry transformation $\psi\to\psi+\varepsilon\sigma_2$.

In order to obtain the localisation of the nonlocal symmetry (7), taking $\psi_1=\psi_x$ and $p=\int\psi^2\mathrm{d}x$, and the corresponding linearised equations are

$$\sigma_{2,x}-\sigma_3=0, \tag{11}$$

$$\sigma_{4,x}-2\sigma_2\psi=0, \tag{12}$$

under the transformation $\psi_1\to\psi_1+\varepsilon\sigma_3$ and $p\to p+\varepsilon\sigma_4$. The equation

$$\sigma_{4,t}-2\mu\sigma_4+12\lambda\sigma_2\psi-8\sigma_3\psi_1+2\sigma_2\psi_{1,x}+2\sigma_{3,x}\psi=0 \tag{13}$$

is a direct result from the Lax pair of KdV equation (1).

Therefore, the linear system (5), (9)–(13) has a solution

$$\sigma_1=\psi\psi_1e^{-2\mu t},\ \sigma_2=-\frac{1}{4}\psi pe^{-2\mu t},\ \sigma_3=-\frac{1}{4}(\psi^3+\psi_1p)e^{-2\mu t},$$
$$\sigma_4=-\frac{1}{4}p^2e^{-2\mu t}. \tag{14}$$

The above-mentioned solution (14) indicates that the nonlocal symmetry is localised in the properly prolonged system with the Lie point symmetry vector of (14), namely

$$\mathbf{V}_1=\psi\psi_1e^{-2\mu t}\frac{\partial}{\partial u}-\frac{1}{4}\psi pe^{-2\mu t}\frac{\partial}{\partial\psi}-\frac{1}{4}(\psi^3+\psi_1p)e^{-2\mu t}$$
$$\frac{\partial}{\partial\psi_1}-\frac{1}{4}p^2e^{-2\mu t}\frac{\partial}{\partial p}. \tag{15}$$

Using the following initial condition

$$\begin{aligned}
&\frac{\mathrm{d}\hat{u}(\varepsilon)}{\mathrm{d}\varepsilon}=\psi(\varepsilon)\psi_1(\varepsilon)\mathrm{e}^{-2\mu t},\ \hat{u}(0)=u,\\
&\frac{\mathrm{d}\hat{\psi}(\varepsilon)}{\mathrm{d}\varepsilon}=-\frac{1}{4}\psi(\varepsilon)p(\varepsilon)\mathrm{e}^{-2\mu t},\ \hat{\psi}(0)=\psi,\\
&\frac{\mathrm{d}\hat{\psi}_1(\varepsilon)}{\mathrm{d}\varepsilon}=-\frac{1}{4}(\psi^3(\varepsilon)+\psi_1(\varepsilon)p(\varepsilon))\mathrm{e}^{-2\mu t},\ \hat{\psi}_1(0)=\psi_1,\\
&\frac{\mathrm{d}\hat{p}(\varepsilon)}{\mathrm{d}\varepsilon}=-\frac{1}{4}p^2(\varepsilon)\mathrm{e}^{-2\mu t},\ \hat{p}(0)=p,
\end{aligned} \tag{16}$$

the corresponding finite transformation reads

$$\hat{u}=u-\frac{2(\varepsilon\psi^3 e^{-2\mu t}-(8+2\varepsilon p e^{-2\mu t})\psi_1)\varepsilon\psi e^{-2\mu t}}{(4+\varepsilon p e^{-2\mu t})^2},$$

$$\hat{\psi}=\frac{4\psi}{4+\varepsilon p e^{-2\mu t}},$$

$$\hat{\psi}_1=\frac{4\psi_1}{4+\varepsilon p e^{-2\mu t}}-\frac{4\varepsilon\psi^3 e^{-2\mu t}}{(4+\varepsilon p e^{-2\mu t})^2},$$

$$\hat{p}=\frac{4p}{4+\varepsilon p e^{-2\mu t}}. \tag{17}$$

Here, $\{u, \psi, \psi_1, p\}$ is a solution of the prolonged system (1) with $\psi_1=\psi_x$ and $p=\int\psi^2 \mathrm{d}x$.

Under the transformation

$$\{x, t, u, \psi, \psi_1, p\}\to\{x+\varepsilon\xi,\ t+\varepsilon\tau,\ u+\varepsilon U,\ \psi+\varepsilon\Psi,\ \psi_1+\varepsilon\Psi_1,\ p+\varepsilon P\}, \tag{18}$$

the general Lie point symmetry of the prolonged system is

$$\mathbf{V}_2=\xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+U\frac{\partial}{\partial u}+\Psi\frac{\partial}{\partial\psi}+\Psi_1\frac{\partial}{\partial\psi_1}+P\frac{\partial}{\partial p}, \tag{19}$$

or has the following form

$$\sigma_1=\xi u_x+\tau u_t-U(x,t,u,\psi,\psi_1,p)\equiv\xi u_x+\tau u_t-U,$$

$$\sigma_2=\xi\psi_x+\tau\psi_t-\Psi(x,t,u,\psi,\psi_1,p)\equiv\xi\psi_x+\tau\psi_t-\Psi,$$

$$\sigma_3=\xi\psi_{1,x}+\tau\psi_{1,t}-\Psi_1(x,t,u,\psi,\psi_1,p)\equiv\xi\psi_{1,x}+\tau\psi_{1,t}-\Psi_1,$$

$$\sigma_4=\xi p_x+\tau p_t-P(x,t,u,\psi,\psi_1,p)\equiv\xi p_x+\tau p_t-P. \tag{20}$$

With the help of computer programs, such as Maple, the solution can be obtained

$$\xi=-\frac{c_1 x}{2}-6c_1\lambda t+c_5,$$

$$\tau=-\frac{3c_1 t}{2}+c_4,$$

$$U=c_1(u-\lambda)+c_2\psi\psi_1 e^{-2\mu t},$$

$$\Psi=\left(c_6-\frac{3c_1\mu t}{2}\right)\psi-\frac{c_2 p\psi}{4}e^{-2\mu t},$$

$$\Psi_1=\left(c_6+\frac{c_1}{2}-\frac{3c_1\mu t}{2}-\frac{c_2 p}{4}e^{-2\mu t}\right)\psi_1-\frac{c_2\psi^3}{4}e^{-2\mu t},$$

$$P=\left(2c_6-\frac{c_1}{2}-3c_1\mu t\right)p+c_3 e^{2\mu t}-\frac{c_2 p^2}{4}e^{-2\mu t}. \tag{21}$$

From (21), one can obtain the following six operators

$$M_1=-\left(\frac{x}{2}+6\lambda t\right)\frac{\partial}{\partial x}-\frac{3t}{2}\frac{\partial}{\partial t}+(u-\lambda)\frac{\partial}{\partial u}-\frac{3\mu t}{2}\psi\frac{\partial}{\partial\psi}+\frac{1-3\mu t}{2}\psi_1\frac{\partial}{\partial\psi_1}-\frac{1+6\mu t}{2}p\frac{\partial}{\partial p},$$

$$M_2=\psi\psi_1 e^{-2\mu t}\frac{\partial}{\partial u}-\frac{p\psi}{4}e^{-2\mu t}\frac{\partial}{\partial\psi}-\left(\frac{p\psi_1+\psi^3}{4}\right)e^{-2\mu t}\frac{\partial}{\partial\psi_1}-\frac{p^2}{4}e^{-2\mu t}\frac{\partial}{\partial p},$$

$$M_3=e^{2\mu t}\frac{\partial}{\partial p},\ M_4=\frac{\partial}{\partial t},\ M_5=\frac{\partial}{\partial x},\ M_6=\psi\frac{\partial}{\partial\psi}+\psi_1\frac{\partial}{\partial\psi_1}+2p\frac{\partial}{\partial p}. \tag{22}$$

Hence, we obtain the commutator table listed in Table 1 with the (i, j)-th entry indicating $[M_i, M_j]$ according to the commutator operators $[M_p, M_q]=M_pM_q-M_qM_p$.

4 Similarity Reduction and the Wave Structures of the KdV Equation

Consider the following characteristic equation

$$\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}u}{U}=\frac{\mathrm{d}\psi}{\Psi}=\frac{\mathrm{d}\psi_1}{\Psi_1}=\frac{\mathrm{d}p}{P}, \tag{23}$$

where ξ, τ, U, Ψ, Ψ_1, and P are dedicated by (21). Without loss of generality, taking $c_1=0$, $c_2=-4k_1$, $c_3=-\frac{A^2-c_6^2}{k_1}$, $c_4=1$, $c_5=k$ and $\mu=0$, one has

Table 1: Lie bracket.

$[M_i, M_j]$	M_1	M_2	M_3	M_4	M_5	M_6
M_1	0	$-\frac{1}{2}M_2$	$\frac{1}{2}M_3$	$\frac{3}{2}M_4+6\lambda M_5$	$\frac{1}{2}M_5$	0
M_2	$\frac{1}{2}M_2$	0	$\frac{1}{4}M_6$	0	0	$-2M_2$
M_3	$-\frac{1}{2}M_3$	$-\frac{1}{4}M_6$	0	0	0	$2M_3$
M_4	$-\frac{3}{2}M_4-6\lambda M_5$	0	0	0	0	0
M_5	$-\frac{1}{2}M_5$	0	0	0	0	0
M_6	0	$2M_2$	$-2M_3$	0	0	0

$$
\begin{aligned}
u=&\frac{2k_1^2G(X)^4\operatorname{sech}^2(A(t+F(X)))}{A^2}\\
&-\frac{4k_1G(X)G_1(X)\tanh(A(t+F(X)))}{A}+F_1(X),\\
\psi=&G(X)\operatorname{sech}(A(t+F(X))),\\
\psi_1=&\left(\frac{k_1G(X)^3\tanh(A(t+F(X)))}{A}+G_1(X)\right)\operatorname{sech}(A(t+F(X))),\\
p=&-\frac{c_6+A\tanh[A(t+F(X))]}{k_1},
\end{aligned}\tag{24}
$$

where $X=x-kt$, A, k, k_1, and c_6 are arbitrary constants. Substituting (24) into the prolonged system, one can derive

$$
\begin{aligned}
F_X(X)=&-\frac{k_1G^2(X)}{A^2},\ G_1(X)=G_X(X),\ F_1(X)\\
=&\frac{-A^2G(X)_{XX}-k_1^2G(X)^5+\lambda A^2G(X)}{A^2G(X)},
\end{aligned}\tag{25}
$$

$$
\begin{aligned}
G(X)=&\frac{1}{\sqrt{a_0+a_2\operatorname{sn}^2(cX,m)}},\\
G_1(X)=&-\frac{a_2c\operatorname{sn}(cX,m)\operatorname{cn}(cX,m)\operatorname{dn}(cX,m)}{(a_0+a_2\operatorname{sn}^2(cX,m))^{\frac{3}{2}}},\\
F(X)=&-\frac{k_1\operatorname{EllipticPi}\left(\operatorname{sn}(cX,m),-\frac{a_2}{a_0},m\right)}{ca_0A^2},\\
F_1(X)=&-\frac{a_0a_2^2(a_0+a_2)(a_2+a_0m^2)(k_2-8\lambda)+4a_0^2K_0^2(3a_0m^2+a_2m^2+a_2)}{2a_0(a_0+a_2)(a_0m^2+a_2)(a_0+a_2\operatorname{sn}^2(cX,m))^2}\operatorname{sn}^4(cX,m)\\
&+\frac{2a_0^2a_2(a_0+a_2)(a_2+a_0m^2)(k_2-8\lambda)+4a_2K_0^2(3m^2a_0^2-a_2^2)}{2a_0(a_0+a_2)(a_0m^2+a_2)(a_0+a_2\operatorname{sn}^2(cX,m))^2}\operatorname{sn}^2(cX,m)\\
&\frac{a_0^3(a_0+a_2)(a_2+a_0m^2)(k_2-8\lambda)+4K_0^2a_0a_2(a_0+a_2)+4K_0^2a_0^2m^2(2a_0+a_2)}{2a_0(a_0+a_2)(a_0m^2+a_2)(a_0+a_2\operatorname{sn}^2(cX,m))^2},
\end{aligned}\tag{30}
$$

and $G(X)$ satisfies

$$
\begin{aligned}
&2k_1A^2G(X)G(X)_{XX}-4k_1A^2G(X)_X^2\\
&\quad-2k_1G(X)^6+k_1A^2(6\lambda-k)G(X)^2-A^4=0.
\end{aligned}\tag{26}
$$

One can simplify (26) using $\frac{1}{\sqrt{W(X)}}$ to replace $G(X)$, and the reduced equation is

$$
\begin{aligned}
&2k_1A^2W(X)W(X)_{XX}-k_1A^2W(X)_X^2+2A^4W(X)^3\\
&\quad+2k_1A^2(k-6\lambda)W(X)^2+4k_1^3=0.
\end{aligned}\tag{27}
$$

Equation (27) can be transformed to

$$
W_X^2(X)=-\frac{A^2W^3(X)}{k_1}+(12\lambda-2k)W^2(X)+c_0W(X)+\frac{4k_1^2}{A^2}\tag{28}
$$

or

$$
W_X^2(X)=K_3W^3(X)+K_2W^2(X)+K_1W(X)+4K_0^2,\tag{29}
$$

where $K_3=-\frac{A^2}{k_1}$, $K_2=12\lambda-2k$, $K_1=c_0$, $K_0=-\frac{k_1}{A}$.

One know that the general solution of (29) can be written out in terms of Jacobi elliptic functions. Hence, the solution expressed by (24) is just the explicit exact interaction between the soliton and the cnoidal periodic wave.

We assume $W(X)=a_0+a_2\operatorname{sn}^2(cX,m)$ as a simple solution of (29). Then

where c satisfies $c^2=-\frac{a_2K_0^2}{a_0(a_0^2m^2+a_0a_2m^2+a_0a_2+a_2^2)}$. Here the terms sn, cn, and dn are three usual Jacobian elliptic functions with modulus m, whereas a_0 and a_2 are two independent constants. The incomplete elliptic integral EllipticPi is defined by EllipticPi$(z,\nu,k)=\int_0^z\frac{1}{(1-\nu t^2)\sqrt{1-t^2}\sqrt{1-k^2t^2}\mathrm{d}t}$. For the term K_0,

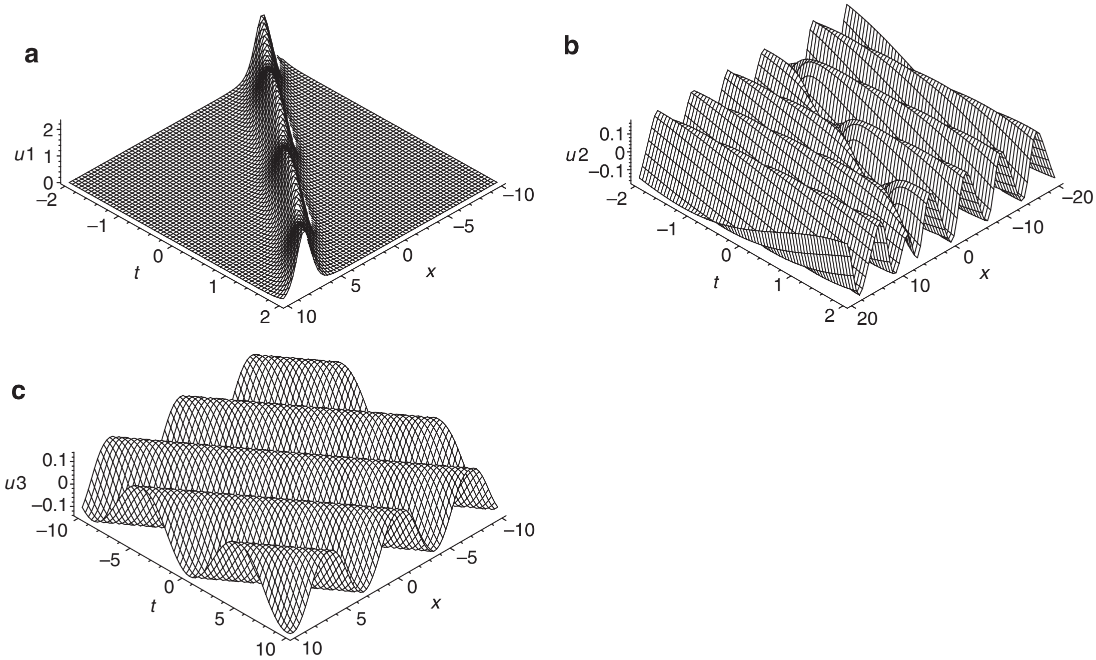

Figure 1: Evolution of the wave distribution solution as a function of propagation distance *x* and time *t*. (a–c) show the corresponding variables u_1, u_2, and u_3 expressed by (33)–(35), respectively.

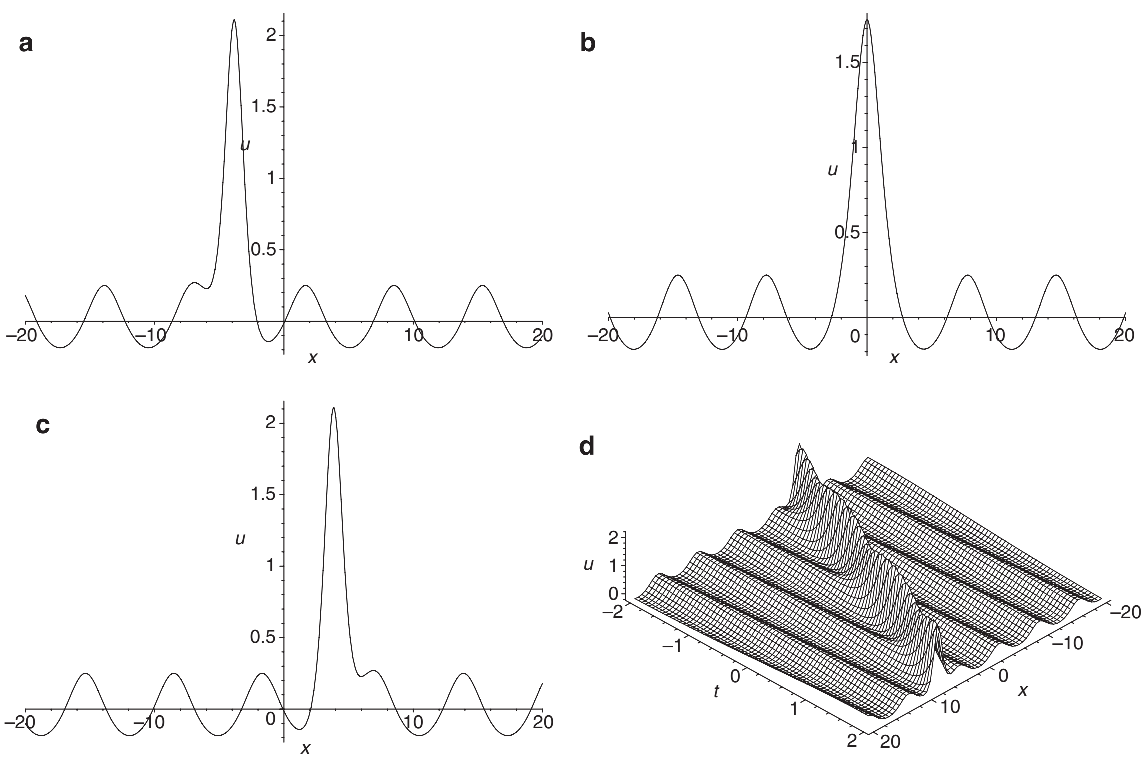

Figure 2: Interaction graphs to the KdV equation. (a–c) show the time evolution view at $t=-1$, 0 and 1, respectively. (d) Evolution of the wave distribution solution (32) as a function of propagation distance *x* and time *t*.

it is connected with K_1, K_2, K_3, and the relation reads as follows

$$\begin{aligned} K_1&=-\frac{4(3a_0^2m^2+2a_0a_2m^2+2a_0a_2+a_2^2)K_0^2}{a_0(a_0^2m^2+a_0a_2m^2+a_0a_2+a_2^2)},\\ K_2&=\frac{4(3a_0m^2+a_2m^2+a_2)K_0^2}{a_0(a_0^2m^2+a_0a_2m^2+a_0a_2+a_2^2)},\\ K_3&=-\frac{4m^2K_0^2}{a_0(a_0^2m^2+a_0a_2m^2+a_0a_2+a_2^2)}. \end{aligned} \tag{31}$$

Therefore, the formula of (24) can be rewritten

$$\begin{aligned} u&=\frac{2}{A^2}k_1^2G^4(X)\mathrm{sech}^2[A(t+F(X))]\\ &\quad-\frac{4}{A}k_1G(X)G_1(X)\tanh[A(t+F(X))]+F_1(X)\\ &\equiv u_1+u_2+u_3, \end{aligned} \tag{32}$$

with

$$u_1=\frac{2}{A^2}k_1^2G^4(X)\mathrm{sech}^2[A(t+F(X))], \tag{33}$$

$$u_2=-\frac{4}{A}k_1G(X)G_1(X)\tanh[A(t+F(X))], \tag{34}$$

$$u_3=F_1(X). \tag{35}$$

In order to study the structure of this solution, some figures which corresponds to soliton solutions found above are given. Figure 1 shows the expression (33–35), respectively. Figure 2 shows the evolution of intensity profile u of soliton solution (32) of KdV equation, where the parameters are fixed as $\{m, \lambda, a_0, a_2, K_0, K_1, K_2, K_3, c\}=\{0.8, 1, 2, -0.3333, 1.7961, -22.9169, -13.4666, -2.6170, 0.5837\}$. One can see that the component u exhibits a soliton propagating on a cnoidal wave background. In fact, it is of interest to study such types of analytical solutions. As we know, solitary waves and cnoidal periodic waves are two typical nonlinear waves widely appearing in many physical fields such as ocean. Here we mainly devote to obtain the exact form of soliton–cnoidal waves solution from the original model equation. It is expected to the realistic physical interpretation and experiment observation. For instance, in a recent work [13], the oblique propagation of ion-acoustic soliton–cnoidal waves was reported in a magnetised electron-positron-ion plasma with superthermal electrons. For this kind of soliton–cnoidal waves solution, it is shown that every peak of a cnoidal wave elastically interacts with a usual soliton except for some phase shifts. The authors discussed the influence of the electron superthermality, positron concentration, and magnetic field obliqueness on the soliton–cnoidal wave in detail.

5 Summary and Conclusion

Based on the original Lie symmetry method and some changes of the assumption form of the symmetry, we derive not only the local symmetry, but the nonlocal symmetry which is just one of the destinations of this article. With the introduction of two auxiliary variables $\psi_1=\psi_x$, $p=\int\psi^2\mathrm{d}x$, we localise the nonlocal symmetry into the Lie point ones with the closed prolonged system successfully. Starting from the prolonged system, the explicit analytic interaction solutions related to the hyperbolic and Jacobi elliptic functions are obtained. The physical interaction between the cnoidal waves and a solitary wave is illustrated through the image simulation.

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